

Dimer Statistics and Phase Transitions

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After the introduction of the concept of lattice graph and a brief discussion of its role in the theory of the Ising model, a related combinatorial problem is discussed, namely that of the statistics of non-overlapping dimers, each occupying two neighboring sites of a lattice graph. It is shown that the configurational partition function of this system can be expressed in terms of a Pfaffian, and hence calculated explicitly, if the lattice graph is planar and if the dimers occupy all lattice sites. By the examples of the quadratic and the hexagonal lattice, it is found that the dimer system may show a phase transition similar to that of a two-dimensional Ising model, or one of a different nature, or no transition at all, depending on the activities of various classes of bonds. The Ising problem is then shown to be equivalent to a generalized dimer problem, and a rederivation, of Onsager's expression for the Ising partition function of a rectangular lattice graph is sketched on the basis of this equivalence.

I. INTRODUCTION

MANY aspects of the Ising model have been investigated by exact methods¹ since the publication of Onsager's solution of the two-dimensional problem.² In addition to their direct significance for the theory of phase transitions, the results thus obtained were important as a test for the various approximation methods which have been developed for the study of cooperative systems. It was found that approximate theories, though describing the overall properties of the model in a rather satisfactory way, fail to yield an adequate description of its most characteristic behavior, viz. that in the neighborhood of the transition temperature. This knowledge has, in turn, led to an increased interest in exact solutions of other combinatorial problems on crystal lattices. By studying a variety of problems one might hope to get some insight into such questions as: (a) Why, by the methods developed thus far, can the Ising problem be exactly solved for a one-dimensional lattice in an arbitrary magnetic field and for two-dimensional lattices with nearest neighbor interaction in the absence of a magnetic field, but not for any other system? (b) What determines the mathematical form of the singularities in the properties of the Ising model, and can the behavior at the singular point be predicted if the lattice and the interactions are known?

(a) The first question has been answered only partially thus far. It is easy to see that the applicability of the existing methods depends not only on the geometrical arrangement of the lattice sites

(and thus, e.g., the dimensionality, in the usual, geometrical sense), but also on the way in which the sites combine to interacting pairs. On the other hand, it is obvious that when we represent the resulting structure by a *graph*, connecting any two interacting sites by a line, only the topological and not the metrical properties of this *lattice graph* are relevant.³ We now know the determining property, at least for the success of the determinantal approach to the Ising problem introduced by Kac and Ward⁴: the partition function of a system can be expressed in terms of a single determinant only if its lattice graph is *planar*, i.e. can be imbedded (drawn without intersecting lines) in a plane. For lattice graphs which can be imbedded in a torus but not in a plane, one needs four determinants to express the partition function; more generally, for a graph which can be drawn on a surface of genus g ,⁵ but not on one of genus $g - 1$ (and which is, therefore, called⁶ a *graph of genus g*), one needs 4^g determinants. In the limit $g \rightarrow \infty$, application of the method is impossible even in principle. It can easily be seen that the lattice graphs for which the Ising problem could not be solved are all of infinite genus. We thus see that the genus of the lattice graph

³ A system in a magnetic field is equivalent to a system with a more complicated lattice graph in zero field; it is this graph which will then be called the lattice graph of the original system.

⁴ M. Kac and J. C. Ward, Phys. Rev. **88**, 1332 (1952); R. B. Potts and J. C. Ward, Progr. Theoret. Phys. (Kyoto) **13**, 38 (1955); S. Sherman, J. Math. Phys. **1**, 202 (1960).

⁵ The *genus g of a surface* is the maximum number of nonintersecting closed curves which one can draw on the surface without disconnecting it. A plane and a sphere have $g = 0$, a torus $g = 1$, etc.

⁶ D. König, *Theorie der endlichen und unendlichen Graphen* (Chelsea Publishing Company, New York, 1936, and 1950), p. 198.

¹ For a general review, see C. Domb, Advan. Phys. **9**, 149 (1960).

² L. Onsager, Phys. Rev. **65**, 117 (1944).

plays a dominant role in the solubility of the Ising problem.

On the other hand, it is well known that several other combinatorial problems defined with respect to a lattice graph can be solved for *arbitrary graphs*, both planar and nonplanar. Among these we mention the problems of random walks on a lattice,⁷ the distribution of electric current and potential in an infinite resistance network,⁸ and the enumeration of trees, or branched polymers, in a lattice graph.⁹

It might be a real step forward if we understood clearly *why* in the Ising model the genus of the lattice graph plays such an important part, whereas in other problems it does not.

(b) There are various reasons for conjecturing that the behavior in the critical region is primarily determined by the dimensionality of the lattice graph.¹ The dimensionality $\dim L$ of an infinite lattice graph L can be defined with the aid of the asymptotic behavior, for $n \rightarrow \infty$, of the number N_n of lattice sites whose shortest path to a fixed site consists of n bonds of L : $\dim L = 1 + \lim_{n \rightarrow \infty} (\ln N_n / \ln n)$. It is not necessarily equal to the geometrical dimensionality of the underlying point lattice. Thus, if in a two-dimensional, quadratic point lattice only horizontal neighbors are connected by a bond, the resulting (nonconnected) lattice graph has dimensionality 1. In those cases where exact solutions are available, they generally bear out the conjecture; numerical calculations seem to support its validity for three-dimensional lattices, and a comparison with other combinatorial problems shows that analogous quantities have singularities of the same mathematical form. However, no general relations between dimensionality and critical behavior of the Ising model have been rigorously established thus far.

In order to contribute to the discussion on these two points, we consider in this paper a cooperative problem which has recently been solved for a certain class of systems. It concerns the statistics of dimers on a crystal lattice, when each dimer occupies two neighboring lattice sites, and no dimers overlap. This "dimer problem" arises in the theory of liquids consisting of molecules of different size,¹⁰ in the cell-cluster theory of the liquid state,¹¹ and in the

theory of adsorption of diatomic molecules.¹² We concentrate mainly on the case of the dimers completely filling the lattice. A method for solving this problem for a rather general class of lattice graphs is developed in Sec. 2; as no use is made of the regular structure (periodicity) of a lattice graph, the terms "graph" and "lattice graph" are interchangeable in this section. In Secs. 3 and 4, two applications to particular lattices are given, and the phase transitions occurring there are discussed. A connection between the dimer problem and the Ising problem is derived in Sec. 5. The case of a quadratic lattice has been dealt with in detail in an earlier paper;¹³ an alternative approach, leading to results identical to ours, has been independently developed by Temperley and Fisher.¹⁴

II. DIMER CONFIGURATIONS AND PFAFFIANS

Consider a *lattice graph* L consisting of N sites (points, vertices), connected by R bonds (lines, edges), which are divided into several classes, C_1, C_2, \dots, C_h (e.g. the "horizontal" and "vertical" bonds in a simple quadratic lattice). *Dimers* (figures consisting of two points linked by a line) can be placed on L so as to occupy two sites connected by a bond. A (*close-packed*) *dimer configuration* on L is an arrangement of dimers on L such that all sites are singly occupied. Obviously no such arrangement is possible if the number of sites is odd; therefore let N be even, $N = 2M$. Let $g_L(N_1, N_2, \dots, N_h)$ be the number of dimer configurations occupying N_1 bonds from the class C_1 , N_2 from C_2 , \dots , N_h from C_h ($N_1 + \dots + N_h = \frac{1}{2} N = M$). We want to derive an expression for the *generating function for dimer configurations on L* ,

$$Z_L(z_1, z_2, \dots, z_h) = \sum_{N_1, \dots, N_h} g_L(N_1, \dots, N_h) z_1^{N_1} \dots z_h^{N_h}. \quad (1)$$

The variables z_α ($\alpha = 1, \dots, h$) may be considered as activities, and Z_L as the *configurational partition function* of the dimer system. If all z_α are set equal to 1, Z_L reduces to the number of ways in which L can be filled with dimers.

It is not difficult to find a certain analogy with the Ising problem. There the determinantal method of Kac and Ward⁴ was based on the fact that, by their cyclic character, configurations of polygons on a lattice are reminiscent of the terms in the

⁷ E. W. Montroll, J. Soc. Ind. Appl. Math. **4**, 241 (1956).

⁸ H. Davies, Quart. J. Math. (Oxford) **6**, 232 (1955).

⁹ H. N. V. Temperley, Discussions Faraday Soc. **25**, 92 (1958).

¹⁰ E. A. Guggenheim, *Mixtures* (Clarendon Press, Oxford, England, 1952) Chap. X.

¹¹ E. G. D. Cohen, J. de Boer, and Z. W. Salsburg, Physica **21**, 137 (1955).

¹² T. S. Chang, Proc. Roy. Soc. (London) **A169**, 512 (1939).

¹³ P. W. Kasteleyn, Physica **27**, 1209 (1961).

¹⁴ H. N. V. Temperley and M. E. Fisher, Phil. Mag. **6**, 1061 (1961); M. E. Fisher, Phys. Rev. **124**, 1664 (1961).

expansion of a determinant. Here, the pairwise connection of lattice sites in a dimer configuration reminds one of the terms of another mathematical form, the Pfaffian.¹⁵ The Pfaffian of a skew-symmetric $n \times n$ matrix \mathbf{A} with elements $a(k, k')$ (n even $= 2m$) is defined by¹⁶

$$\text{Pf } \mathbf{A} = \sum_{\mathbf{P}} \epsilon_{\mathbf{P}} a(k_1, k_2) a(k_3, k_4) \cdots a(k_{n-1}, k_n) \quad (2a)$$

$$= [m! 2^m]^{-1} \sum_{\mathbf{P}} \epsilon_{\mathbf{P}} a(k_1, k_2) \cdots a(k_{n-1}, k_n). \quad (2b)$$

The first sum runs over those permutations $\mathbf{P} = k_1 k_2 \cdots k_n$ which satisfy $k_1 < k_2$; $k_3 < k_4$; \cdots , $k_{n-1} < k_n$, and $k_1 < k_3 < \cdots < k_{n-1}$; the second sum runs freely over all permutations, and $\epsilon_{\mathbf{P}}$ is the signature of the permutation \mathbf{P} ($\epsilon_{\mathbf{P}} = +1$ or -1 according as \mathbf{P} is even or odd). The equivalence of Eqs. (2a) and (2b) follows from the skew symmetry of \mathbf{A} . An important property of Pfaffians is their relation to determinants:

$$[\text{Pf } \mathbf{A}]^2 = \det \mathbf{A}. \quad (3)$$

By analogy with the Kac-Ward method we shall now try to construct a skew-symmetric $N \times N$ matrix \mathbf{D} with elements $d(k, k')$ such that $\text{Pf } \mathbf{D}$ is, in absolute value, equal to the generating function for close-packed dimer configurations on \mathbf{L} :

$$|\text{Pf } \mathbf{D}| = Z_{\mathbf{L}}(z_1, \cdots, z_h). \quad (4)$$

This will have been achieved if (a) there is a one-to-one correspondence between the nonvanishing different terms of $\text{Pf } \mathbf{D}$ and the dimer configurations on \mathbf{L} ; (b) to a dimer configuration occupying N_1 bonds from \mathbf{C}_1 , N_2 bonds from \mathbf{C}_2 , etc., there corresponds a term in Eq. (2a) of absolute magnitude $z_1^{N_1} z_2^{N_2} \cdots z_h^{N_h}$; (c) all terms have the same sign (either $+$ or $-$).

To satisfy condition (a), let the row and column indices of \mathbf{D} correspond to the sites of \mathbf{L} (numbered in an arbitrary way), and let

$$d(k, k') = 0 \text{ if the sites } k \text{ and } k' \text{ are not connected by a bond.} \quad (5a)$$

Condition (b) can be fulfilled by defining

$$d(k, k') = -d(k', k) = \pm z_{\alpha} \text{ if } k \text{ and } k' \text{ are connected by a bond of the class } \mathbf{C}_{\alpha}. \quad (5b)$$

From Eqs. (5a) and (5b),

$$\text{Pf } \mathbf{D} = \sum_{N_1, \dots, N_h} \sum_{\mathbf{C}}^{(N_{\alpha})} (\pm) z_1^{N_1} \cdots z_h^{N_h}, \quad (6)$$

¹⁵ For earlier references to this connection, cf. G. Brunel, *Mém. Soc. Sci. Bordeaux* (4) 5, 165 (1895); W. T. Tutte, *J. London Math. Soc.* 22, 107 (1947).

¹⁶ T. Muir, *A Treatise on the Theory of Determinants* (Cambridge University Press, New York, 1904), p. 92.

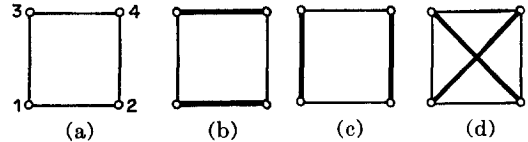


FIG. 1. (a) Example of a lattice graph; (b) and (c) Close-packed dimer configurations on this graph; (d) Representation of the vanishing term in the corresponding Pfaffian.

where the second sum runs over all configurations \mathbf{C} with given values of N_{α} ($\alpha = 1, \dots, h$). If, e.g., \mathbf{L} is the lattice graph of Fig. 1a, and

$$\mathbf{D} = \begin{bmatrix} 0 & z_1 & z_2 & 0 \\ -z_1 & 0 & 0 & z_2 \\ -z_2 & 0 & 0 & z_1 \\ 0 & -z_2 & -z_1 & 0 \end{bmatrix},$$

then

$$\begin{aligned} \text{Pf } \mathbf{D} &= \epsilon_{1234} d(1, 2) d(3, 4) + \epsilon_{1324} d(1, 3) d(2, 4) \\ &+ \epsilon_{1423} d(1, 4) d(2, 3) = +z_1^2 - z_2^2 + 0; \end{aligned}$$

the three terms correspond successively to Figs. 1b and 1c (representing dimer configurations on \mathbf{L}) and 1d (not representing a dimer configuration on \mathbf{L}).

The crux of the problem lies in condition (c). For an arbitrary choice of signs in Eq. (5b), the terms in Eq. (6) will not have equal signs, owing to the occurrence of the signature $\epsilon_{\mathbf{P}}$ in Eq. (2) and the skew symmetry of \mathbf{D} . It is formally possible, of course, to avoid all minus signs by working with a symmetric matrix, and its hafnian and permanent¹⁷ (in whose definitions $\epsilon_{\mathbf{P}}$ does not occur). However, this is of no use if we want to evaluate the partition function for a large periodic lattice, since we do not have a "permanent calculus" by which to calculate permanents of cyclic matrices. Therefore it has to be investigated whether it is possible to choose the signs in Eq. (5b) such as to satisfy condition (c). To facilitate the discussion, we represent the signs of the matrix elements by arrows along the bonds of \mathbf{L} : an arrow pointing from site k to site k' will indicate that $d(k, k') > 0$, and hence $d(k', k) < 0$. To each choice of signs in Eq. (5b), then, corresponds an *orientation* of the lattice graph \mathbf{L} .

Consider two configurations, \mathbf{C} and \mathbf{C}' , and further the graph consisting of the sites of \mathbf{L} , the bonds of \mathbf{L} occupied by \mathbf{C} , and the bonds occupied by \mathbf{C}' (Fig. 2); it is made up of *double bonds* and *cycles* (closed paths without double points). Clearly, \mathbf{C}' can be obtained from \mathbf{C} (and vice versa) by shifting,

¹⁷ E. R. Caianiello, *Nuovo Cimento* 10, 1634 (1953).

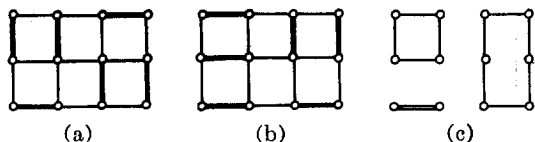


FIG. 2. (c) Transition graph of two close-packed dimer configurations (a) and (b).

in each of these cycles, all dimers by one step in either direction; therefore we call this graph the *transition graph* of C and C' , and its cycles *transition cycles*. As each cycle contains an *even* number of sites, the transition from C to C' involves as many *odd* permutations of lattice sites as there are cycles in the transition graph. Now the sign of a term in $\text{Pf } D$ is the product of the signature of a permutation and the signs of M matrix elements. From this, one readily sees that the terms representing C and C' have equal signs if in all cycles of the transition graph the number of bonds oriented in either direction is odd. If, for a cycle of even length (*even cycle*), the parity of the number of bonds oriented in either direction is called the *orientation parity*, we thus have the following theorem:

- (A) If the orientation of L which corresponds to D is such that the orientation parity of all transition cycles is odd, then $|\text{Pf } D| = Z_L$.

From now on, the discussion will be restricted to *planar* lattice graphs, although for the next theorem a more general formulation can be given which holds for nonplanar graphs as well. A plane representation of a planar graph L is a *net* whose *meshes* are bounded by cycles (*mesh cycles*). The following theorems can easily be proved:

- (B) A planar graph can be oriented in such a way that for all even mesh cycles the orientation parity is odd.
- (C) If in an oriented planar graph the orientation parity of all even mesh cycles is odd, then the orientation parity of an arbitrary even cycle enclosing an even (odd) number of sites is odd (even).
- (D) In a planar graph all transition cycles enclose an even number of sites.

From (A), (B), (C) and (D) we conclude that for planar lattice graphs, a matrix D can be defined such that $|\text{Pf } D| = Z_L$. It is only in the proof of (C) and (D) that the planarity of the graph plays an essential role: if a nonplanar graph is represented in a plane (*with intersecting lines*, of course), neither (C) nor (D) is true.

When the dimers do not completely fill the lattice, i.e. when vacancies or monomers are present, the

theory also ceases to be true. It is not difficult to construct a matrix whose Pfaffian generates all monomer-dimer configurations on a lattice graph. However, except for graphs such as that of Fig. 1(a), which have no "interior points," it is impossible to get all configurations correctly counted. The reason is that in this case a transition cycle may enclose an *odd* number of sites, e.g. one site occupied by a monomer, contrary to Theorem (D).

As in the case of the Ising model, the above method can be extended formally to cover nonplanar lattice graphs as well. For a toroidal graph, however, the partition function is a linear combination of *four* Pfaffians, etc., and for lattice graphs of infinite genus the method fails completely. The only use of this extension lies in the application to two-dimensional lattice graphs with periodic boundary conditions.¹³

III. THE QUADRATIC LATTICE

For a quadratic $m \times n$ lattice graph with $N = mn$ even, (suppose: m is even), the orientation sketched in Fig. 3 satisfies the condition of Theorem (A). According to (3), the evaluation of $\text{Pf } D$ can be reduced to that of $\det D$. If the class C_1 contains all "horizontal" and C_2 all "vertical" bonds, the matrix D has a periodic structure, so that it can be diagonalized, and its determinant evaluated. For the total partition function (generation function) $Z_{mn}(z_1, z_2)$ and the partition function per site for the infinite quadratic lattice, $Z(z_1, z_2)$, we then find

$$Z_{mn}(z_1, z_2) = \prod_{k=1}^m \prod_{l=1}^n 2 \left[z_1^2 \cos^2 \frac{k\pi}{m+1} + z_2^2 \cos^2 \frac{l\pi}{n+1} \right]^{\frac{1}{2}}, \quad (7)$$

$$Z(z_1, z_2) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} [Z_{mn}(z_1, z_2)]^{1/mn} \\ = \exp \left\{ (2\pi)^{-2} \int_0^\pi d\omega_1 \int_0^\pi d\omega_2 \ln 2 [z_1^2 + z_2^2 + z_1^2 \cos \omega_1 + z_2^2 \cos \omega_2] \right\}. \quad (8)$$

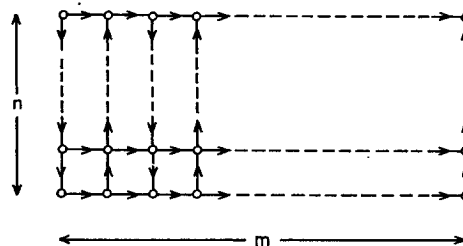


FIG. 3. Orientation of the $m \times n$ quadratic lattice graph, (m even).

The molecular freedom φ_2 of dimers in an infinite lattice defined as the number of arrangements per dimer, is given by

$$\varphi_2 = Z^2(1, 1) = \exp(2G/\pi) = 1.791\,622\,812 \dots, \quad (9)$$

where $G = 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \dots = 0.915\,965\,594 \dots$ (Catalan's constant).

The expression (8) has a remarkable resemblance to Onsager's expression for the Ising partition function per site for an infinite rectangular lattice.² In fact, when z_1 and z_2 are replaced by $\sinh(2J/kT)$ and $\sinh(2J'/kT)$, where J and J' are the horizontal and vertical Ising coupling constants, Eq. (8) reduces, apart from uninteresting factors, to the Ising partition function at the critical point (which varies with J and J'). This is consistent with the fact that for no values of z_1/z_2 has Eq. (8) a singular point.¹⁴

It is possible to extend the dimer problem on a quadratic lattice so that a correspondence to the Ising problem at all temperatures is established. To achieve this, make a further distinction between various types of horizontal and vertical bonds, e.g. by introducing four activities, z_1, z_2, z_3 and z_4 , as indicated in Fig. 4. The dimer partition has then, for real values of the activities, a singularity at $z_1 = z_2, z_3 = z_4$. The mathematical form of the singularity is the same as in the Ising partition function. If $z_1/z_2 = z_3/z_4 = u$, and if u is increased from 0 to ∞ , a "phase transition" takes place at $u = 1$. We are used to relate such a transition to the appearance or disappearance of a certain ordering. It is not immediately evident, however, in which respect the phase with $u > 1$ or that with $u < 1$ is ordered. What is more, the symmetry of the system with respect to the transformation $u \rightarrow u^{-1}$ seems to exclude the possibility that one phase is ordered, and the other disordered. The solution to this dilemma is simple. It is possible to find two order criteria such that the phase with $u > 1$ is ordered with respect to the first criterion, but disordered with respect to the second, whereas

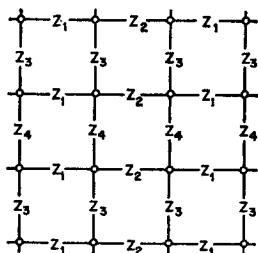


FIG. 4. Generalized quadratic lattice exhibiting a phase transition.

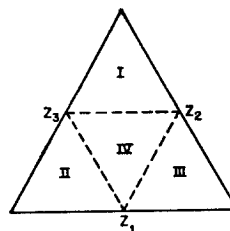


FIG. 5. The four phases of a close-packed system of dimers on a hexagonal lattice.

the phase with $u < 1$ is disordered with respect to the first criterion, but ordered with respect to the second. A full discussion of these two kinds of order would take us too far, but it is intended to publish the details elsewhere.

IV. THE HEXAGONAL LATTICE

When in the infinite hexagonal lattice graph three classes of bonds are distinguished, each one containing all bonds parallel to one of the three principal directions, a close-packed system of dimers on this lattice shows an interesting behavior.¹⁸ The partition function per site, $Z(z_1, z_2, z_3)$, is a smoothly varying function of the activities z_1, z_2 and z_3 if these are in a "triangle relation", i.e. if $z_1 < z_2 + z_3, z_2 < z_3 + z_1, z_3 < z_1 + z_2$. If, on the other hand, one of the activities is larger than the sum of the other two, Z is identically equal to the square root of the largest activity, e.g., for $z_1 > z_2 + z_3$ we find $Z(z_1, z_2, z_3) \equiv z_1^{1/2}$, independent of z_2 and z_3 . In words, when " z_1 bonds" are favored strongly enough, only one dimer configuration, viz. that with dimers only on z_1 bonds is realized. This is related to the fact that all transition cycles connecting this ordered configuration with other configurations have infinite length, i.e. that no configurations exist which deviate from it in the position of only a finite number of dimers. Obviously, a phase transition takes place when $z_1 = z_2 + z_3$; it can be shown that at the transition point $Z(z_1, z_2, z_3)$ behaves as $z_1^{1/2}[1 + \alpha(z_2 + z_3 - z_1)^{1/2}]$. Likewise we find a phase transition when $z_2 = z_3 + z_1$ and when $z_3 = z_1 + z_2$. The four phases of this system of dimers are best represented in a triangular diagram where the distances of a point to the three axes are proportional to z_1, z_2 and z_3 . In Fig. 5, the regions I, II and III represent successively the

¹⁸ A similar behavior is found for a dimer system on a quadratic lattice graph if a suitable, but somewhat more artificial classification of bonds is introduced. This is not surprising, as the hexagonal lattice graph can be considered as a quadratic lattice graph with certain bonds removed (or having zero activity). On the other hand, there is also an (again less "natural") classification of the bonds of the hexagonal lattice graph for which the partition function shows the analytical behavior sketched in the previous section.

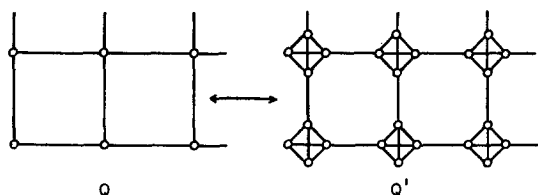


FIG. 6. Part of a finite simple quadratic lattice graph Q and the corresponding part of the cluster lattice graph Q' .

ordered phases in which Z is identically equal to $z_1^{\frac{1}{2}}$, $z_2^{\frac{1}{2}}$ and $z_3^{\frac{1}{2}}$; region IV represents the disordered phase, in which Z is a more complicated, continuous function of all three activities. It is interesting to compare this behavior of a system of dimers on a hexagonal lattice with the peculiar properties of the Ising antiferromagnetic triangular lattice.¹⁹

A system like this will be realized when diatomic molecules with an electric dipole moment p fixed to the axis are adsorbed on a hexagonal surface (leaving no vacancies) in the presence of an electric field E . This system exhibits a phase transition at a temperature T_c given by $pE/kT_c = 2 \ln [\frac{1}{2}(1 + 3^{\frac{1}{2}} + 2^{\frac{1}{2}}3^{\frac{1}{2}})]$. A remarkable feature of this phase transition is that at the transition point the system shows a perfect (directional) ordering; when the temperature is lowered from $T = T_c$ to $T = 0$, this ordering exhibits no further change.

V. THE ISING PROBLEM

In Sec. II, we have seen that the dimer problem is analogous to the Ising problem in that it admits of a combinatorial solution involving one single determinant for planar lattices only. The relation between the two problems is much clarified by the fact that the Ising problem can be formulated as a dimer problem. This was implied already in the recent approach to the Ising problem by Hurst and Green.²⁰ Whereas these authors arrived at a Pfaffian by an algebraic method,²¹ and did not introduce dimers at all, we shall reduce the Ising problem directly to a (generalized) dimer problem, which can then be solved by the method developed in Sec. II. In this way another purely combinatorial solution of the Ising problem is obtained.

It is well known¹ that the solution of the Ising problem for a simple quadratic lattice graph Q can be reduced to the determination of the number of ways in which a given number of horizontal and of vertical bonds can be selected so as to form closed

polygons, i.e. configurations in which the number of selected bonds incident with any site is even (0, 2 or 4).

Now associate with Q another lattice graph Q' in the following way: replace each site of Q by a cluster containing four sites, and each bond between two sites of Q by a bond between the opposite sites of two corresponding clusters; within the cluster, connect any site with any other (Fig. 6).

Consider a dimer configuration on the "cluster lattice" Q' . Since each cluster contains an even number of sites, the total number of dimers connecting it to neighboring clusters is even (0, 2 or 4). The analogy of these dimer configurations with the polygon configurations on Q , and therefore the possibility of translating them into each other, is obvious. More precisely, we have the following correspondence: (1) With each dimer configuration on Q' , we can uniquely associate a polygon configuration on Q . (2) With each polygon configuration on Q we can associate a dimer configuration on Q' ; this association, however, is not unique. For a site of Q , where four or two bonds of a polygon meet, there is only one way of translating this local configuration with the aid of dimers, placed on the corresponding cluster of sites of Q' (cf. Fig. 7(a), 7(b); in the latter case an extra dimer has to be introduced to connect the two sites which are not connected by a dimer to the sites of neighboring clusters. For a site of Q that does not lie on any polygon, on the other hand, there are three ways of translating, i.e. of connecting the four corresponding sites of Q' [Fig. 7(c)].

If this correspondence had been purely one-to-one or purely one-to-three, we should have been able to enumerate polygon configurations on Q by

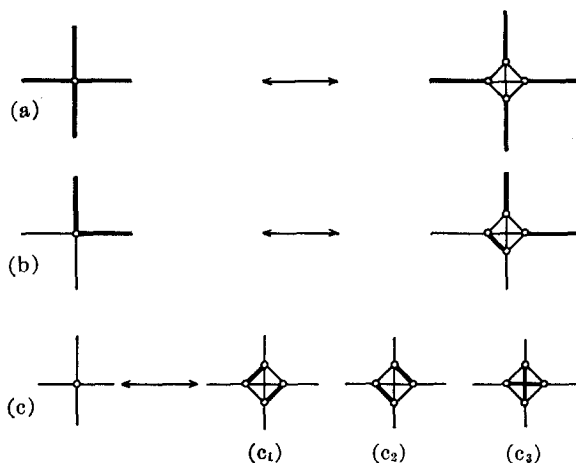


FIG. 7. Correspondence between local polygon configurations on Q and local dimer configurations on Q' .

¹⁹ R. M. F. Houtappel, *Physica* **16**, 425 (1950).

²⁰ C. A. Hurst and H. S. Green, *J. Chem. Phys.* **33**, 1059 (1960).

²¹ Essentially the same method has been used by Fisher¹⁴ in his solution of the dimer problem.

enumerating dimer configurations on Q' . Still, the mixed character of the correspondence is an advantage rather than a disadvantage. This becomes evident when we try to apply the method developed in Sec. II to the lattice graph Q' . It turns out that we cannot find the proper generating function for dimer configurations on Q' . Q' is nonplanar (unless it is a $2 \times n$ lattice, but this restriction is not essential to what follows), and no orientation satisfying the condition of theorem (A) can be found. We can, however, easily orient the bonds of Q' in such a way that the orientation parity of all even cycles *without self-intersections* is odd (Fig. 8). Then, the orientation parity of the transition cycles connecting the local dimer configurations (c_1) and (c_3), and (c_2) and (c_3), of Fig. 7 is even. This implies that the configuration (c_3) is counted with a different sign from that of (c_1) and (c_2). If then we put $z_\alpha = 1$ for all bonds connecting sites of the same cluster, the three local configurations on Q' corresponding to a single isolated site in the polygon configuration on Q , are counted as $1 + 1 - 1 = 1$. So the nonplanarity of Q' just compensates for the one-to-three correspondence between isolated sites on Q and isolated clusters on Q' . Apart from this effect, all dimer configurations on Q' are correctly counted.

Therefore, if we let $|d(k, k')|$ be equal to $x, y, 1$ or 0 , accordingly as k and k' are connected by a horizontal intercluster bond, a vertical intercluster bond, an intracluster bond, or no bond, and we choose the signs of the $d(k, k')$ in accordance with the orientation of Fig. 8, $|\text{Pf } \mathbf{D}|$ is the generating function for polygon configurations on Q . \mathbf{D} is then equivalent to the matrix introduced by Hurst and Green.²⁰ The Pfaffian, and hence the Ising partition function, can, in principle, be calculated in the standard way. However, there is no simple way of diagonalizing \mathbf{D} when the lattice graph has edges, i.e., is planar. For a toroidal lattice graph (lattice with periodic boundary conditions), on the other hand, the calculation is easy, although, of course, four Pfaffians (i.e., four matrices) must be introduced. Instead of carrying out the calculations, one may equally well prove that the four corresponding determinants are equal to those introduced by Potts and Ward⁴ in their analysis of the Ising model.

The generalization of the above to arbitrary planar (toroidal) graphs is straightforward. The only care to be taken is that a site where an *odd* number of bonds, say p , meet, is replaced by a cluster of $p + 1$ sites in order that the number of dimers connecting the cluster to neighboring clusters be

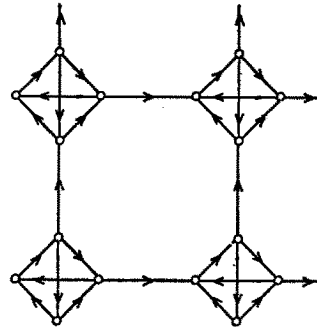


FIG. 8. An orientation of the cluster lattice graph Q' .

even. The addition of *two* extra sites has no effect.

Finally, the Ising partition function of a lattice graph L in the presence of a magnetic field involves the enumeration of configurations of polygons *and open chains* on L . This is equivalent to the enumeration of configurations of dimers *and monomers* on the cluster lattice L' , for which the present method is not adequate. Only if the ends of the open chains are fixed to certain lattice sites of L , can the above method be used; the clusters of L' corresponding to these sites should then consist of an odd number of sites. This is important for the calculation of the correlations and the spontaneous magnetization of an infinite lattice.

VI. CONCLUDING REMARKS

It has been shown that the dimer problem has many features in common with the Ising problem. Like the latter, it can only be solved for planar (or toroidal) lattices, under a restrictive condition (no monomers present). It should be added that, here too, the partition function for a one-dimensional lattice can be exactly calculated even if this condition is not fulfilled.^{11,14} Moreover, it has been shown that the Ising problem is a special case of a generalized dimer problem (where dimer configurations are not counted as different when they differ within a "cluster"). This approach to the Ising problem has the advantage of making more natural the appearance of square roots of determinants as the fundamental quantities (e.g. in the analysis of toroidal lattices, where linear combinations of Pfaffians rather than of determinants occur).

The question of why the genus plays a part in these problems and not in others, has not been answered. The form of the singularity in the partition function, however, has been found to be of the well-known type again, although in particular cases (e.g. the hexagonal lattice), the specific form of the problem may lead to the appearance of additional singularities of a different mathematical form.