Outline for 2011 AARMS Summer Course:

Analysis and geometry of partial differential equations

Instructors

Part I – Optimal transportation: Analysis, Geometry and PDE (weeks 1–2)
Prof. Robert J. McCann
Department of Mathematics
University of Toronto – Canada

Part II: Geometric properties of solutions of PDE's (weeks 3–4)
Prof. Andrea Colesanti
Dipartimento di Matematica "U. Dini"
Università di Firenze – Italy

Brief description. This is a course on the geometry and analysis of (mostly second-order nonlinear) partial differential equations.

The first two weeks will concentrate on equations which arise in connection with the optimal transportation problem of Monge and Kantorovich — a celebrated problem in the calculus of variations and the subject of much active research (surveyed in two books by 2010 Fields' medallist Cédric Villani).

The last two weeks will be devoted to geometric properties of solutions to boundary–value problems for elliptic PDE's, such as convexity, quasi–convexity, star-shapedness of level sets, radial symmetry, and so on. Moreover geometric type inequalities will be presented, such as isoperimetric or Brunn–Minkowski inequalities, both in their classic formulation and for functionals related to such equations. The following topics will be covered.

Week 1 – Introduction to optimal transportation. Existence, uniqueness and characterization of optimal maps. Connection to geometric inequalities such as isoperimetry, Monge-Ampére type equations, linear programming and game theory. Smoothness and counterexamples. Differential geometry and topology; roles of curvature: sectional, Ricci and mean.

Week 2 – Dynamical problems related to transportation; dissipative and conservative fluid flow; gradient descent; nonlinear heat equations; geometric flows, such as Ricci flow. Applications to economics and to meteorology. Student presentations begin.

Week 3 – Basic Theory of second order elliptic PDE's. Classical examples. The notion of solution of a PDE: classical and weak solutions (in the viscosity sense). Maximum principles. Existence and uniqueness results for solutions of boundary–value problems.

Week 4 – Geometric properties of solutions and geometric inequalities. How does the shape of the domain may influence the shape of the solution? Techniques to prove convexity, power-convexity and convexity of level sets of solutions to boundary–value problems. The Brunn-Minkowski and the isoperimetric inequalities. Brunn–Minkowski type inequalities for "energies" of solutions of PDE's.

Prerequisites. The course is intended to be introductory, but will require active student participation. The only prerequisites are an undergraduate degree in Mathematics. Some previous exposure to partial differential equations and to measure theory will be helpful.

Course notes:

Part I: *Five lectures on optimal transportation: geometry, regularity and applications,* R. McCann and N. Guillen. [47] at `http://www.math.toronto.edu/mccann/publications`

Part II: Lecture notes of the second part of the course will be made available on the home page of Andrea Colesanti (`www.math.unifi.it\users\colesant`) about one month before the beginning of the course.

Additional references

- L. C. Evans, *Partial differential equations*, American Mathematical Society, 2010.
- D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1983.
- Q. Han, F. Lin, *Elliptic partial differential equations*, American Mathematical Society, Providence, 1997.
- B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture notes in Mathematics 1150, Springer, Berlin, 1985.
- C. Villani, *Topics in Optimal Transportation*, Publisher = "American Mathematical Society, Providence, 2003
- C. Villani, *Optimal Transport. Old and New*, Springer, New York, 2009.
- The instructor's homepages (`URL`s given above)

Evaluation. As part of the course, students working in pairs will be expected to research and present assigned readings, mostly short and significant original sources for classic contributions to the field.

1) Readings will be assigned and scheduled during the first two class meetings.

2) All students will be responsible for doing the primary reading assigned in advance of each presentation. They are responsible for familiarizing themselves with the issues addressed by the article and its main conclusions.

3) Before each presentation, all students other than the presenting pair are required to submit a one paragraph written summary of the article to be discussed, along with two questions they have formulated about the article. This summary can also be submitted by E-mail in advance.

4) The presenting pair is responsible for having read and digested their article as completely as possible, at least three days prior to the presentation. It is MANDATORY that they meet with the instructor, no later than three days in advance of their presentation to resolve difficulties. They should come to this meeting prepared with a written outline of the structure of their presentation. Each student should be prepared to carry out the full presentation, alone if necessary. Ordinarily the instructor will select one of the two students to rehearse the presentation during the meeting, and the other student to present it in class. Failure to arrange this meeting or inadequate preparation will be reflected in the presentation grade.

5) Some of these readings are DIFFICULT. Begin preparing for your presentation AS FAR AHEAD AS POSSIBLE. You are welcome to consult the instructors at any time.

Grading Scheme:

50% Presentations (generally based on both individual and pair contributions)

30% Written Summaries of Other Readings

20% Class Attendance and Active Participation

Late summaries will not be accepted; except in case of documented medical absence, a grade of zero will be assigned.

SOURCES FOR STUDENT PROJECTS *(FIRST HALF)*

(c.f. `www.math.toronto.edu/mccann/AARMS`)

Student Pair

0 Introduction and Overview of Optimal Transportation. By the second meeting we will have decided who presents what.

L. Kantorovich. On the translocation of masses. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **37**, 199–201 (1942). T.C. Koopmans. Optimum utilization of the transportation system. *Econometrica (Supplement)*, 17:136–146, 1949.

1a Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C.R. Acad. Sci. Paris Sér. I Math.* **305**, 805–808 (1987).

This short but challenging French paper proves a nice theorem with a lot of nice connections, to e.g. Hodge theory. The basic argument got expanded in English [3] but it is probably easier to follow in Caffarelli [4] or the following paper of Gangbo and McCann:

1b W. Gangbo and R.J. McCann. Optimal maps in Monge's mass transport problem. *C.R. Acad. Sci. Paris Sér. I Math.* **321**, 1653–1658 (1995).

3 A.D. Aleksandrov. Existence and uniqueness of a convex surface with a given integral curvature. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **35**, 131–134 (1942).

Spectacular, but not easy. Some background in geometry (the Gauss map, curvature) is a plus. The uniqueness argument is spelled out in more detail in:

2 D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. **182** (2004) 307–332.

Combines entropy and transportation to recover this fundamental inequality in analysis. Although the paper is long, the key theorem and its proof are more succinctly covered on pages 200-203 of [17]. See also McCann [11] for an antecedent.

4 S. Alesker, S. Dar and V. Milman. A remarkable measure preserving diffeomorphism between two convex bodies in $\mathbf{R}^n$. *Geom. Dedicata* **74** (1999) 201–212.

More than you ever wanted to know about inequalities governing vector sums of convex bodies. The basic inequality is Brunn-Minkowski [8, §3.2.41]: domination of the geometric by the arithemetic mean gone bananas. See also McCann [11].

5 H. Tanaka. An inequality for a functional of probability distributions and its application to Kac's one-dimensional model of a Maxwellian gas. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **27** (1973) 47–52.

Used of a transportation distance to prove the central limit theorem (and convergence to equilibrium of the velocity profile in a model for a one-dimensional gas).

If needed R.J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* **80**, 309–323 (1995).

Brenier's theorem put into perspective by Aleksandrov and Rockafellar [16].

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] A. Aleksandrov. Smoothness of the convex surface of bounded Gaussian curvature. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **36** (1942) 195–199.

[3] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44**, 375–417 (1991).

[4] L. Caffarelli. Allocation maps with general cost functions. In P. Marcellini et al, editors, *Partial Differential Equations and Applications*, number 177 in Lecture Notes in Pure and Appl. Math., pages 29–35. Dekker, New York, 1996.

[5] L.C. Evans. Partial differential equations and Monge-Kantorovich mass transfer. In R. Bott et al., editors, *Current Developments in Mathematics*, pages 26–78. International Press, Cambridge, 1997.

[6] L.C. Evans. *Measure Theory and Fine Properties of Functions.* CRC Press 1992.

[7] W. Gangbo and R.J. McCann. The geometry of optimal transportation. *Acta Math.* **177** (1996) 113–161.

[8] H. Federer. *Geometric Measure Theory.* Springer-Verlag, New York, 1969.

[9] H. Hotelling. Stability in competition. *Economic Journal*, 39:41–57, 1929.

[10] S.G. Krantz. *Complex Analysis: the Geometric Viewpoint. Second edition.* Mathematical Association of America, Washington, DC, 2004.

[11] R.J. McCann A convexity principle for interacting gases. *Adv. Math.* **128** (1997) 153–179.

[12] R.J. McCann. Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* **11** (2001) 589-608.

[13] J. Milnor. *Morse Theory.* Princeton University Press, Princeton NJ 1963.

[14] F. Otto. The geometry of dissipative evolution equations. *Comm. Partial Differential Equations* **26** (2001) 101–174.

[15] M. Reed and B. Simon. *Functional Analysis* (volume 1 of Methods of Modern Mathematical Physics; Revised and Enlarged Edition). Academic Press, San Diego, 1980.

[16] R.T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pacific J. Math.* **17**, 497–502 (1966).

[17] C. Villani. *Topics in Optimal Transportation.* American Mathematical Society (2003) Providence. c.f. `http://math.univ-lyon1.fr/homes-www/villani/surveys.html#tot`

[18] C. Villani. *Optimal Transport, Old and New.* Springer (2009) New York. Downloadable from `www.umpa.ens-lyon.fr/∼cvillani/Cedrif/surveys.html#oldnew`.