AN ELLIPTIC PROOF OF THE SPLITTING THEOREMS FROM LORENTZIAN GEOMETRY

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ABSTRACT. We provide a new proof of the splitting theorems from Lorentzian geometry, in which simplicity is gained by sacrificing linearity of the d'Alembertian to recover ellipticity. We exploit a negative homogeneity (non-uniformly) elliptic p-d'Alembert operator for this purpose. This allows us to bring the Eschenburg, Galloway, and Newman Lorentzian splitting theorems into a framework closer to the Cheeger-Gromoll splitting theorem from Riemannian geometry.

CONTENTS

1. Introduction	1
1.1. History and results	1
1.2. Overview of strategy	3
1.3. Technical challenges	6
1.4. Outlook	7
2. Equi-semiconcavity of the Busemann limits	7
3. <i>p</i> -harmonicity of the Busemann function	12
4. A Bochner-Ohta identity of homogeneity $2p - 2 < 0$	17
5. Local splitting in Newman's setting	21
6. Local splitting in Galloway's setting	24
7. Global splitting	28
Acknowledgments	30
References	31

1. INTRODUCTION

1.1. **History and results.** Splitting theorems play a vital role in both Riemannian [13] and Lorentzian [5] geometry. Under the strong energy

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condition from general relativity, they confirm the intuition expressed, e.g. by Geroch [25], following the discovery of the singularity theorems [50, 33, 34] by Penrose and Hawking, that a geodesically complete spacetime ought to be exceptional: if even one of its complete geodesics is timelike and maximizing, then the space is a stationary, static, geometric product. This is made precise by the following theorem, in which spacetime refers to a connected smooth *n*-dimensional manifold M carrying a smooth Lorentzian metric tensor g_{ij} of signature $(+, -, \ldots, -)$ and a continuous timelike vector field which distinguishes future from past. Our methods can be adapted to yield Lorentzian splitting theorems for low regularity metrics (e.g. $g_{ij} \notin C_{loc}^2(M)$) with the option of Bakry-Émery style weights [11]; this will be addressed in a forthcoming work. The present manuscript focuses on the classical smooth case to exploit existing theory [23, 43], and to prevent technicalities from obscuring the power and simplicity of the new ideas.

Theorem 1 (Lorentzian splitting theorem). If a spacetime (M, g) satisfies the strong energy condition

(1) $R(v,v) \ge 0$ for all timelike vectors v,

contains an isometrically embedded (timelike) copy γ of the Euclidean line **R**, and is either (a) globally hyperbolic or (b) timelike geodesically complete, then (M, g) is isometric to $(\mathbf{R} \times S, dt^2 - h)$ where (S, h) is a complete Riemannian manifold with nonnegative Ricci curvature.

For smooth metric tensors g, the theorem was conjectured under hypothesis (b) by Yau [54] and proved by Newman [46], after Galloway had established the conclusion for spacetimes with compact Cauchy surfaces [21], Beem, Ehrlich, Markvorsen and Galloway [6] under (a) plus the stronger hypothesis of a suitable sign on all timelike *sectional* curvatures, Eschenburg [18] under (a) plus (b), and Galloway [22] under (a) using a result of Bartnik [3]. Under a *nonsmooth* sectional curvature hypothesis relaxing [6], the result has been extended to the Lorentzian length space [38] setting by Beran, Ohanyan, Rott and Solis [8]. Given recent interest in establishing Hawking and/or Penrose type singularity theorems in very low regularity settings [39, 40, 32, 31, 36, 1, 12], the generalization of Theorem 1 to such a setting remains a tantalizing challenge for future research; c.f. [7] and analogous developments in positive signature [26, 27, 28].

Existing proofs are rather complicated relative to that of the analogous splitting theorem from Riemannian geometry by Cheeger and Gromoll [14] or its simplification by Eschenburg and Heintze [17]. The reason for this is that the Lorentzian Laplacian (also known as the d'Alembertian or wave operator), although linear, is hyperbolic rather than elliptic. The purpose of the present article is to simplify the proof of the Lorentzian splitting theorems, by replacing the Laplacian with the nonlinear *p*-d'Alembert operator $-\Box_p u := \nabla \cdot (|du|^{p-2} du) =$ $\delta E/\delta u$ given by the variational derivative of the functional E(u) := $\int_M H(du) dvol_g$ with

(2)
$$H(v) := \begin{cases} -\frac{1}{p} |v|^p := -\frac{1}{p} (g^{ij} v_i v_j)^{p/2} & v \text{ is future-directed, } p \neq 0, \\ -\log |v| & v \text{ is future-directed, } p = 0, \\ +\infty & \text{else.} \end{cases}$$

In the range p < 1 this operator turns out to be (non-uniformly) elliptic [7] as a consequence of convexity of the Hamiltonian integrand H(v) introduced for $p \neq 0$ in [43]. Thus we sacrifice linearity to gain ellipticity. Our signs have been chosen to make the linearization of \Box_p around any suitable u with $E(u) < \infty$ a negative semidefinite operator, in analogy with the analyst's rather than the (Riemannian) geometer's Laplacian. Combining this insight with optimal transport ideas, elliptic techniques [29], a Bochner-Ohta identity of homogeneity 2p-2 < 0, c.f. [48, 45], and the simplification of Eschenburg's and Newman's arguments by Galloway and Horta [23], we are able to arrive at a much simpler proof of the splitting theorem. We give a brief outline of our strategy below. Curiously, this new approach to the smooth theorem relies on a d'Alembert comparison estimate for p < 1 first established in a much less smooth setting with the octet [7].

1.2. Overview of strategy. The geometry of a spacetime is wellknown to be encoded in a time-separation (or Lorentzian distance) function $\ell : M^2 \longrightarrow \{-\infty\} \cup [0,\infty]$ defined by a Lagrangian action principle

(3)
$$\ell(x,y) := \sup_{\sigma(a)=x, \sigma(b)=y} \mathcal{L}(\sigma),$$

(4)
$$\mathcal{L}(\sigma) := \int_a^b g(\sigma'(r), \sigma'(r))^{1/2} dr,$$

where the supremum is over future-directed Lipschitz curves σ connecting $x, y \in M$. It satisfies, for all $x, y, z \in M$, the reverse triangle inequality

(5)
$$\ell(x,z) \ge \ell(x,y) + \ell(y,z),$$

(with the convention $\infty - \infty = -\infty$ under hypothesis (b)). (Under hypothesis (a), it also satisfies the reflexivity property $\ell(x, x) = 0$ and

the antisymmetry condition

$$\min\{\ell(x,y), \ell(y,x)\} = -\infty \text{ unless } x = y.)$$

The chronological and causal relations are then defined by $I^+ = \{\ell > 0\} \subset M^2$ and $J^+ = \{\ell \ge 0\} \subset M^2$, and we also write any of the equivalent conditions (i) $x \ll y$ or (ii) $y \in I^+(x)$ or (iii) $x \in I^-(y)$ if and only if (iv) $(x, y) \in I^+$; similarly, we write (i') $x \le y$ or (ii') $y \in J^+(x)$ or (iii') $x \in J^-(y)$ if and only if (iv') $(x, y) \in J^+$. Also $I^{\pm}(X) := \bigcup_{x \in X} I^{\pm}(x)$ and $I(X, Y) := I^+(X) \cap I^-(Y)$ for $X, Y \subset M$ and analogously for J^{\pm} . Note that, on any spacetime, the reverse triangle inequality implies the following push-up property for the timelike relation: whenever $x \le y \ll z$ or $x \ll y \le z$, then already $x \ll z$, for all $x, y, z \in M$.

Given $[a, b] \subset [-\infty, \infty]$, a ray will refer to an inextendible, maximizing, causal geodesic $\gamma : [a, b) \longrightarrow M$. Here *inextendible* means $\lim_{s\to b} \gamma(s)$ does not converge to any point in M, and maximizing means the Lorentzian length functional (4) satisfies $\mathcal{L}(\gamma|_{[a,s]}) = \ell(\gamma(a), \gamma(s))$ for each $s \in [a, b]$. Being a causal geodesic, γ is affinely parameterized by default and either future- or past-directed; we say γ is (future or past) complete precisely when $b = \infty$ in this parameterization. A *line* will refer to a doubly inextendible, maximizing, causal geodesic $\gamma: (a, b) \longrightarrow M$, where now maximizing means $\mathcal{L}(\gamma|_{[r,s]}) = \ell(\gamma(r), \gamma(s))$ for each $[r, s] \subset (a, b)$. When future-directed, the (affinely parameterized) line is said to be *future-complete* if $b = \infty$, *past-complete* if $a = -\infty$, and *complete* if both hold. For causal lines and rays, inextendibility follows from completeness assuming (a) or (b). For timelike rays in case (b) only, the converse is true. Thus one of the technical challenges surmounted by Galloway working under (a) without (b) was the fact that rays and lines need not be complete, because the exponential map may not be globally defined [22]. Working under (b) without (a), Newman instead had to contend with (x, y) for which the supremum (3) defining $\ell(x,y) \geq 0$ need neither be attained nor continuous nor finite [46]. Henceforth, when we speak of lines or rays, we will mean future-directed, timelike, and proper-time parameterized, unless explicitly stated otherwise. With this convention, any (line or) ray defined on a (doubly) unbounded domain may be inferred to be complete unless otherwise noted. We write $I^{\pm}(\gamma) := I^{\pm}(\gamma((a, b)))$ and $I(\gamma) := I^+(\gamma) \cap I^-(\gamma)$ for a line, and similarly for a ray.

As in more traditional approaches to the splitting theorems (except [21] and [4]), the central objects of interest are the forward and backward Busemann functions $\pm b^{\pm}$ analogous to those introduced in [9]

and rediscovered by Cheeger and Gromoll [14]. The Busemann function $b^+: M \longrightarrow [-\infty, \infty]$ associated to a complete future-directed ray γ is defined as the limit $b^+(x) = \lim_{r \to \infty} b_r^+(x)$, where

(6)
$$b_r^+(x) = \ell(\gamma(0), \gamma(r)) - \ell(x, \gamma(r));$$

the negated Busemann function of a complete past-directed ray is defined by $b^{-}(x) = \lim_{r \to -\infty} b_{r}^{-}(x)$, where

(7)
$$b_r^-(x) = \ell(\gamma(r), x) - \ell(\gamma(r), \gamma(0)).$$

Thus a complete future-directed line γ has two Busemann functions $\pm b^{\pm}$ associated with it. From the triangle inequality (5) and propertime parameterization of γ it easily follows and is well-known that the limits above converge monotonically and r > 0 implies

(8)
$$b_r^+ \ge b^+ \ge b^- \ge b_{-r}^-$$

with all four functions coinciding on the intersection of the line γ with the open diamond $I(\gamma(-r), \gamma(r))$. Note that db^{\pm} does not depend on which point on the line γ is chosen as the base $\gamma(0)$, because the limiting functions b^+ differ only by additive constants.

In the linear case p = 2, where b_r^+ is smooth the strong energy condition (1) is well-known to yield the following bound [18, §5] on the d'Alembertian

(9)
$$\Box b_r^+ = \Box_p b_r^+ \le \frac{n-1}{\ell(\cdot, \gamma(r))} \quad \text{on } I^-(\gamma(r));$$

the same inequality extends across the timelike cutlocus when derivatives are interpreted in a suitable weak sense [7], as in the Riemannian case [10]. Since $|db_r^+| = 1$, it is perhaps not surprising that we are able to show the same inequality holds weakly for the p-d'Alembertian if p < 1; in the current smooth setting, we give a new proof of this fact that is logically independent of our original argument with the octet [7]. With our sign conventions, the Busemann functions $b^+ \ge b^-$ associated to a line are therefore *p*-super- and *p*-subharmonic: $\Box_p b^+ \leq 0 \leq \Box_p b^-$. For p < 1 however, (non-uniform) ellipticity of $-\Box_p$ and the strong maximum principle suggest the touching super and subsolutions must coincide, so that $b^+ = b^-$ is p-harmonic. In that case the negatively homogeneous Bochner-Ohta formula established below (which is a variant of formulas appearing in [48, 45]) implies b^+ is smooth and its Hessian vanishes identically; by contrast, the more familiar linear/quadratic Bochner-Weitzenböck identity is unhelpful since in Lorentzian signature it controls only a signed difference of Hessian terms (analogous to the failure of the usual d'Alembertian to be elliptic). At this point we

are essentially done: the level sets $S_r := \{x \in M \mid b^+(x) = r\}$ of b^+ are spacelike surfaces whose normal vector $N = db^+$ is locally parallel. From here it is easy to deduce the second fundamental form of S_r vanishes and that N is a Killing vector field generating a local isometry between S_r and S_0 for each $r \in \mathbf{R}$. An argument is still needed to globalize this result, achieved through a simplification of [18, 22, 46].

1.3. **Technical challenges.** Unfortunately, the strong maximum principle is delicate to establish for non-uniformly elliptic nonlinear equations; see e.g. the more standard case p > 1 treated in [30, 51] and their references for the *p*-Laplacian in Riemannian signature; examples of Krol and Lewis discussed in [41] show the $C^{1,\alpha}$ regularity established, e.g. in [16, 42], is best possible when p > 2. Our equation with p < 1in Lorentzian signature is worse. We must first establish that the equation becomes uniformly elliptic when linearized around the Busemann functions in question. To do this requires more regularity than Busemann functions generally possess. (As observed in [23], de-Sitter space is both (a) globally hyperbolic and (b) (timelike) geodesically complete, but its Busemann functions exhibit discontinuities and are not globally real-valued.) However, under the hypotheses of Theorem 1, Galloway and Horta gave a simple demonstration of Eschenburg's observation that the Busemann functions are Lipschitz continuous in a neighbourhood of the line. By combining their approach with Proposition 3.4 of McCann [43], we are able to improve this conclusion by showing the limiting Busemann function b^+ inherits semiconcavity from the approximate Busemann functions b_r^+ . In this neighbourhood then, we can pass to the limit of the (weakly reformulated) nonlinear comparison inequality (9) and show the linearization of the resulting operator around the Busemann function becomes uniformly elliptic. Unlike the Riemannian case p = 2, the weak form of (9) involves only one integration by parts so to obtain the limit $r \to \infty$ requires a.e. convergence not only of b_r^+ to b^+ but also of db_r^+ to db^+ . This is enough regularity to obtain $b^+ = b^-$ from the strong maximum principle; moreover, it is enough regularity to apply our Bochner-Ohta identity and conclude that the geometry splits locally and smoothly on the neighbourhood in question. To globalize this splitting theorem we are able to simplify the strategies of [18, 23] by using the knowledge that the Hessian of $b^+ = b^-$ vanishes on a neighbourhood of the line, and not just on a spacelike hypersurface. We still need to propagate this information outside the neighbourhood in question, first along nearby asymptotes to the line, and then throughout M by connectedness.

1.4. **Outlook.** While in this work we focus on demonstrating the elliptic *p*-d'Alembertian methods to give significant simplifications to the classical Lorentzian splitting theorem [18, 22, 46, 23], our techniques lend themselves to generalizations both to weighted spacetimes as well as to Lorentzian metric tensors with regularity below C^2 (see [37] for analogous developments in Riemannian signature). The rich literature on spacetime geometry with non-smooth Lorentzian metrics [39, 40, 32, 31, 36] highlights the relevance of such results, as nonsmooth metrics arise naturally in general relativity, e.g. as solutions to the Einstein equations. These generalizations of the Lorentzian splitting theorems require a more sophisticated analysis of the *p*-d'Alembert operator, and will be addressed in an upcoming work.

2. Equi-semiconcavity of the Busemann limits

A function on a Lorentzian manifold will be called *linear* if its covariant Hessian vanishes. Our goal is to prove linearity of the Busemann function b^+ . In contrast to the Riemannian setting, where the set of linear functions of slope 1 which vanish at a given point is compact, this is not the case in Lorentzian geometry, as the limits $\theta \to \pm \infty$ of the following elementary family of functions on the Minkowski (t, x)plane shows:

(10)
$$\{t\cosh\theta + x\sinh\theta\}_{\theta\in\mathbf{R}}$$

This divergence is due to noncompactness of the pseudosphere, and reflects the non-uniform ellipticity of the *p*-d'Alembert operator for p < 1. A similar mechanism is responsible for the poor regularity of Lorentzian Busemann functions more generally. However, in a neighbourhood of the line defining the Busemann function, this non-uniformity of ellipticity will be ruled out using a series of simple but subtle observations by Eschenburg [18] and Galloway [22] in case (a) and by Galloway and Horta [23] following Newman [46] in case (b), summarized in the following theorem. Here Lipschitz refers to any auxiliary Riemannian metric \tilde{g} on M. Such an auxiliary metric plays a useful role throughout, and can be taken to be complete on M [47, 24].

Theorem 2 (Busemann functions are Lipschitz near the line). In an (a) globally hyperbolic or (b) timelike geodesically spacetime M, let γ : $(-\infty, \infty) \longrightarrow M$ be a complete future-directed timelike line. Then (i) the Busemann functions b^+ and $-b^-$ associated with γ are locally Lipschitz continuous in a neighbourhood U of γ . Given $X \subset U$ compact, there exists $R, \tilde{C} > 0$ such that such that (ii) a maximizing timelike proper-time parameterized geodesic segment connects x to $\gamma(r)$ for each pair $x \in X$ and $|r| \geq R$. Moreover, (iii) for $x \in X$ any such segment σ satisfies $\dot{\sigma}(0) \in K_{\tilde{C}}(x) := \{v \in T_x M \mid |v|_g = 1, |v|_{\tilde{g}} \leq \tilde{C}\}.$

Proof. Under hypothesis (b), Theorem 3.7 and Corollary 5.2 of [23] together yield the statement for a neighbourhood of $\{\gamma(r)\}_{r>0}$, assuming only that γ is a ray. However, in the case of a line the Busemann function b^+ is independent of which point on γ is selected to be $\gamma(0)$, apart from an additive constant. In case (b) this establishes (i); while (ii)-(iii) follow from Lemma 3.3–3.4 (ibid), completeness of γ , and compactness of X, using a covering argument. As Galloway and Horta note, (iii) and (i) are even easier to prove in case (a), as was first done by Eschenburg [18, Lemmas 3.2–3.3] and exploited in [22]; (ii) holds as soon as $X \subset I^-(\gamma(R)) \cap I^+(\gamma(-R))$ in case (a), and the existence of such R follows from compactness of $X \subset U$ by a covering argument since finiteness of b^{\pm} implies $U \subset I(\gamma)$.

Remark 3 (Conic intersections). Near any point in this neighbourhood, the Lipschitz bound combines with the monotonicity (c.f. (13)) of b^+ to prevent db^+ from escaping to infinity asymptotically to the light cone: it must lie in the intersection of the solid future hyperboloid $g(v, v) \ge 1$ with the ellipsoid $\tilde{g}(v, v) \le \tilde{C}$. Moreover, in Corollary 6 below, we will show equality holds in $|db^+| \ge 1$, hence $|db^+| = 1 = |db_r^+|$.

Heuristically at least, reexpressing the p-d'Alembertian in nondivergence form

(11)

$$\frac{\Box_p u}{|du|^{p-2}} = -\frac{1}{|du|^{p-2}} \nabla \cdot (|du|^{p-2} du)$$

$$= (2-p)(\operatorname{Hess} u)(\frac{du}{|du|}, \frac{du}{|du|}) + \Box_2 u$$

$$= \left[(2-p) \frac{\nabla^i u \nabla^j u}{|\nabla u|^2} - g^{ij} \right] \nabla_i \nabla_j u$$

where ∇ denoted the Levi-Civita connection of g, demonstrates this ellipticity. Recalling the Lorentzian signature $(+, -, \dots, -)$, by choosing suitable coordinates we will eventually show that the term in square brackets — presently expressed using abstract index notation [53] becomes positive-definite for p < 1 as soon as du is timelike. Indeed, one could regard the term in square brackets as a (nonsmooth) Riemannian metric h^+ induced by $u = b^+$. This suggests the associated metric tensor $h_{ij}^+ \in L^{\infty}(U)$ will become uniformly positive-definite in suitable coordinates — corresponding to uniform ellipticity of the nondivergence form operator. Notice however, since the coefficients are merely bounded and measurable, the passage between divergence and nondivergence form is not automatic.

We will frequently make use of [23, Lemma 2.5], which we shall prove now for non-smooth metrics (in anticipation of future applications). To this end, note that a (future) *S-ray*, for some subset $S \subset M$, is a future inextendible causal geodesic $\gamma : [0, a) \to M$ maximizing the time separation to *S*, i.e. $\mathcal{L}(\gamma|_{[0,t]}) = \sup_{x \in S} \ell(x, \gamma(t))$, for every $t \in [0, a)$. Here, \mathcal{L} denotes the Lorentzian arclength functional.

Lemma 4 (Upper supports to approximate Busemann functions). If $\gamma : [0, \infty) \longrightarrow M$ is a future complete S-ray in a spacetime (M, g) with $g \in C_{\text{loc}}^{0,1}$ and $\sigma_r : [0, s_r] \longrightarrow M$ is timelike and maximizing between $\sigma_r(0) \in I^+(S) \cap I^-(\gamma(r))$ and $\sigma_r(s_r) = \gamma(r)$ for some r > 0, then for each $s \in (0, s_r]$

(12)
$$u_r(x) := b_r^+(\sigma_r(0)) + \ell(\sigma_r(0), \sigma_r(s)) - \ell(x, \sigma_r(s))$$

satisfies $u_r(x) \ge b_r^+(x)$ for all $x \in M$, where b_r^+ is from (6). Equality holds at $x = \sigma_r(a)$ for each $a \in [0, s]$. Moreover, both u_r and b_r^+ are real-valued on the neighbourhood $I^+(S) \cap I^-(\sigma_r(s))$ of $\sigma_r(0)$.

Proof. The definition (6) of b_r^+ , maximality of $\sigma_r : [0, s_r] \longrightarrow M$ between $\sigma_r(0)$ and $\sigma_r(s_r) = \gamma(r)$, and $s \in (0, s_r]$ yield

$$u_r(x) = \ell(\gamma(0), \sigma_r(s_r)) - \ell(\sigma_r(0), \sigma_r(s_r)) + \ell(\sigma_r(0), \sigma_r(s)) - \ell(x, \sigma_r(s))$$
$$= \ell(\gamma(0), \sigma_r(s_r)) - \ell(\sigma_r(s), \sigma_r(s_r)) - \ell(x, \sigma_r(s))$$
$$\ge \ell(\gamma(0), \sigma_r(s_r)) - \ell(x, \sigma_r(s_r))$$
$$= b_r^+(x)$$

as desired, where the reverse triangle inequality (5) has been used.

For $a \in [0, s]$, equality holds at $x = \sigma_r(a)$ holds since maximality of σ_r ensures the triangle inequality is saturated. The fact that γ maximizes time from S to $\gamma(r)$ implies $0 < \ell(x, \sigma_r(s_r)) \le r$ for each $x \in$ $I^+(S) \cap I^-(\sigma_r(s_r))$. The reverse triangle inequality shows $\ell(x, \sigma_r(s))$, $u_r(x)$ and $b_r^+(x)$ are all finite on the smaller neighbourhood $I^+(S) \cap$ $I^-(\sigma_r(s))$ of $\sigma_r(0)$.

We assume $g_{ij} \in C^{\infty}$ hereafter. Our first task will be to improve the regularity of the Busemann function on the neighbourhood mentioned above by showing it is semiconcave. Fixing a smooth Riemannian metric \tilde{g} on M, recall a function $u: M \longrightarrow \mathbf{R}$ is said to be *semiconcave* on $U \subset M$, with *semiconcavity constant* $C < \infty$ if

$$\limsup_{w \to 0} \frac{u(\exp_x^g w) + u(\exp_x^g - w) - 2u(x)}{\tilde{g}(w, w)} \le C$$

for each $x \in U$. It is said to be locally semiconcave on U if it is semiconcave on each compact subset of U. Here $\exp^{\tilde{g}}$ denotes the Riemannian exponential map. Although C depends on \tilde{g} , the property of being locally semiconcave does not. For each $y \in M$, the function $v(x) = -\ell(x, y)$ was shown to be semiconcave near each $x \in I^-(y)$ in Proposition 3.4 of [43], with a semiconcavity constant C(x, y) depending continuously on x and y. The following lemma shows this semiconcavity is inherited by the Busemann function b^+ . A family of functions such as $\{b_r^+\}_{r\geq R}$ is said to be *equi-semiconcave* on $U \subset M$ if they share the same semiconcavity constant on each compact subset Xof U.

Proposition 5 (Equi-semiconcavity of Busemann limits near the line). Let $b^+ = \lim_{r\to\infty} b_r^+$ denote the Busemann function associated by (6) to a complete future-directed timelike line $\gamma : (-\infty, \infty) \longrightarrow M$. Then b^+ is locally semiconcave on the neighbourhood U of γ provided by Theorem 2(a) or (b). On each compact set $X \subset U$, one can find $R = R(M, g, \tilde{g}, X, \gamma, \gamma(0))$ and a single semiconcavity constant C = $C(M, g, \tilde{g}, X, \gamma(\mathbf{R}))$ which works throughout X for all b_r^+ with $r \geq R$.

Proof. For R large enough (depending on $\gamma(0)$) we shall show $(b_r^+)_{r\geq R}$ has a semiconcavity constant C of which depends on γ but not on $\gamma(0)$. Fix a compact set $X \subset U$. Theorem 2(ii) provides R and \tilde{C} such that for each $x \in X$ and $z = \gamma(r)$ with $r \geq R$, there is a (proper-time parameterized) maximimizing geodesic segment σ_r joining $\sigma_r(0) = x$ to $\sigma_r(a_r) = z$. Theorem 2(iii) asserts $|\dot{\sigma}_r(0)|_{\tilde{g}} \leq \tilde{C}$ for all $x \in X$ and $r \geq R$. Since $a_r = \ell(x, \gamma(r)) \geq \ell(x, \gamma(R)) + r - R$ diverges as $r \to \infty$, taking R larger if necessary ensures $a_r \geq 1$ (uniformly with respect to $(x, r) \in X \times [R, \infty)$).

Now $K_{\tilde{C}} := \{(v, y) \in TM \mid |v|_g = 1, |v|_{\tilde{g}} \leq \tilde{C}, y \in X\}$ is a compact set of unit timelike directions. Among timelike geodesics with initial conditions in $K_{\tilde{C}}$, the time to the timelike cut-locus attains its minimum over $K_{\tilde{C}}$ by the lower semicontinuity shown in [5, Proposition 9.7] for case (a). This minimum value is positive; call it $t_0 \in (0, \infty]$ and fix $0 < t < \min\{t_0, 1\}$. The set $G = \{\exp_z sv \mid (v, z, s) \in K_{\tilde{C}} \times [0, t]\}$ is then compact and contains $\sigma_r([0, t])$ for all $x \in X$ and $r \geq R$.

Given σ_r as above, fix $y = \sigma_r(t)$. Lemma 4 asserts $u(x') = b_r^+(x) + \ell(x, y) - \ell(x', y) \ge b_r^+(x')$ holds for all x' in a neighbourhood of x and equality holds at x' = x; in other words, u supports b_r^+ from above at x. Thus b_r^+ inherits from u at x a semiconcavity constant C(x, y) which depends continuously on its arguments in $\{\ell > 0\}$ for case (a), according to Proposition 3.4 of [43]. Taking $C = \max_{(x,y)\in G} C(x, y)$ then concludes the proof of case (a).

Although the proposition last mentioned was proved under hypothesis (a), the conclusions we need can be extended to hypothesis (b) by the following observation. The proof of Proposition 3.4 of [43] provides a semiconcavity constant C of b_r^+ at $x = \sigma_r(0)$ given by a simple integral which depends only on the geometry (\tilde{g}, g) of M along the proper-time parameterized timelike geodesic $\{\sigma_r(s)\}_{s\in[0,t]}$. Since this geodesic lies in the compact set G, this explicit formula is easy to bound, allowing us to conclude the same way whenever $t_0 > 0$. If $t_0 = 0$, we set t = 1and argue as before after observing the resulting set G is again compact since the exponential map is continuous and defined globally on the timelike tangent bundle in case (b).

Corollary 6 (Unit gradients converge a.e. for Busemann limits). If a family $\{b_r^+\}_{r\geq R}$ is equi-semiconcave on an open set $U \subset M$, then pointwise convergence to a real-valued limit $b^+ = \lim_{r\to\infty} b_r^+$ on U implies locally uniform convergence of b_r^+ and pointwise convergence a.e. of db_r^+ on U. Moreover, b^+ inherits the semiconcavity constants of $\{b_r^+\}_{r\geq R}$. If the family consists of approximate Busemann functions (6) and M is an (a) globally hyperbolic or (b) timelike geodesically complete spacetime, then a.e. on U their gradients are future-directed and $|db^+| = 1 = |db_r^+|$.

Proof. Semiconcavity implies that in any smooth coordinate chart, the function becomes a concave function after subtraction of a multiple C of a fixed parabola, depending on the coordinates. Equi-semiconcavity of $\{b_r^+\}_{r\geq R}$ therefore implies locally uniform convergence of $b_r^+ \to b^+$ on U and convergence a.e. of their gradients, as for concave functions; c.f. Theorem 10.9 and the proof of Theorem 25.7 from Rockafellar [52]. The limit b^+ inherits the same constant C of semiconcavity. Semiconcavity also implies differentiability of b_r^+ outside a set of measure zero. For approximate Busemann functions (6), in the globally hyperbolic case (a) it is well-known that $|db_r^{\pm}| = 1$ where defined; Theorem 3.6 of [43]. In case (b) this also holds since we already know that b_r^+ is a.e. differentiable on U and for any $q \in U$ there exists a timelike maximizer to $\gamma(r)$ (for large enough r, uniformly on compact subsets of U, by Theorem 2(ii)). Thus $|db^+| = 1$ a.e. on U. The same estimates apply to $-b^-$. The reverse triangle inequality implies

(13)
$$b_r^{\pm}(y) - b_r^{\pm}(x) \ge \ell(x, y)$$

for all $y \in J^+(x)$, so db_r^{\pm} and their limits are both future-directed (recall that the Lorentzian metric has signature $(+, -, \ldots, -)$).

BRAUN, GIGLI, MCCANN, OHANYAN, AND SÄMANN

3. *p*-harmonicity of the Busemann function

By linearizing the nonlinear inequalities $\Box_p b^+ \leq 0 \leq \Box_p b^-$ for p < 1, in this section we show the sum $b^+ - b^- \geq 0$ of the Busemann functions is a weak supersolution to a linear, uniformly elliptic equation. Since $b^+ - b^-$ vanishes on a line, the strong maximum principle will then yield $b^+ = b^-$ in the connected neighbourhood U of this line provided by Theorem 2. Semiconcavity (and *p*-superharmonicity) of b^+ combines with semiconvexity (and *p*-subharmonicity) of b^- to imply $b^+ = b^- \in C_{loc}^{1,1}(U)$ (and $\Box_p b^+ = 0$ respectively).

Given a symmetric tensor field a^{ij} — bounded and measurable though not necessarily smooth — on a Euclidean domain $\Omega \subset \mathbf{R}^n$, a Lipschitz function u will be called a *weak* solution of $Lu \ge 0$ for the linear operator

(14)
$$Lu := -\partial_j (a^{ij} \partial_i u)$$

(or *weak supersolution*) if and only if

(15)
$$\int_{\Omega} (\partial_i u) a^{ij} \partial_j \phi \, dx \ge 0, \quad \forall \ 0 \le \phi \in C_0^1(\Omega);$$

here $C_0^1(\Omega)$ denotes the set of continuously differentiable functions with compact support in Ω . (We prefer to require Lipschitz regularity of urather than the more customary Sobolev hypothesis $u \in W^{1,2}(\Omega)$.) Similarly, u would be called a weak solution of $Lu \leq 0$ (or weak subsolution) if the first inequality in (15) were reversed. It is called a *weak solution* of Lu = 0 if both inequalities hold. The operator L is uniformly elliptic if $a^{ij} \in L^{\infty}(\Omega)$ and satisfy

(16)
$$a^{ij}v_iv_j \ge \lambda > 0$$

for some $\lambda > 0$ and all covectors $v \in \mathbf{R}^n$ with Euclidean unit norm. For a uniformly elliptic linear operator L on a connected domain $\Omega \subset \mathbf{R}^n$, the remark immediately following Theorem 8.19 of Gilbarg and Trudinger [29] asserts that any continuous weak supersolution $u \geq 0$ which vanishes at an interior point must vanish throughout Ω . In the proposition below we apply this to the sum $u = b^+ - b^-$ of the psuperharmonic Busemann functions b^+ and $-b^-$ to conclude $b^+ = b^$ in a neighbourhood of the line which defines them.

If instead the coefficients $a^{ij} \in L^{\infty}$ are defined on the cotangent bundle, hence depend on du(x) as well as $x \in \Omega$, the operator

(17)
$$Qu := -\partial_i (a^{ij} \partial_j u)$$

becomes quasilinear. The same definition (15) of weak supersolution (and the corresponding definitions of weak subsolution and weak solution), with Q in place of L continue to make sense. This turns out to be the case for the p-d'Alembertian $Qu = -\Box_p u := \nabla \cdot (|du|^{p-2}du)$, whose weak super and subsolutions are called p-superharmonic and psubharmonic respectively. If u satisfies both, i.e. Qu = 0 weakly, we say u is p-harmonic.

On a spacetime satisfying the strong energy condition, we claim b^+ is *p*-superharmonic and b^- is *p*-subharmonic near the line γ . We only prove the claim for b^+ , the statement for b^- follows by time-reversal symmetry.

Corollary 8 and Proposition 9 below require a weak reformulation which extends the smooth d'Alembert comparison theorem of Eschenburg [18] past the cut locus. This is most naturally formulated in terms of our *p*-d'Alembertian; cf. (15). Such an estimate was recently proved in much greater generality [7]. However, for the current smooth setting the following proposition gives a much more direct proof based on the smooth calculation due to Eschenburg [18]. Integrating his estimate by parts requires some care due to the presence of cut points. On the other hand, by employing the semiconcavity of appropriate Lorentzian distance functions (as in the discussion before Proposition 5), we control the sign of any singular contribution to render it harmless. Recall u is semiconvex if -u is semiconcave.

Proposition 7 (Weak d'Alembert comparison). Fix $0 \neq p < 1$ and a point $o \in M$. Suppose that there exists an open set $U \subset I^-(o)$ such that $\ell(\cdot, o)$ is (real and) semiconvex on U, the intersection of the past timelike cut locus of o with U is relatively closed with measure zero in U, and $\ell(\cdot, o)$ is smooth outside that set with unit timelike gradient. Then every nonnegative $\phi \in C_0^1(U)$ satisfies

$$\int_{M} \left[\frac{(n-1)\phi}{\ell(\cdot, o)} + g \left(d\phi, \frac{d\ell(\cdot, o)}{|d\ell(\cdot, o)|^{2-p}} \right) \right] d\mathrm{vol}_{g} \ge 0.$$

Proof. We write $u := \ell(\cdot, o)$. By partition of unity, we may and will assume $\phi \in C_0^1(U)$ is supported on a fixed chart. Since u is semiconvex, partitioning further if necessary, we may also assume the existence of a smooth function v such that u+v is a convex function of the coordinates in the usual Euclidean sense [2, Satz 2.3]. It is also not restrictive to assume that the coordinate representation $(2-p) (\partial^i u) (\partial^j u)/|du|^2 - g^{ij}$ of the tensor from (11) is uniformly positive-definite on the support of ϕ . By [2, p. 312], the constructions below will not depend on the choice of v. To relax our notation, we regard M as being covered by a single chart from which it inherits a Euclidean metric.

Standard distribution theory implies the distributional Euclidean Hessian of u+v is given by a nonnegative-definite matrix-valued Radon measure $D^2(u+v)$. In turn, as v is smooth the distributional Hessian of u is a signed Radon measure D^2u satisfying

$$\int_M \partial_i \partial_j \phi \, u \, dx = \int_M \phi \, d(D^2 u)_{ij}.$$

Moreover, the dx-singular part $(D^2 u)^{\perp}$ of $D^2 u$ is nonnegative-definite.

Given $\varepsilon > 0$ let ρ_{ε} be a standard convolution kernel, and set $u_{\varepsilon} := u * \rho_{\varepsilon}$. By the Lipschitz regularity assumed of u on U, for sufficiently small $\varepsilon > 0$ the matrix $(2-p) \partial_i u_{\varepsilon} \partial_j u_{\varepsilon} / |du_{\varepsilon}|^2 - g_{ij}$ stays uniformly positive definite and $|du_{\varepsilon}| > 0$ on the support of ϕ . Recalling (11), the p-d'Alembertian of u_{ε} is

$$\Box_p u_{\varepsilon} = |du_{\varepsilon}|^{p-2} \left[(2-p)g^{ik}g^{jl} \frac{\partial_k u_{\varepsilon} \partial_l u_{\varepsilon}}{|du_{\varepsilon}|^2} - g^{ij} \right] \left[\partial_i \partial_j u_{\varepsilon} - \Gamma_{ij}^k \partial_k u_{\varepsilon} \right].$$

Since u_{ε} is smooth, we can transform the right-hand side into its divergence form and use integration by parts to obtain

(19)
$$\int_{M} (g^{ij} \partial_{i} \phi \frac{\partial_{i} u_{\varepsilon}}{|du_{\varepsilon}|^{2-p}}) \sqrt{|g|} \, dx = \int_{M} (\phi \Box_{p} u_{\varepsilon}) \sqrt{|g|} \, dx.$$

On the other hand, convolution commutes with differentiation; in conjunction with the nonnegativity of $(D^2u)^{\perp}$ this entails

$$\partial_i \partial_j u_{\varepsilon} = \rho_{\varepsilon} * (D^2 u)_{ij} \ge \rho_{\varepsilon} * (D^2 u)_{ij}^{\text{a.c.}}.$$

Here $(D^2u)^{\text{a.c.}}$ is the dx-absolutely continuous part of D^2u , and the last inequality is understood in the sense of positive-semidefiniteness of matrix-valued distributions. Since the Hessian $\partial_i \partial_j u_{\varepsilon}$ is multiplied by a symmetric and positive-definite matrix in (18) which admits a square-root, the previous observation combines with (19) and the nonnegativity of ϕ to give

$$\int_{M} (g^{ij} \partial_{i} \phi \frac{\partial_{i} u_{\varepsilon}}{|du_{\varepsilon}|^{2-p}}) \sqrt{|g|} \, dx \ge \int_{V} (\phi \Box_{p} u_{\varepsilon}) \sqrt{|g|} \, dx.$$

Here V is the open subset of U where u is smooth. Its complement relative to U is negligible since we assumed the past timelike cut locus of o has measure zero in U.

As $\varepsilon \to 0$, the left-hand side converges to

$$\int_{M} g^{ij} \,\partial_i \phi \, \frac{\partial_i u}{|du|^{2-p}} \, \sqrt{|g|} \, dx$$

14

by the dominated convergence theorem, since $\phi \in C_0^1(U)$. By the same argument, the term containing the Christoffel symbols in (18) in the integral on the right-hand side converges. Lastly, since u differs from a convex function by the addition of a smooth function v whose Hessian is bounded on the support of ϕ , Fatou's lemma applies and yields the following limiting estimate for the right-hand side of the previous inequality:

$$\liminf_{\varepsilon \to 0} \int_{V} (\phi \Box_{p} u_{\varepsilon}) \sqrt{|g|} \, dx \ge \int_{V} (\phi \Box_{p} u) \sqrt{|g|} \, dx.$$

Finally, by Eschenburg's smooth d'Alembert comparison [18, §5] (note the different sign conventions on the metric tensor) and the definition of u, on the given set V its p-d'Alembertian is bounded from above by $(n-1)/\ell(\cdot, o)$. This implies

$$\int_{M} g^{ij} \partial_{i} \phi \, \frac{\partial_{i} u_{\varepsilon}}{|du_{\varepsilon}|^{2-p}} \sqrt{|g|} \, dx \ge -\int_{M} \frac{\phi(n-1)}{l(\cdot, o)} \sqrt{|g|} \, dx,$$

ed. \Box

as desired.

Corollary 8 (Busemann function b^+ is *p*-superharmonic). Fix $0 \neq p < 1$. Under the hypotheses of Theorem 1(a) or (b) on (M,g) and γ , if $V \subset M$ is a domain on which the functions $\{b_r^+\}_{r\geq R}$ of (6) are equi-semiconcave when R is sufficiently large, and if in addition for all $p \in V$ and $r \geq R$ there exists a timelike maximizing geodesic from p to $\gamma(r)$, then $b^+ = \lim_{r\to\infty} b_r^+$ is semiconcave, *p*-superharmonic and $|db^+| = 1$ on V.

Proof. The reverse triangle inequality (5) shows $I^-(\gamma(r))$ increases with r; since $b_r^+(x) = +\infty$ unless $x \in I^-(\gamma(r))$, we have $V \subset \bigcup_{r>0} I^-(\gamma(r))$. Using normal coordinates around $\gamma(r)$ shows $b_r^+(\cdot) = const(r) - \ell(\cdot, \gamma(r))$ to be smooth on V, except on the closure of the timelike cutlocus.

In case (a), the timelike cutlocus of $\gamma(r)$ intersects $I^-(\gamma(r))$ in a relatively closed set [43, Lemma 2.3] which has of zero volume as a consequence of Theorem 3.5 and the approximate second-differentiability a.e. of the semiconvex function $\ell(\cdot, \gamma(r)) = const(r) - b_r^+(\cdot)$ described following Definition 3.8 (ibid). Proposition 7 then applies to the approximate Busemann function b_r^+ for fixed r > 0 and yields

(20)
$$\int_{M} \left[\frac{(n-1)\phi}{\ell(\cdot,\gamma(r))} - g\left(d\phi, \frac{db_{r}^{+}}{|db_{r}^{+}|^{2-p}}\right) \right] d\mathrm{vol}_{g} \ge 0$$

for every nonnegative $\phi \in C_0^1(I^-(\gamma(r)))$. The convergence provided by Corollary 6 allows us to take $r \to \infty$ in (20) to get *p*-superharmonicity of b^+ , semiconcavity and $|db^+| = 1$ on any open subset X with compact closure in V by Lebesgue's dominated convergence theorem, where the monotone limit $\lim_{r\to\infty} \ell(\cdot, \gamma(r)) = \infty$ and the compact support of $\phi \in C_0^1(X)$ have been used. This concludes case (a).

Case (b) will follow similarly once we have verified that the timelike cutolcus of $\gamma(r)$ intersects $I^{-}(\gamma(r))$ in a relatively closed set of zero volume for each $r \geq R$. Although Theorem 3.5 (ibid) is stated under hypothesis (a), inspection of its proof reveals it applies equally well to (b) as soon as $r \geq R$. This is not true of Lemma 2.3 (ibid) however, so we must find another argument to show the intersection in question is relatively closed. Recall the (past) timelike cut locus is contained in the graph of $G(v) := \exp s(v)v \in M$ over the past unit observer bundle $T_1^-M := \{v \in TM \mid H(-v) = 1\}$, where $s: T_1^-M \longrightarrow [0, \infty]$ is upper semicontinuous according to [5, Proposition 9.5] in case (b). We shall complete the proof by arguing that s is also lower semicontinuous on the intersection of T_1^- with $\exp_{\gamma(r)}^{-1} V \subset T_{\gamma(r)}M$ provided $r \geq R$, hence G is continuous there. In case (a) this would follow from [5, 5]Proposition 9.7. However, the proof of that proposition reveals global hyperbolicity is used only to guarantee the existence of a maximizing geodesic linking $\gamma(r)$ to each $x \in V$, which the present context is guaranteed by hypothesis. Case (b) is therefore resolved.

Proposition 9 (Strong tangency principle). If the spacetime (M, g) of Proposition 5 satisfies the strong energy condition, then $b^{\pm} = \lim_{r\to\infty} b_r^{\pm}$ from (6)–(7) satisfy $b^+ = b^- \in C^{1,1}_{loc}(U)$ and $|db^{\pm}| = 1$ and are pharmonic for all $0 \neq p < 1$ on a neighbourhood U of the line γ .

Proof. Proposition 5, Corollary 8 and Theorem 2(ii) combine with Corollary 6 and time-reversal symmetry to show $-b^-$ and b^+ are *p*-superharmonic, semiconcave, and have unit timelike gradients, past-directed in case of $-b^-$ and future-directed in case of b^+ . Thus $u := b^+ - b^-$ and $b(t) := b^- + tu$ with $0 \le \phi \in C_0^1(U)$ yield

$$0 \leq -\int_{M} d\operatorname{vol}_{g} \int_{0}^{1} \frac{d}{dt} g(d\phi, |db|^{p-2} db) dt$$

$$= -\int_{U} d\operatorname{vol}_{g} \int_{0}^{1} |db|^{p-2} g(d\phi, [(p-2)\frac{db}{|db|} \otimes \frac{\nabla b}{|\nabla b|} + I] du) dt$$

$$= +\int_{\Omega} dx \sqrt{|g|} \partial_{i} \phi \partial_{j} u \int_{0}^{1} |db|^{p-2} \left[(2-p)\frac{\partial^{i} b \partial^{j} b}{|db|^{2}} - g^{ij} \right] dt$$

where the last line is expressed in a coordinate chart; we assume spt ϕ is supported in such a chart without loss. Viewing the coefficients as 'frozen' shows u is a weak supersolution $Lu \ge 0$ of the *linear* operator

given in divergence form (14)-(15) by

(21)
$$a^{ij}(x) = \sqrt{|g|} \int_0^1 |db|^{p-2} \left[(2-p) \frac{\partial^i b \partial^j b}{|db|^2} - g^{ij} \right] dt$$

Choosing Fermi coordinates near the line γ , our signature $(+, -, \ldots, -)$ of g with $0 \neq p < 1$ make it easy to check bounded measurability and uniform ellipticity (16) of these coefficients, taking U smaller if necessary: along the line $\gamma(r)$ where $db^{\pm} = d\gamma/dr$, the expression in square brackets becomes the diagonal matrix $diag(1 - p, 1, \ldots, 1)$; near γ , semiconcavity ensures that the gradients db^{\pm} are not very different from $d\gamma/dr$ either, Theorems 24.4 and 25.1 of [52]. On the other hand, $u \geq 0$ throughout U and vanishes on γ , according to (8). Thus u = 0 throughout a neighbourhood U of γ by the strong maximum principle, Theorem 8.19 of [29]. Now $b^+ = b^-$ is both semiconvex and semiconcave, hence $b^+ \in C_{loc}^{1,1}(U)$; similarly $b^+ = b^-$ is both p-super and p-subharmonic, hence p-harmonic.

Remark 10 (Ellipticity). Although it is easier to verify the ellipticity claimed using semiconcavity of $\pm b^{\pm}$, it can alternatively be deduced from Theorem 2 using Remark 3, which ensures the Clarke subdifferential of b(t) remains bounded at each point on γ . Here the Clarke subdifferential [15] of b(t) at x refers to the closed convex hull of limits of $db(t)(x_k)$ along sequences $x_k \to x$ of points in M of differentiability of b(t).

Remark 11 (Higher regularity). Although it might be possible to obtain higher regularity for p-harmonic $C_{loc}^{1,1}$ functions using Evans's [20] or Krylov's [35] techniques, it does not follow from their stated results and we were not successful in adapting their methods. Instead we shall establish it in the next section using a radically simpler approach.

4. A Bochner-Ohta identity of homogeneity 2p - 2 < 0

In this section we establish a nonlinear Bochner-Ohta formula on Lorentzian spacetimes; this will eventually imply that the Levi-Civita Hessian of our *p*-harmonic Busemann function b^+ vanishes. After discovering this formula, we realized that a large family of similar identities were previously established in Theorem 4.4 of Ohta [48] for Hamiltonians which are smooth and strongly convex away from the zero section of a manifold admitting a Riemannian structure. Although our result is similar in spirit, it complements them in the sense that our Hamiltonian, being adapted to the Lorentzian setting, satisfies neither the smoothness nor uniform convexity stipulated there. In particular, our more specialized setting allows us to give a simple statement and proof in terms of standard differential geometric concepts; c.f. Remark 13.

On our *n*-dimensional, signature $(+, -, \dots, -)$ spacetime (M, g), the Levi-Civita connection is denoted by ∇ . Equip the cotangent bundle with a Hamiltonian H(v, x) = f(g(v, v)/2) on the timelike future, where it depends smoothly on the Lorentz norm of v, and is irrelevant elsewhere. Denote derivatives of H with respect to $v = (v_1, \dots, v_n)$ by $DH \in TM$, having components $H^i = \frac{\partial H}{\partial v_i}$, and the Hessian $D^2H =$ $(H^{ij})_{1 \leq i,j \leq n}$ and higher v derivatives of H similarly. For a function $u \in C^3(M)$ whose gradient is future-directed and timelike everywhere, the identity we derive in (23) is the following:

(22)
$$\nabla \cdot [(D^2 H|_{du})d(H|_{du})] - (DH)d[\nabla \cdot (DH|_{du})] = \operatorname{Tr}[(D^2 H)(\nabla^2 u)(D^2 H)(\nabla^2 u)] + R(DH, DH),$$

where H and its derivatives are all tacitly composed with du, adjacent tensors are contracted in the obvious way (see (23)), and $\nabla^2 u = (\nabla_i \nabla_j u)_{1 \leq i,j \leq n}$ denotes the Levi-Civita Hessian of u. (We use d here simply to denote the differential of a function. On an orientable spacetime, we can alternately use the adjoint operator d^* to denote divergence instead of ∇ .)

The relevance of this identity is the following. Choose H(v, x) from (2) with $0 \neq p < 1$, in which case the identity we establish has homogeneity 2p - 2 < 0. If u is such that du has constant Lorentz norm and $\Box_p u = 0$, the first two terms vanish, since $\nabla \cdot (DH|_{du}) = \Box_p u = 0$. The strong energy condition (1) combined with convexity of H on timelike future covectors v makes the right hand side of (22) strictly positive unless the Hessian of u vanishes identically (and the timelike Ricci curvature vanishes in direction DH or equivalently, du). Applied to the p-harmonic function $u = b^+$, this identity yields the desired linearity and smoothness of the Busemann function.

Lemma 12 (A Lorentzian Bochner-Ohta identity).

If $H(v, x) = f(v_i v_j g^{ij}(x)/2)$ on the timelike future bundle of covectors to a spacetime (M, g) with $f \in C^3((0, \infty))$, and the differential of $u \in C^3(M)$ is future-directed and timelike everywhere, then evaluating Hand all its derivatives at du yields

(23)

$$\nabla_i (H^{ij}|_{du} \nabla_j (H|_{du})) - H^i \nabla_i (\nabla_j (H^j|_{du})) = H^{ij} u_{jk} H^{kl} u_{li} + R_{ij} H^i H^j,$$

where superscripts denote derivatives of H(v, x) with respect to components of the covector $v = (v_1, \ldots, v_n)$, subscripts denote covariant derivatives with respect to the Levi-Civita connection, except R_{ij} is the Ricci curvature tensor, and the Einstein summation convention holds.

Proof. Evaluating the left hand side of (23) using the chain rule (and the fact that v derivatives of H all commute with each other since the cotangent space at each point is flat, while x derivatives of H vanish due to the form of our Hamiltonian) yields

$$\nabla_i (H^{ij} H^k u_{jk}) - H^k \nabla_k (H^{ij} u_{ji})$$

= $H^{ijl} H^k (u_{il} u_{jk} - u_{kl} u_{ji}) + H^{ij} u_{jk} H^{kl} u_{il} + H^{ij} H^k (u_{ijk} - u_{kji})$

Here $u_{ijk} := \nabla_i \nabla_j \nabla_k u$. The terms involving third derivatives of H cancel each other (since superscripts on H can be freely permuted and the Levi-Civita connection is torsion free). It remains to see only that the terms involving third derivatives of u constitute the Ricci curvature term on the right hand side of (23). But this follows by combining consequences

(24)
$$H^{i} = (\nabla^{i} u) f'_{|du|^{2}/2} \quad \text{and} \quad$$

(25)
$$H^{ij} = (\nabla^i u)(\nabla^j u)f''_{|du|^2/2} + g^{ij}f'_{|du|^2/2}$$

of the structure of our Hamiltonian with the defining property (and antisymmetry) of the Riemann tensor

$$u_{ikj} - u_{kij} = R_{ikj}{}^{\iota}u_l.$$

Remark 13 (A simpler but longer formula). Formula (23) can also be written in terms of f instead of H using (24)–(25) in which superscripts on the right hand side now correspond to standard tensor indices raised using the Lorentzian metric tensor, an even more substantial simplification relative to the analogous expressions in [48].

The next corollary applies our result to the power-law Hamiltonian (2); the formula (26) it contains also seems simpler to us than the variant developed in [45, Appendix].

Corollary 14 (Linearity and smoothness of Busemann functions). From the strong energy condition and the conclusions of Proposition 9 it follows for $g_{ij} \in C^{\infty}(M)$ that $b^+ \in C^{\infty}(U)$ and has vanishing Hessian $\nabla_i \nabla_j b^+ = 0$ throughout U. *Proof.* Specializing (23) to the Hamiltonian (2) with $0 \neq p < 1$, under the strong energy condition (1), Lemma 12 yields

$$g(|du|^{p-2}du, d(\Box_p u)) - \nabla \cdot ((D^2 H)d(|du|^p/p))$$

$$(26) = \operatorname{Tr}\left[\sqrt{D^2 H}(\nabla^2 u)D^2 H(\nabla^2 u)\sqrt{D^2 H}\right] + |du|^{2p-4}R(du, du),$$

$$\geq \operatorname{Tr}\left[\sqrt{D^2 H}(\nabla^2 u)(D^2 H)(\nabla^2 u)\sqrt{D^2 H}\right]$$

where we have used convexity of the Hamiltonian to take the matrix square-root: strict positive-definiteness of D^2H on the timelike future cone was shown in Lemma 3.1 of McCann [43]. Proposition 9 provides a neighbourhood U of the line γ such that $b^+ \in C^{1,1}_{loc}(U)$ satisfies $|db^+| = 1$ and $\Box_p b^+ = 0$ a.e. on U. If $b^+ \in C^3(U)$, the left-hand side of (26) vanishes when $u = b^+$; in this case we conclude $\sqrt{D^2 H}(\nabla^2 b^+)\sqrt{D^2 H} = 0$ throughout U, and positive-definiteness of D^2H yields the desired linearity $\nabla_i \nabla_i b^+ = 0$ of b^+ .

If instead $b^+ \in C^{1,1}_{loc}(U) = W^{2,\infty}_{loc}(U)$, there exists a sequence $u_{\epsilon} \in C^{\infty}(U)$ with $|du_{\epsilon}| \geq 1 - \epsilon$ and $||u_{\epsilon} - b^+||_{W^{2,r}(X)} \to 0$ for each $r \in [1,\infty)$ and compact $X \subset U$; moreover $\nabla^2 u_{\epsilon} \to \nabla^2 b^+$ (hence $\Box_p u_{\epsilon} \to \Box_p b^+$) vol_g-a.e. [19, §5.3.1 and C.4]. For each test function $0 \leq \phi \in C^1_0(U)$, evaluating (26) at u_{ϵ} and integrating against ϕ yields

$$\int_{U} [(\Box_{p} u_{\epsilon}) \nabla \cdot (\phi DH_{\epsilon}) - D^{2} H_{\epsilon}(d\phi, d(H_{\epsilon}))] d\mathrm{vol}_{g}$$

$$\geq \int_{U} \phi \mathrm{Tr} \left[\sqrt{D^{2} H_{\epsilon}} (\nabla^{2} u_{\epsilon}) (D^{2} H_{\epsilon}) (\nabla^{2} u_{\epsilon}) \sqrt{D^{2} H_{\epsilon}} \right] d\mathrm{vol}_{g}$$

where $D^2 H_{\epsilon} := D^2 H|_{du_{\epsilon}}$ and similarly $DH_{\epsilon} = DH|_{du_{\epsilon}}$ and $H_{\epsilon} = H|_{du_{\epsilon}}$. Setting $u_0 := b^+$ and $X = \operatorname{spt} \phi$ yields $D^2 H_{\epsilon} \to D^2 H_0$ and $\nabla \cdot (\phi DH_{\epsilon}) \to \nabla \cdot (\phi DH_0)$ in $L^r(X)$ for all $r \in [1, \infty)$. Since the sequences $dH_{\epsilon} \to dH_0 = 0$ and $\Box_p u_{\epsilon} \to \Box_p u_0 = 0$ and $\nabla^2 u_{\epsilon} \to \nabla^2 u_0$ converge vol_g -a.e. on X and are bounded, they also converge in $L^r(X)$ for all $r \in [1, \infty)$. Choosing r = 2, the $\epsilon \to 0$ limit yields

$$0 \ge \int_U \phi \operatorname{Tr} \left[\sqrt{D^2 H_0} (\nabla^2 u_0) (D^2 H_0) (\nabla^2 u_0) \sqrt{D^2 H_0} \right] d \operatorname{vol}_g.$$

Since $b^+ \in C^{1,1}_{loc}(U)$, the covariant Hessian $\nabla^2 b^+$ is absolutely continuous with respect to vol_g ; positive-definiteness of $D^2 H_0$ implies $\nabla^2 b^+$ vanishes vol_g -a.e. — hence everywhere — on X. Arbitrariness of $0 \leq \phi \in C^1_0(U)$ concludes the proof that b^+ is smooth and linear throughout U.

5. Local splitting in Newman's setting

Having established linearity of the Busemann function b^+ in a neighbourhood of the line γ , we can prove a local version of the splitting (Theorem 1). Although our strategy is inspired by that of Eschenburg [18], Galloway [22], and Galloway and Horta [23], it is much simpler than these (as well as the textbook proof [5]), due to the fact we already know that the Busemann functions $b^+ = b^-$ are linear in an entire neighbourhood U of γ , and not merely on the intersection of this neighbourhood with some well-chosen spacelike surface such as the zero level set S_0 , where $S_r = \{x \in M \mid b^+(x) = r\}$ for each $r \in \mathbf{R}$. The vanishing Hessian of b^+ already shows S_0 to be totally geodesic in U, so we need no recourse to Bartnik's existence result for surfaces of zero mean curvature [3] either. However, we shall still need to show the vanishing of this Hessian propagates to all asymptotes of γ that pass through S_0 in this neighbourhood.

To define asymptotes, we recall the terminology of Galloway and Horta [23]: Let $\mathcal{L}(\sigma)$ denote the Lorentzian length (i.e. proper-time) along a future-directed (hence causal) curve $\sigma : [s,t] \longrightarrow M$ from (4) above. Given $t \in (0,\infty]$, a set $S \subset M$, a ray $\sigma : [0,t) \longrightarrow M$ is called an *S*-ray if γ maximizes distance to *S*, i.e. if

$$\mathcal{L}(\sigma|_{[0,s]}) = \ell(S, \sigma(s)) \qquad \forall 0 \le s < t,$$

where $\ell(S, x) := \sup\{\ell(y, x) \mid y \in S\}$. Thus a ray σ is also a $\{\sigma(0)\}$ -ray. A sequence of future-directed curves $\sigma_k : [s_k, t_k] \longrightarrow M$ is called *limit* maximizing if

$$0 \leq \liminf_{k \to \infty} [\mathcal{L}(\sigma_k) - \ell(\sigma_k(s_k), \sigma_k(t_k))].$$

Given a complete S-ray γ , a generalized co-ray refers to a ray constructed as a limit curve $\sigma : [0,t) \longrightarrow M$ — in the sense of [23] — of a limit maximizing sequence $\sigma_k : [0,s_k] \longrightarrow M$ with $\lim_{k\to\infty} \sigma_k(0) = \sigma(0) \in I^-(\gamma) \cap I^+(S)$ and $\sigma_k(s_k) = \gamma(r_k)$ and $r_k \to \infty$. If $\mathcal{L}(\sigma_k) = \ell(\sigma_k(s_k), \sigma_k(t_k))$, meaning the curves σ_k are all maximizing, then σ is called a *co-ray*. If $\sigma_k(0) = \sigma(0)$ for each k, the co-ray is called an *asymptote*.

We are now ready to give a simple proof of the following local splitting theorem, proved for $g_{ij} \in C^{\infty}(M)$ by Eschenburg [18, Proposition 6.3] under hypothesis (a) plus (b), Galloway [22] under (a) and by Newman [46] (also Galloway and Horta [23, §5]) under (b). The initial argument covers both cases but then bifurcates: we complete the proof of Newman's case (b) in the present section and defer the completion of Galloway's case (a) to the following section. **Theorem 15** (Local splitting). Under the hypotheses of Theorem 1(a) or (b), there is a neighbourhood $W \subset M$ of γ which splits: There is a smooth spacelike hypersurface $S \subset S_0 \subset M$ containing $\gamma(0)$ and a diffeomorphism $E : \mathbf{R} \times S \longrightarrow W$ given by $E(r, x) = \exp_x rdb^+$ which is a local isometry in the sense that E pulls the metric g back to the product metric $dr^2 - h$, where -h is the restriction of g to S. Moreover, $r \in \mathbf{R} \mapsto E(r, x)$ is a line maximizing time to $S_0 := \{b^+ = 0\}$ for each $x \in S$, and $\gamma(\cdot) = E(\cdot, \gamma(0))$.

Proof in case (b). Taking the neighbourhood U of γ from Corollary 14 — on which the Hessian of $b^+ \in C^{\infty}(U)$ vanishes — smaller if necessary, ensures each $x \in S := S_0 \cap U$ forms the base $x = \sigma^{\pm}(0)$ of forward and backward asymptotes $\sigma^+ : [0, s_+) \longrightarrow M$ and $\sigma^- : (s_-, 0] \longrightarrow M$ to γ (with $s_+ = \infty = -s_-$ in case (b)), both proper-time reparameterized to be future-directed; this follows by using Lemmas 2.1–2.4 of Galloway and Horta [23] to extract timelike subsequential limits $r \to \infty$ of the maximizing segments from x to $\gamma(\pm r)$ provided for r sufficiently large by our Theorem 2(ii)-(iii). Together they form a future-directed geodesic σ , potentially broken at $\sigma(0) \in S$. Proposition 2.6 of the same reference shows $b^+(\sigma(r)) = r$ for all $r \in [0, s_+)$; then

$$b^{-}(\sigma(r)) = b^{-}(\sigma(r)) - b^{-}(\sigma(0))$$
$$\geq r = b^{+}(\sigma(r))$$

couples with (8) $b^+ \ge b^-$ to yield $b^-(\sigma(r)) = r$ in the same range of $r \in [0, s_+)$. Combining the preceding argument with time-reversal symmetry shows both Busemann functions increase along the full asymptote at unit rate: $b^{\pm}(\sigma(r)) = r$ for all $r \in (s_-, s_+)$. Because

$$b^+(y) - b^+(x) \ge \ell(x, y)$$

for all $x \ll y$, with equality at $(x, y) = (\sigma(r), \sigma(s))$ for all $r < s \in (s_-, s_+)$, it follows that (i) that σ is a timelike maximizer; (ii) its tangent $\sigma'(r) = N(\sigma(r))$ agrees with the direction $N = db^+$ of slowest increase of b^+ on U; and (iii) σ^{\pm} are S_0 -rays. Thus σ is not broken at $\sigma(0)$ after all. Moreover, σ is normal to the zero level set $S \subset S_0$ of b^+ , which is spacelike since N is unit timelike, and smooth by the implicit function theorem.

The arguments above show that $E(r, x) := \exp_x rN$, with maximal domain of definition $\mathcal{D} \subset \mathbf{R} \times S$, has the following properties: The trajectories $E(\cdot, x)$ and $E(\cdot, y)$, for $x, y \in S$, are timelike lines which do not cross (unless x = y), contain no conjugate points nor focal points to S, and maximize the time separation to S. In case (a) \mathcal{D} is open by [5, Proposition 9.7], while in case (b), $\mathcal{D} = \mathbf{R} \times S$ holds automatically. In both cases the implicit function theorem shows E is a diffeomorphism from $\mathcal{D} \subset \mathbf{R} \times S$ onto a set $W \subset M$ which is open, and $E(r, \gamma(0)) = \gamma(r)$ for all $r \in \mathbf{R}$.

We have now shown $b^+ = b^-$ throughout W (in addition to U). Since E is a smooth diffeomorphism and $b^+(E(r, x)) = r$ for each $(r, x) \in \mathcal{D}$, the continuous differentiability of $b^{\gamma} := b^{\pm}$ propagates throughout W. On U, recall b^{γ} is smooth and its Hessian vanishes, from Corollary 14. Thus $N = db^{\gamma}$ satisfies Killing's equation in U, hence is parallel throughout U, so that any geodesic in $U \subset M$ has constant inner product with N. In particular, since $N = db^{\gamma}$ is normal to the zero level $S^{\gamma} := S_0 \cap U = S_0 \cap W$ of b^{γ} , we see S is totally geodesic: its second fundamental form (or shape operator) vanishes.

At this point we specialize to case (b), defering the completion of case (a) to the following section. In case (b) the line σ is complete, so applying the same logic to it as to γ yields a neighbourhood V of σ on which the forward Busemann function b^{σ} is smooth and has vanishing Hessian. Thus its zero set $S^{\sigma} := \{x \in V \mid b^{\sigma}(x) = 0\}$ is also totally geodesic in V. Taking V smaller if necessary again ensures each $x \in S^{\sigma}$ lies on a complete line $\beta^x : (-\infty, \infty) \longrightarrow M$ with $\beta^x(0) = x$ which is (future and past) asymptotic to σ . Since both S^{γ} and S^{σ} are orthogonal to σ and are totally geodesic, we conclude they coincide in the neighbourhood $V \cap W$ of $\sigma(0)$. Now the asymptotes to σ and to γ both intersect S orthogonally in $W \cap V$, so for each $x \in S \cap V$ these asymptotes $(E(\cdot, x)$ to γ and $\beta^{x}(\cdot)$ to σ) also coincide. Moreover both b^{γ} and b^{σ} increase at rate one along these asymptotes, hence $b^{\gamma} = b^{\sigma}$ throughout $W \cap V$. Thus b^{γ} inherits the linearity of b^{σ} : its Hessian vanishes throughout $W \cap V$. Since the asymptote σ of γ had arbitrary base $\sigma(0) \in S$, we conclude the Hessian of b^{γ} vanishes globally on W.

We have now shown the flow map F(r, x) of $N = db^{\gamma}$ satisfying $\frac{dF}{dr} = N(F(r, x))$ and F(0, x) = x coincides with E on $\mathbf{R} \times S$. The fact that the Hessian of b^{γ} vanishes means $N = db^{\gamma}$ is parallel throughout W and satisfies Killing's equation. Thus $F(r, \cdot) : W \longrightarrow W$ pulls-back the metric g to itself for each $r \in \mathbf{R}$, which shows E(r, S) to be isometric to S = E(0, S). Along the totally geodesic surface S normal to N, the metric g therefore splits into the direct sum of its restriction -h to S plus dr^2 in the orthogonal direction N. Thus E gives the desired local isometry between $(\mathcal{D} = \mathbf{R} \times S, dr^2 - h)$ and (W, g).

Corollary 16 (Local to global isometry). With the hypotheses and notation of Theorem 15, the time-separation function ℓ on M^2 satisfies

(27)
$$\ell(E(s,x), E(t,y)) \ge \begin{cases} \sqrt{(t-s)^2 - d_h^2(x,y)} & \text{if } t-s \ge d_h(x,y), \\ -\infty & \text{else,} \end{cases}$$

where $s, t \in \mathbf{R}$ and d_h denotes the Riemannian distance in S between $x, y \in S$. Moreover, if W = M then this estimate becomes an equality so the isometry becomes global.

Proof. This follows from Theorem 15, the definition of the product metric $dr^2 - h$, and of ℓ from (3).

6. LOCAL SPLITTING IN GALLOWAY'S SETTING

To adapt the local splitting from the (b) timelike geodesically complete setting of Newman [46] to the (a) globally hyperbolic setting of Galloway [22] requires additional arguments to rule out any possible incompleteness of the inextendible asymptotes constructed in its proof. Readers interested only in timelike geodesically complete spacetimes (b) can skip this section. We begin with a criterion for asymptotes to be timelike, which can be viewed as a partial converse to Theorem 2.

Lemma 17 (Differentiability criterion for asymptotes to be timelike). Let (M, g) be a strongly causal spacetime. Let $b^+ = \lim_{r\to\infty} b_r^+$ denote the Busemann function associated by (6) to a future-complete S-ray $\gamma: [0, \infty) \longrightarrow M$. Let $\alpha: [0, a_+) \longrightarrow M$ be an asymptote to γ . If b^+ is differentiable at $\alpha(0)$ and remains Lipschitz nearby, then α is timelike.

Proof. The definition of asymptote asserts α is a (subsequential) limit curve of a sequence of maximizing geodesics $\sigma_r : [0, s_r] \longrightarrow M$ with $\sigma_r(0) = \alpha(0) \in I^+(S) \cap I^-(\gamma)$ and $\sigma_r(s_r) = \gamma(r)$ as $r \to \infty$. In other words, as $r \to \infty$ along a subsequence, $\tilde{\sigma}_r$ converges uniformly to $\tilde{\alpha}$ on compact subsets of $[0, \infty)$, where $\tilde{\sigma}_r$ and $\tilde{\alpha}$ denote the \tilde{g} -arclength reparameterizations of σ and α respectively. Such an asymptote is a ray by [23, Lemma 2.4], hence $\tilde{\alpha}$ can be extended to $(-\infty, \infty)$ and affinely reparameterized as a future- and past-inextendible geodesic $\alpha : (a_-, a_+) \longrightarrow M$ for some $a_- \in [-\infty, 0)$, and $a_+ \in (0, \infty]$ with $\alpha(0) = \tilde{\alpha}(0)$.

Strong causality of (M, g) and [44, Thms. 2.9, 2.35] imply for $\epsilon > 0$ sufficiently small, that $\tilde{\alpha}(-\epsilon) \in I^{-}(S)$ and $\tilde{\alpha}|_{[-\epsilon,\epsilon]}$ is maximizing. As in the last paragraph of the proof of Proposition 5 (and of Proposition 18), setting $x = \alpha(0)$ and $y_r = \tilde{\sigma}_r(\epsilon)$ yields

$$b_r^+(\tilde{\alpha}(s)) \le b_r^+(x) + \ell(x, y_r) - \ell(\tilde{\alpha}(s), y_r)$$

24

for all $|s| < \epsilon$. To derive a contradiction, assume α is null. Letting $r \to \infty$ along the relevant subsequence yields

$$b^{+}(\tilde{\alpha}(s)) \leq b^{+}(x) + \ell(x, \tilde{\alpha}(\epsilon)) - \ell(\tilde{\alpha}(s), \tilde{\alpha}(\epsilon))$$
$$= b^{+}(\tilde{\alpha}(0))$$

by the nullity of $\tilde{\alpha}$ and our choices of ϵ and $|s| < \epsilon$. Differentiation at s = 0 then shows $db^+|_x(\tilde{\alpha}'(0)) = 0$, so $db^+|_x$ is vanishing, spacelike or null. On the other hand, the Lipschitz continuity hypothesized of b^+ around $x = \alpha(0)$ yields $db^+|_x$ timelike (and future-directed, with Lorentzian magnitude at least 1) as in Remark 3. This contradiction forces α to be timelike as claimed. \Box

Our next result is a variant on the semiconcavity Proposition 5. As before, the equi-semiconcavity of the approximate Busemann functions $(b_r^+)_{r\geq R}$ is essential; semiconcavity of the limiting function b^+ alone does not yield the desired corollary.

Proposition 18 (Equi-semiconcave Busemann limits on asymptotes). Let (M, g) be an (a) globally hyperbolic spacetime. Let $b^+ = \lim_{r\to\infty} b_r^+$ denote the Busemann function associated by (6) to a future-directed ray γ : $[0, \infty) \longrightarrow M$. Let α : $[0, a_+) \longrightarrow M$ be a (future inextendible) timelike asymptote to γ . Let $b^+ \in C^1(W)$ on a neighbourhood of W of $x_0 = \alpha(a)$ for some $a \in [0, a_+)$. Then for large enough $C = C(M, g, \tilde{g}, x_0, db^+(x_0))$ and R depending on the same parameters as well as on γ , the functions $(b_r^+)_{r\geq R}$ have semiconcavity constant Con some smaller neighbourhood $X \subset I^-(\gamma(R))$ of x_0 .

Proof. Let $\alpha : [0, a_+) \longrightarrow M$ be a (future inextendible, proper-time parameterized) asymptote to γ with b^+ continuously differentiable in a neighbourhood of $x_0 = \alpha(a)$ for some $a \in [0, a_+)$. Since [23, Proposition 2.6] asserts b^+ increases at its minimal rate along α , it follows from (13) that $\alpha'(a) = db^+(x_0)$. Fixing $0 < \epsilon < a_+ - a$ yields $y_0 := \alpha(a + \epsilon) =$ $\exp_{x_0} \epsilon db^+(x_0)$. Since the asymptote is timelike and maximizing, [5, Proposition 9.7] combines with [43, Theorem 3.6] to yield a compact neighbourhood $V \subset TM$ of $(x_0, \epsilon db^+(x_0))$ in the timelike future bundle on which the exponential map is defined and ℓ is smooth on $\exp V$. Take C large enough that $C\tilde{g}$ dominates the \tilde{g} -covariant Hessian (with respect to x) of $v(x) := -\ell(x, y)$ for all $(x, y) \in \exp V$. We claim C is the desired semiconcavity constant.

Let $X \times Y \subset \exp V$ be a neighbourhood of (x_0, y_0) , where exp here denotes the map taking a vector v_p to $(p, \exp_p(v)) \in M^2$. Since $b^+(x_0) < \infty$, taking R sufficiently large ensures $x_0 \in I^-(\gamma(R))$ (by the push-up property). Taking X smaller if necessary guarantees $X \subset$ $W \cap I^-(\gamma(R))$ as well. For each $x \in X$ and $r \geq R+1$, global hyperbolicity provides a proper-time parameterized maximizing geodesic segment $\sigma_r : [0, s_r] \longrightarrow M$ from $\sigma_r(0) = x$ to $\sigma_r(s_r) = \gamma(r)$. The reverse triangle inequality shows $s_r \to \infty$, since $\ell(\gamma(R), \gamma(r)) \to \infty$ as $r \to \infty$. Set $y_r = \sigma_r(\epsilon)$. By Lemmas 2.1 (the limit curve theorem) and 2.4 of [23], one can extract an asymptote σ to γ as a subsequential limit curve of σ_r as $r \to \infty$. Since $x \in W$ and $b^+ \in C^1(W)$, Lemma 17 asserts this asymptote is timelike. Now [23, Proposition 2.6] again asserts b^+ increases at its minimal rate along σ , thus $\sigma(s) = \exp_x sdb^+(x)$ by the same logic as above, so the full sequence $\sigma_r \to \sigma$ on $[0, a+\epsilon]$ and $y_r \to y$ as $r \to \infty$. Our hypothesis $b^+ \in C^1$ yields $y \to y_0$ as $x \to x_0$, so taking X smaller and compact if necessary ensures $y \in Y$ for all $x \in X$ and all $r \geq R+1$. Since $y_r \to y$ as $r \to \infty$, taking R larger (independently of x within the compact set X) then ensures $y_r \in Y$.

For each $x \in X$ and $r \geq R$, Lemma 4 asserts $u(x') := b_r^+(x) + \ell(x, y_r) - \ell(x', y_r) \geq b_r^+(x')$ for all x' near x. Since equality holds at x' = x, this means u(x') supports b_r^+ from above at x. Thus b_r^+ inherits the asserted semiconcavity constant C chosen above from u at x. \Box

Corollary 19 (*p*-superharmonicity of b^+ near asymptotes). Assume the (a) globally hyperbolic spacetime (M, g) satisfies the strong energy condition (1). Then inside the neighbourhood X of $\alpha(a)$ identified in Proposition 18, b^+ is p-superharmonic, semiconcave, and $|db^+| = 1$.

Proof. Proposition 18 provides the equi-semiconcavity of $\{b_r^+\}_{r\geq R}$ necessary for Corollary 8 to imply $|db^+| = 1$ and semiconcavity and *p*-superharmonicity of b^+ on the interior of X; global hyperbolicity ensures timelike maximizing geodesics connect each $x \in X \subset I^-(\gamma(R))$ to $\gamma(r)$ for $r \geq R$.

We are now ready to conclude the proof of Theorem 15 in case (a).

Proof of Theorem 15 in case (a). Recall from the proof of case (b) that in both cases we had identified a neighbourhood U of γ on which the Busemann functions $b = b^{\pm}$ coincide and have vanishing Hessian (hence db is a parallel Killing vector field on U), and a neighbourhood W of γ foliated by timelike lines $r \in (r_x^+, r_x^-) \mapsto E(r, x)$ which are future- and past-asymptotic to γ for each $x \in S$, where $S = U \cap S_0 = W \cap S_0$ is spacelike and totally geodesic with timelike unit normal N = db and $S_0 = \{b = 0\}$. In fact W was the image of an open set $\mathcal{D} \subset \mathbf{R} \times S$ under the smooth diffeomorphism $E(r, s) = \exp_x r db(x)$; moreover, $b^{\pm}(E(r, x)) = r$ on \mathcal{D} , so $b = b^{\pm}$ coincide and are smooth on W. We now argue their Hessians vanish there. Corollary 19 and its time-reversal show $\pm b^{\pm}$ to be *p*-superharmonic on W and $|db^{\pm}| = 1$, thus $b = b^{\pm}$ is *p*-harmonic and satisfies the conclusions of Proposition 9 on W. Corollary 14 then asserts that the Hessian of *b* vanishes on W. Thus *db* is a parallel Killing vector field throughout W (as was already known on U). This shows E gives a local isometry between the metrics $dr^2 - h$ on \mathcal{D} and g on W, where -h is the restriction of g to S.

Taking U (and thus W) smaller if necessary ensures S is a geodesic ball centered at $\gamma(0)$. It remains to deduce that $E(\cdot, x)$ is complete for each $x \in S$, so that $\mathcal{D} = \mathbf{R} \times S$. This is shown as in Galloway [22, p. 383], but we recall the argument for the convenience of the reader; we only argue future-completeness. Let $\hat{\sigma} : [0, R) \to S$ be any radial geodesic starting from $\hat{\sigma}(0) = \gamma(0)$, where R is the radius of the ball S. Let $\alpha_s(\cdot) := E(\cdot, \hat{\sigma}(s))|_{[0,l_s)}$ be future-inextendible, and $l_s = r^+_{\hat{\sigma}(s)} \in (0, \infty]$ its Lorentzian arclength. Fix any r > R. We will now show that

$$(28) l_s > r - s$$

for all $s \in [0, R)$, thus (by arbitrariness of r > R and $\hat{\sigma}$) establishing the claimed future completeness of the asymptotic lines to γ based in S. Denote by A the set of $t \in [0, R)$ such that (28) holds for all $s \in [0, t]$. Clearly, A is an interval containing 0, since $\alpha_0 = \gamma|_{[0,\infty)}$. Let $a := \sup A$. We deduce a = R by showing the non-empty interval A is both relatively open and closed in [0, R). First observe that since \mathcal{D} is open (or equivalently, recalling lower semicontinuity of $l_s = r^+_{\hat{\sigma}(s)}$ from [5, Proposition 9.7]), it follows that A is a relatively open subset of [0, R). On the other hand, we will derive a contradiction by showing a < R implies $a \in A$. We may also assume a > 0 since $0 \in A$ has already been checked. By definition, $l_s > r - s$ for any $s \in [0, a)$. The geometry we have established on W shows $\eta(u) := E(r - u, \hat{\sigma}(u))$ to be a past-directed null geodesic $\eta: [0, a) \to M$ from $\eta(0) = \gamma(r)$. We claim that $l_a > r - a$. If not, then $l_s > r - s > r - a \ge l_a$ for any $s \in [0, a)$. Thus, $\alpha_s(t)$ is well-defined for $s \in [0, a)$ and $t \in [0, l_a)$, and $\alpha_a(t) = \lim_{s \to a} \alpha_s(t)$ by continuity of E. It follows that for $s \in [0, a)$,

$$\alpha_s(t) \ll \alpha_s(r-s) = E(r-s, \hat{\sigma}(s)) = \eta(s) \le \gamma(r).$$

Taking $s \to a$ above and using the closedness of the causal relation guaranteed by global hyperbolicity (a), we conclude that $\alpha_a \subset J^+(\hat{\sigma}(a)) \cap J^-(\gamma(r))$, a contradiction to non-total imprisonment. Thus $l_a > r - a$ and hence $a \in A$. To avoid this contradiction, a = R and (28) holds for all $s \in [0, R)$, to establish the claim. \Box

7. Global splitting

Finally, to globalize the local splitting using connectedness of M, we follow the strategy of Eschenburg [18] (augmented by an observation of Galloway [22] when (b) fails to hold). We detail the argument for completeness. Recall a *flat strip* refers to a totally geodesic isometric immersion $F : (\mathbf{R} \times [0, s_0], dr^2 - ds^2) \longrightarrow (M, g)$ such that $r \in \mathbf{R} \mapsto$ F(r, s) is a (complete) line for any $s \in [0, s_0]$. Two such lines γ and $\tilde{\gamma}$ are called *strongly parallel* if they bound a flat strip, so that $\gamma(r) =$ F(r, 0) and $\tilde{\gamma}(r) = F(r, s_0)$ for all $r \in \mathbf{R}$ and some F as above. To globalize Theorem 15, it is elementary to recall that [18, Lemma 7.2] holds without assuming (a) or (b):

Lemma 20 (Strongly parallel lines share their Busemann functions). Under the hypotheses of Theorem 2: if $\tilde{\gamma}$ and γ are strongly parallel lines, then $I(\tilde{\gamma}) = I(\gamma) := I(\gamma(\mathbf{R}), \gamma(\mathbf{R}))$ and their forward Busemann functions \tilde{b}^+ and b^+ coincide.

Proof of Theorem 1. Let $W \subset M$ be the largest connected open subset (ordered by inclusion) on which the conclusion of Theorem 15 holds, meaning $E(r, x) := \exp_x r db^+$ gives a local isometry from $(\mathbf{R} \times S, dr^2 - h)$ onto its image W in (M, g), and $r \in \mathbf{R} \mapsto E(r, x)$ is an S_0 -line for each $x \in S = W \cap S_0$ (and $\gamma(\cdot) = E(\cdot, \gamma(0))$), and h is the restriction of -g to S. Such a subset exists by Zorn's lemma, and is non-empty by Theorem 15. We claim W is a connected component of M.

Whenever $\hat{\sigma} : [0, s_0] \longrightarrow S$ is an *h*-geodesic in *S*, then $F(r, s) = E(r, \hat{\sigma}(s))$ is a flat strip, so its boundaries are strongly parallel. Lemma 20 shows the Busemann functions associated with the lines $E(\cdot, x)$ and $E(\cdot, y)$ through $x = \hat{\sigma}(0)$ and $y = \hat{\sigma}(s_0)$ coincide. Since *W* is an open and connected Lorentzian product of (\mathbf{R}, dr^2) with (S, h), it follows that the hypersurface *S* is totally geodesic and path-connected. Any path in *S* from $\gamma(0)$ to $x \in S$ can therefore be replaced by a broken geodesic consisting of arbitrarily short geodesic segments, so iterating the argument above yields a finite sequence of lines starting with γ and ending with $\beta(\cdot) = E(\cdot, x)$ such that each adjacent pair of lines in the sequence is strongly parallel. Thus $I(\gamma) = I(\beta)$ and the Busemann functions associated with γ and β coincide.

To derive a contradiction, suppose W has a boundary point $y \in \partial W$. Fix a coordinate chart around y. Every Euclidean ball of sufficiently small radius in these coordinates centered near y will be Lorentziangeodesically convex. Choose such a ball $B_{2\delta}(y)$ centered at y and then $x \in W \cap B_{\delta}(y)$ sufficiently close to y that the largest Euclidean ball $B_{\epsilon}(x)$ in W is also Lorentzian-geodesically convex. This maximality of $\epsilon < 2\delta$ implies there exists $z \in \partial B_{\epsilon}(x) \cap \partial W$. Moreover, there is a Lorentzian geodesic $\sigma : [0, s_0] \longrightarrow M$ in $B_{\epsilon}(x)$ from $x = \sigma(0)$ to $z = \sigma(s_0)$ of the form $\sigma(s) = ((1 - s/s_0)r_0 + sr_1/s_0, \hat{\sigma}(s))$ where $\hat{\sigma}(s)$ is a (nonconstant) *h*-geodesic in *S*. For each $s < s_0$, the preceding paragraph shows the line $E(\cdot, \hat{\sigma}(s))$ through $\sigma(s)$ shares the same Busemann function as γ . The product geometry guarantees that the tangent X(s) to this line at $\sigma(s)$ is the parallel translate along σ of the tangent X(0) to the analogous line through $x = \sigma(0)$. Letting $s \to s_0$, [23, Lemma 2.4] provides a subsequence of these lines which converge in the limit curve sense to a line through $z = \sigma(s_0)$. Since the parallel transport $X(s_0)$ of X(0) along σ is tangent to this line, and timelike, we can proper-time reparameterize the limiting line as $\tilde{\gamma} : (a_-, a_+) \longrightarrow M$. By the time-translation symmetry of W we can assume $r_1 = 0$, and choose $z = \tilde{\gamma}(0)$ to lie in the closure of *S*.

In case (b) the line $\tilde{\gamma}$ is complete: $a_+ = \infty = -a_-$. Denote its forward Busemann function by \tilde{b}^+ . Theorem 15 provides a neighbourhood \tilde{W} of $\tilde{\gamma}$ on which $\tilde{E}(r, x) := \exp_{x} r d\tilde{b}$ gives a local isometry $\tilde{E}: (\mathbf{R} \times \tilde{S}, dr^2 - \tilde{h}) \longrightarrow (M, g)$ such that $r \in \mathbf{R} \mapsto \tilde{E}(r, x)$ is an \tilde{S}_0 -line for each $x \in \tilde{S} = \tilde{W} \cap \tilde{S}_0$ (and $\tilde{\gamma}(\cdot) = \tilde{E}(\cdot, \tilde{\gamma}(0))$), and \tilde{h} is the restriction of -g to \tilde{S} . Since the product geometry shows both S_0 and \tilde{S}_0 are totally geodesic and orthogonal to $X(s_0)$, they must coincide on the nonempty set $W \cap \tilde{W}$ — as must E and \tilde{E} . Since $\tilde{F}(r,s) = \exp_{\sigma(s)} rX(s)$ is a flat strip bounded by β and $\tilde{\gamma}$, Lemma 20 shows $b^+ = \tilde{b}^+$ hence $S_0 = \tilde{S}_0$. Note for $w \in S$ and $\tilde{w} \in \tilde{S}$, the lines $E(\cdot, w)$ and $E(\cdot, \tilde{w})$ cannot cross (unless they coincide), since both are assumed to maximize time to S_0 . Thus $W \cup W$ provides an enlargement of W on which the conclusion of Theorem 15 holds, contradicting the assumed maximality of W. This contradiction forces ∂W to be empty; connectedness of M yields M = W, and the splitting becomes global by Corollary 16. Ricci nonnegativity of (S, h) follows from the strong energy condition (1) by the tensorization of the Ricci tensor in product geometries, as in [49, Corollary 7.43] with trivial warping factor f = 1.

The logic and conclusion of the preceding paragraph will apply to case (a) also as soon as completeness of the timelike line $\tilde{\gamma}$ is established. We'll show future-completeness $a_+ = \infty$ as in [22]; past-completeness $a_- = -\infty$ can be shown similarly (or by time-reversal symmetry). Defining $\tilde{F}(r,s) = \exp_{\sigma(s)} rX(s)$ as above, it remains true that the restriction of \tilde{F} to $\mathbf{R} \times [0,s]$ is a flat strip for each $s < s_0$. Moreover, for each $t \in (a_-, a_+)$ there is a sequence $(r_i, s_i) \in \mathbf{R} \times [0, s_0)$ such that $\tilde{F}(r_i, s_i) \to \tilde{\gamma}(t)$. Since $\tilde{\gamma}(t)$ is separated from $\tilde{\gamma}(0) \in \bar{S}$ by time $t < a_+$ and $F(r_i, s_i)$ is separated from $\tilde{\gamma}(0)$ by time not much less than $r_i + (1 - s_i/s_0)r_0$, continuity of the time-separation function ℓ implies $r_i < a_+ + 1$ for *i* sufficiently large. Also $R > |r_0| + a_+ + 1 + |\hat{\sigma}'(0)|_h$ ensures $\tilde{F}(R, 0)$ lies in the future of $\tilde{F}(r_i, s_i)$ for all *i* sufficiently large, by (27). Since $t < a_+ < \infty$ was arbitrary, the restriction $\tilde{\gamma}|_{[0,a_+)}$ — being future-inextendible — is a timelike curve of unbounded \tilde{g} -length in the compact diamond $J(\tilde{\gamma}(0), \tilde{F}(R, 0))$, contradicting the nontotal imprisonment which global hyperbolicity (a) implies. We therefore conclude future-completeness of $\tilde{\gamma}$: $a_+ = +\infty$ (and past-completeness $a_- = -\infty$ similarly).

Apart from (metric) completeness of (S, h), Theorem 1 has now been established. To see completeness of (S, h), let $x_k \in S$ denote a Cauchy sequence. Then the entire sequence $\{x_k\}_{k\in\mathbb{N}}$ lies in an open *h*-ball $B_r(x_1)$ of radius *r* sufficiently large. From (3) it follows that x_k also lies in the diamond $J(E(-r, x_1), E(r, x_1))$, which is compact assuming (a) global hyperbolicity. In this case x_k admits a subsequential limit x_{∞} in *M*. We claim $d_h(x_k, x_{\infty}) \to 0$. This follows from the facts (i) that d_h metrizes the topology *S* inherits from *M*, and (ii) that whenever a Cauchy sequence has a convergent subsequence then the full sequence also converges (to the same limit).

If instead (M, g) is (b) timelike geodesically complete, we will assume incompleteness of (S, h) to derive a contradiction. In this case the Hopf-Rinow theorem provides an *h*-geodesic $\tau : (s, t) \longrightarrow S$ which is inextendible (say at *t*). Lifting τ produces a timelike geodesic $\beta(r) :=$ $(r, \tau(r/2))$ in the product metric $dr^2 - h$, which is future-inextendible at r = 2t: the desired contradiction to (b). So (S, h) is complete and Theorem 1 is established.

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