Continuity and injectivity of optimal maps for non-negatively cross-curved costs*

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Abstract

Consider transportation of one distribution of mass onto another, chosen to optimize the total expected cost, where cost per unit mass transported from x to y is given by a smooth function $c(x, y)$. If the source density $f^+(x)$ is bounded away from zero and infinity in an open region $U' \subset \mathbf{R}^n$, and the target density $f^-(y)$ is bounded away from zero and infinity on its support $\overline{V} \subset \mathbf{R}^n$, which is strongly c-convex with respect to U', and the transportation cost c is non-negatively cross-curved, we deduce continuity and injectivity of the optimal map inside U' (so that the associated potential u belongs to $C^1(U')$). This result provides a crucial step in the low/interior regularity setting: in a subsequent paper [15], we use it to establish regularity of optimal maps with respect to the Riemannian distance squared on arbitrary products of spheres. The present paper also provides an argument required by Figalli and Loeper to conclude in two dimensions continuity of optimal maps under the weaker (in fact, necessary) hypothesis **(A3w)** [17]. In higher dimensions, if the densities $f^\pm$ are Hölder continuous, our result permits continuous differentiability of the map inside U' (in fact, $C^{2,\alpha}_{loc}$ regularity of the associated potential) to be deduced from the work of Liu, Trudinger and Wang [33].

Contents

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1 Introduction

Given probability densities $0 \le f^{\pm} \in L^1(\mathbf{R}^n)$ with respect to Lebesgue measure $\mathscr{L}^n$ on $\mathbf{R}^n$, and a cost function $c : \mathbf{R}^n \times \mathbf{R}^n \longmapsto [0, +\infty]$, Monge's transportation problem is to find a map $G : \mathbf{R}^n \longmapsto \mathbf{R}^n$ pushing $d\mu^+ = f^+ d\mathscr{L}^n$ forward to $d\mu^- = f^- d\mathscr{L}^n$ which minimizes the expected transportation cost [38]

$$\inf_{G_{\#}\mu^+ = \mu^-} \int_{\mathbf{R}^n} c(x, G(x)) d\mu(x), \tag{1.1}$$

where $G_{\#}\mu^+ = \mu^-$ means $\mu^-[Y] = \mu^+[G^{-1}(Y)]$ for each Borel $Y \subset \mathbf{R}^n$.

In this context it is interesting to know when a map attaining this infimum exists; sufficient conditions for this were found by Gangbo [20] and by Levin [31], extending work of a number of authors described in [21] [46]. One may also ask when G will be smooth, in which case it must satisfy the prescribed Jacobian equation $|\det DG(x)| = f^+(x)/f^-(G(x))$, which turns out to reduce to a degenerate elliptic partial differential equation of Monge-Ampère type for a scalar potential u satisfying $Du(\tilde{x}) = -D_x c(\tilde{x}, G(\tilde{x}))$. Sufficient conditions for this were discovered by Ma, Trudinger and Wang [37] and Trudinger and Wang [43] [44], after results for the special case $c(x, y) = |x - y|^2/2$ had been worked out by Brenier [4], Delanöe [12], Caffarelli [6] [5] [7] [8] [9], and Urbas [45], and for the cost $c(x, y) = -\log|x - y|$ and measures supported on the unit sphere by Wang [48].

If the ratio $f^+(x)/f^-(y)$ — although bounded away from zero and infinity — is not continuous, the map G will not generally be differentiable, though one may still hope for it to be continuous. This question is not merely of academic interest, since discontinuities in $f^\pm$ arise unavoidably in applications such as partial transport problems [10] [3] [13] [14]. Such results were established for the classical cost $c(x,y) = |x-y|^2/2$ by Caffarelli [5] [7] [8], for its restriction to the product of the boundaries of two strongly convex sets by Gangbo and McCann [22], and for more general costs satisfying the strong regularity hypothesis **(A3)** of Ma, Trudinger and Wang [37] — which excludes the cost $c(x,y) = |x-y|^2/2$ — by Loeper [34]; see also [27] [32] [44]. Under the weaker and degenerate hypothesis **(A3w)** of Trudinger and Wang [43], which includes the cost $c(x,y) = |x-y|^2/2$ (and whose necessity for regularity was shown by Loeper [34]), such a result remains absent from the literature; we aim to provide it below under a slight strengthening of their condition (still including the quadratic cost) which appeared in Kim and McCann [28][29], called *non-negative cross-curvature.* (Related but different families of strengthenings were investigated by Loeper and Villani [36] and Figalli and Rifford [18].) Our main result is stated in Theorem 2.1. A number of interesting cost functions do satisfy non-negative cross-curvature hypothesis, and have applications in economics [16] and statistics [40]. Examples include the Euclidean distance between two convex graphs over two sufficiently convex sets in $\mathbf{R}^n$ [37], the Riemannian distance squared on multiple products of round spheres (and their Riemannian submersion quotients, including products of complex projective spaces $\mathbf{CP}^n$) [29], and the simple harmonic oscillator action [30]. In a sequel, we apply the techniques developed here to deduce regularity of optimal maps in the latter setting [15]. Moreover, Theorem 2.1 allows one to apply the higher interior regularity results established by Liu, Trudinger and Wang [33], ensuring in particular that the transport map is C^∞-smooth if f^+ and f^- are.

Most of the regularity results quoted above derive from one of two approaches. The continuity method, used by Delanoë, Urbas, Ma, Trudinger and Wang, is a time-honored technique for solving nonlinear equations. Here one perturbs a manifestly soluble problem (such as $|\det DG_0(x)| = f^+(x)/f_0(G_0(x))$ with $f_0 = f^+$, so that $G_0(x) = x$) to the problem of interest ($|\det DG_1(x)| = f^+(x)/f_1(G_1(x))$, $f_1 = f^-$) along a family $\{f_t\}_t$ designed to ensure the set of $t \in [0,1]$ for which it is soluble is both open and closed. Openness follows from linearization and non-degenerate ellipticity using an implicit function theorem. For the non-degenerate ellipticity and closedness, it is required to establish estimates on the size of derivatives of the solutions (assuming such solutions exist) which depend only on information known a priori about the data (c, f_t). In this way one obtains smoothness of the solution $y = G_1(x)$ from the same argument which shows G_1 to exist.

The alternative approach relies on first knowing existence and uniqueness of a Borel map which solves the problem in great generality, and then deducing continuity or smoothness by close examination of this map after imposing additional conditions on the data $(c, f^\pm)$. Although precursors can be traced back to Alexandrov [2], in the present context this method was largely developed and refined by Caffarelli [5] [7] [8], who used convexity of u crucially to localize the map $G(x) = Du(x)$ and renormalize its behaviour near a point $(\tilde{x}, G(\tilde{x}))$ of interest in the borderline case $c(x,y) = -\langle x, y\rangle$. For non-borderline **(A3)** costs, simpler estimates suffice to deduce continuity of G, as in [22] [11] [34] [44]; in this case Loeper was actually able to deduce an explicit bound $\alpha = (4n-1)^{-1}$ on the Hölder exponent of G when $n > 1$, which was recently improved to its sharp value $\alpha = (2n-1)^{-1}$ by Liu [32] using a technique related to the one we develop below and discovered independently from us; both Loeper and Liu also obtained explicit exponents $\alpha = \alpha(n,p)$ for $f^+ \in L^p$ with $p > n$ [34] or $p > (n+1)/2$ [32] and $1/f^- \in L^\infty$. Explicit bounds on the exponent are much worse in the classical case $c(x,y) = -\langle x, y\rangle$ [19], when such exponents do not even exist unless $\log \frac{f^+(x)}{f^-(y)} \in L^\infty$ [8] [47].

Below we extend the approach of Caffarelli to non-negatively cross-curved costs, a class which includes the classical quadratic cost. Our idea is to add a null Lagrangian term to the cost and exploit diffeomorphism (i.e. gauge) invariance to choose coordinates which depend on the point of interest that restore convexity of $u(x)$; our strengthened hypothesis then permits us to exploit Caffarelli's approach more systematically than Liu was able to do [32]. However, we still need to overcome serious difficulties, such as getting an Alexandrov estimate for c-subdifferentials (see Section 7) and dealing with the fact that the domain of the cost function (where it is smooth and satisfies appropriate cross-curvature conditions) may not be the whole of $\mathbf{R}^n$. (This situation arises, for example, when optimal transportation occurs between domains in Riemannian manifolds for the distance squared cost or similar type.) The latter is accomplished using Theorem 5.1, where it is first established that optimal transport does not send interior points to boundary points, and vice versa, under the strong c-convexity hypothesis **(B2u)** described in the next section. (For this result to hold, the cost needs not to satisfy the condition **(A3w)**.) Without our strengthening of Trudinger and Wang's hypothesis [43] (i.e. with only **(A3w)**), we obtain the convexity of all level sets of $u(x)$ in our chosen coordinates as Liu also did; this yields some hope of applying Caffarelli's method and the full body of techniques systematized in Gutierrez [23], but we have not been successful at overcoming the remaining difficulties in such generality. In two dimensions however, there is an alternate approach to establishing continuity of optimal maps which applies to this more general case; it was carried out by Figalli and Loeper [17], but relies on Theorem 5.1, first proved below.

2 Main result

Let us begin by formulating the relevant hypothesis on the cost function $c(x,y)$ in a slightly different format than Ma, Trudinger and Wang [37]. For each $(\tilde{x},\tilde{y}) \in \overline{U} \times \overline{V}$ assume:
(B0) $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^n$ are open and bounded and $c \in C^4\big(\overline{U} \times \overline{V}\big)$;
(B1) (bi-twist) $\left.\begin{array}{l} x \in \overline{U} \longmapsto -D_y c(x,\tilde{y}) \\ y \in \overline{V} \longmapsto -D_x c(\tilde{x},y) \end{array}\right\}$ are diffeomorphisms onto their ranges;
(B2) (bi-convex) $\left.\begin{array}{l} U_{\tilde{y}} := -D_y c(U,\tilde{y}) \\ V_{\tilde{x}} := -D_x c(\tilde{x},V) \end{array}\right\}$ are convex subsets of $\mathbf{R}^n$;
(B3) (non-negative cross-curvature)

$$\mathrm{cross}_{(x(0),y(0))}[x'(0),y'(0)] := -\frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{(s,t)=(0,0)} c(x(s),y(t)) \geq 0 \tag{2.1}$$

for every curve $t \in [-1,1] \longmapsto \big(D_y c(x(t), y(0)), D_x c(x(0), y(t))\big) \in \mathbf{R}^{2n}$ which is an affinely parameterized line segment.

If the convex domains $U_{\tilde y}$ and $V_{\tilde x}$ in **(B2)** are all strongly convex, we say **(B2u)** holds. Here a convex set $U \subset \mathbf{R}^n$ is said to be *strongly* convex if there exists a radius $R < +\infty$ (depending only on U,) such that each boundary point $\tilde x \in \partial U$ can be touched from outside by a sphere of radius R enclosing U; i.e. $U \subset B_R(\tilde x - R\hat n_U(\tilde x))$ where $\hat n_U(\tilde x)$ is an outer unit normal to a hyperplane supporting U at $\tilde x$. When U is smooth, this means all principal curvatures of its boundary are bounded below by $1/R$. Hereafter $\overline{U}$ denotes the closure of U, $\operatorname{int} U$ denotes its interior, $\operatorname{diam} U$ its diameter, and for any measure $\mu^+ \geq 0$ on $\overline{U}$, we use the term *support* and the notation $\operatorname{spt}\mu^+ \subset \overline{U}$ to refer to the smallest closed set carrying the full mass of μ^+.

Condition **(B3)** is the above-mentioned strengthening of Trudinger and Wang's criterion **(A3w)** guaranteeing smoothness of optimal maps in the Monge transportation problem (1.1); unlike us, they require (2.1) only if, in addition [43],

$$\frac{\partial^2}{\partial s \partial t}\bigg|_{(s,t)=(0,0)} c(x(s), y(t)) = 0. \tag{2.2}$$

Necessity of Trudinger and Wang's condition for continuity was shown by Loeper [34], who noted its covariance (as did [28] [41]) and some relations to curvature. Their condition relaxes the hypothesis **(A3)** proposed earlier with Ma [37], which required strict positivity of (2.1) when (2.2) holds. The strengthening considered here was first studied in a different but equivalent form by Kim and McCann [28], where both the original and the modified conditions were shown to correspond to pseudo-Riemannian sectional curvature conditions induced by the cost c on $U \times V$, highlighting their invariance under reparametrization of either U or V by diffeomorphism; see [28, Lemma 4.5]. The convexity of $U_{\tilde y}$ required in **(B2)** is called *c-convexity of U with respect to $\tilde y$* by Ma, Trudinger and Wang (or strong c-convexity if **(B2u)** holds); they call curves $x(s) \in U$, for which $s \in [0,1] \longmapsto -D_y c(x(s), \tilde y)$ is a line segment, *c-segments with respect to $\tilde y$*. Similarly, V is said to be strongly c^*-convex with respect to $\tilde x$ — or with respect to $\overline{U}$ when it holds for all $\tilde x \in \overline{U}$ — and the curve $y(t)$ from (2.1) is said to be a c^*-segment with respect to $\tilde x$. Such curves correspond to geodesics $(x(t), \tilde y)$ and $(\tilde x, y(t))$ in the geometry of Kim and McCann. Here and throughout, *line segments* are always presumed to be affinely parameterized.

We are now in a position to summarize our main result:

Theorem 2.1 (Interior continuity and injectivity of optimal maps). *Let $c \in C^4\big(\overline{U} \times \overline{V}\big)$ satisfy* **(B0)**–**(B3)** *and* **(B2u)**. *Fix probability densities $f^+ \in L^1\big(U\big)$ and $f^- \in L^1\big(V\big)$ with $(f^+/f^-) \in L^\infty\big(U \times V\big)$ and set $d\mu^\pm := f^\pm d\mathscr{L}^n$. If the ratio $(f^-/f^+) \in L^\infty(U' \times V)$ for some open set $U' \subset U$, then the minimum (1.1) is attained by a map $G : \overline{U} \longmapsto \overline{V}$ whose restriction to U' is continuous and one-to-one.*

Proof. As recalled below in Section 3 (or see e.g. [46]) it is well-known by Kantorovich duality that the optimal joint measure $\gamma \in \Gamma(\mu^+, \mu^-)$ from (3.1) vanishes outside the c-subdifferential

(3.3) of a potential $u = u^{c^*c}$ satisfying the c-convexity hypothesis (3.2), and that the map $G : \overline{U} \longmapsto \overline{V}$ which we seek is uniquely recovered from this potential using the diffeomorphism **(B1)** to solve (3.5). Thus the continuity claimed in Theorem 2.1 is equivalent to $u \in C^1(U')$.

Since $\mu^\pm$ do not charge the boundaries of U (or of V), Lemma 3.1(e) shows the c-Monge-Ampère measure defined in (3.6) has density satisfying $|\partial^c u| \le \|f^+/f^-\|_{L^\infty(U\times V)}$ on $\overline{U}$ and $\|f^-/f^+\|^{-1}_{L^\infty(U'\times V)} \le |\partial^c u| \le \|f^+/f^-\|_{L^\infty(U'\times V)}$ on U'. Thus $u \in C^1(U')$ according to Theorem 9.2. Injectivity of G follows from Theorem 9.1, and the fact that the graph of G is contained in the set $\partial^c u \subset \overline{U} \times \overline{V}$ of (3.3). □

Note that in case $f^+ \in C_c(U)$ is continuous and compactly supported, choosing $U' = U'_\varepsilon = \{f^+ > \varepsilon\}$ for all $\varepsilon > 0$, yields continuity and injectivity of the optimal map $y = G(x)$ throughout U'_0.

Theorem 2.1 provides a necessary prerequisite for the higher interior regularity results established by Liu, Trudinger and Wang in [33] — a prerequisite which one would prefer to have under the weaker hypotheses **(B0)**–**(B2)** and **(A3w)**. Note that these interior regularity results can be applied to manifolds, after getting suitable stay-away-from-the-cut-locus results: this is accomplished for multiple products of round spheres in [15], to yield the first regularity result that we know for optimal maps on Riemannian manifolds which are not flat, yet have some vanishing sectional curvatures.

3 Background, notation, and preliminaries

Kantorovich discerned [25] [26] that Monge's problem (1.1) could be attacked by studying the linear programming problem

$$\min_{\gamma\in\Gamma(\mu^+,\mu^-)} \int_{\overline{U}\times\overline{V}} c(x,y)\, d\gamma(x,y). \tag{3.1}$$

Here $\Gamma(\mu^+,\mu^-)$ consists of the joint probability measures on $\overline{U} \times \overline{V} \subset \mathbf{R}^n \times \mathbf{R}^n$ having $\mu^\pm$ for marginals. According to the duality theorem from linear programming, the optimizing measures γ vanish outside the zero set of $u(x) + v(y) + c(x,y) \ge 0$ for some pair of functions $(u,v) = (v^c, u^{c^*})$ satisfying

$$v^c(x) := \sup_{y\in\overline{V}} -c(x,y) - v(y), \qquad u^{c^*}(y) := \sup_{x\in\overline{U}} -c(x,y) - u(x); \tag{3.2}$$

these arise as optimizers of the dual program. This zero set is called the c-subdifferential of u, and denoted by

$$\partial^c u = \{(x,y) \in \overline{U} \times \overline{V} \mid u(x) + u^{c^*}(y) + c(x,y) = 0\}; \tag{3.3}$$

we also write $\partial^c u(x) := \{y \mid (x,y) \in \partial^c u\}$, and $\partial^{c^*} u^{c^*}(y) := \{x \mid (x,y) \in \partial^c u\}$, and $\partial^c u(X) := \cup_{x\in X} \partial^c u(x)$ for $X \subset \mathbf{R}^n$. Formula (3.2) defines a generalized Legendre-Fenchel transform called the *c-transform*; any function satisfying $u = u^{c^*c} := (u^{c^*})^c$ is said to be *c-convex*, which reduces to ordinary convexity in the case of the cost $c(x,y) = -\langle x,y\rangle$. In that case $\partial^c u$ reduces to the ordinary subdifferential ∂u of the convex function u, but more generally we define

$$\partial u := \{(x,p) \in \overline{U} \times \mathbf{R}^n \mid u(\tilde{x}) \geq u(x) + \langle p, \tilde{x} - x\rangle + o(|\tilde{x} - x|) \text{ as } \tilde{x} \to x\}, \tag{3.4}$$

$\partial u(x) := \{p \mid (x,p) \in \partial u\}$, and $\partial u(X) := \cup_{x \in X} \partial u(x)$. Assuming $c \in C^2\big(\overline{U} \times \overline{V}\big)$ (which is the case if **(B0)** holds), any c-convex function $u = u^{c^* c}$ will be semi-convex, meaning its Hessian admits a bound from below $D^2 u \geq -\|c\|_{C^2}$ in the distributional sense; equivalently, $u(x) + \|c\|_{C^2}|x|^2/2$ is convex on each ball in U [21]. In particular, u will be twice-differentiable $\mathscr{L}^n$-a.e. on U in the sense of Alexandrov.

As in [20] [31] [37], hypothesis **(B1)** shows the map $G : \operatorname{dom} Du \longmapsto \overline{V}$ is uniquely defined on the set $\operatorname{dom} Du \subset \overline{U}$ of differentiability for u by

$$D_x c(\tilde{x}, G(\tilde{x})) = -Du(\tilde{x}). \tag{3.5}$$

The graph of G, so-defined, lies in $\partial^c u$. The task at hand is to show continuity and injectivity of G — the former being equivalent to $u \in C^1(U)$ — by studying the relation $\partial^c u \subset \overline{U} \times \overline{V}$.

To this end, we define a Borel measure $|\partial^c u|$ on $\mathbf{R}^n$ associated to u by

$$|\partial^c u|(X) := \mathscr{L}^n(\partial^c u(X)) \tag{3.6}$$

for each $X \subset \mathbf{R}^n$; it will be called the *c-Monge-Ampère measure* of u. (Similarly, we define $|\partial u|$.) We use the notation $|\partial^c u| \geq \lambda$ on U' as a shorthand to indicate $|\partial^c u|(X) \geq \lambda \mathscr{L}^n(X)$ for each $X \subset U'$; similarly, $|\partial^c u| \leq \Lambda$ indicates $|\partial^c u|(X) \leq \Lambda \mathscr{L}^n(X)$. As the next lemma shows, uniform bounds above and below on the marginal densities of a probability measure γ vanishing outside $\partial^c u$ imply similar bounds on $|\partial^c u|$.

Lemma 3.1 (Properties of c-Monge-Ampère measures)**.** *Let c satisfy* **(B0)**-**(B1)**, *while u and u_k denote c-convex functions for each $k \in \mathbf{N}$. Fix $\tilde{x} \in \overline{X}$ and constants $\lambda, \Lambda > 0$.*
(a) Then $\partial^c u(\overline{U}) \subset \overline{V}$ and $|\partial^c u|$ is a Borel measure of total mass $\mathscr{L}^n\big(\overline{V}\big)$ on $\overline{U}$.
(b) If $u_k \to u_\infty$ uniformly, then u_∞ is c-convex and $|\partial^c u_k| \rightharpoonup |\partial^c u_\infty|$ weakly-$*$ in the duality against continuous functions on $\overline{U} \times \overline{V}$.
(c) If $u_k(\tilde{x}) = 0$ for all k, then the functions u_k converge uniformly if and only if the measures $|\partial^c u_k|$ converge weakly-$*$.
(d) If $|\partial^c u| \leq \Lambda$ on $\overline{U}$, then $|\partial^{c^*} u^{c^*}| \geq 1/\Lambda$ on $\overline{V}$.
(e) If a probability measure $\gamma \geq 0$ vanishes outside $\partial^c u \subset \overline{U} \times \overline{V}$, and has marginal densities $f^{\pm}$, then $f^+ \geq \lambda$ on $U' \subset \overline{U}$ and $f^- \leq \Lambda$ on $\overline{V}$ imply $|\partial^c u| \geq \lambda/\Lambda$ on U', whereas $f^+ \leq \Lambda$ on U' and $f^- \geq \lambda$ on $\overline{V}$ imply $|\partial^c u| \leq \Lambda/\lambda$ on U'.

Proof. (a) The fact $\partial^c u(\overline{U}) \subset \overline{V}$ is an immediate consequence of definition (3.3). Since $c \in C^1(\overline{U} \times \overline{V})$, the c-transform $v = u^{c^*} : \overline{V} \longmapsto \mathbf{R}$ defined by (3.2) can be extended to a Lipschitz function on a neighbourhood of $\overline{V}$, hence Rademacher's theorem asserts $\operatorname{dom} Dv$ is a set of full Lebesgue measure in $\overline{V}$. Use **(B1)** to define the unique solution $F : \operatorname{dom} Dv \longmapsto \overline{U}$ to

$$D_y c(F(\tilde{y}), \tilde{y}) = -Dv(\tilde{y}).$$

As in [20] [31], the vanishing of $u(x)+v(y)+c(x,y) \geq 0$ implies $\partial^{c^*} v(\tilde{y}) = \{F(\tilde{y})\}$, at least for all points $\tilde{y} \in \operatorname{dom} Dv$ where $\overline{V}$ has Lebesgue density greater than one half. For Borel $X \subset \mathbf{R}^n$, this shows $\partial^c u(X)$ differs from the Borel set $F^{-1}(X) \cap \overline{V}$ by a $\mathscr{L}^n$ negligible subset of $\overline{V}$, whence $|\partial^c u| = F_{\#}(\mathscr{L}^n \lfloor_{\overline{V}})$ so claim (a) of the lemma is established.

(b) Let $\|u_k - u_\infty\|_{L^\infty(\overline{U})} \to 0$. It is not hard to deduce c-convexity of u_∞, as in e.g. [16]. Define $v_k = u_k^{c^*}$ and F_k on $\operatorname{dom} Dv_k \subset \overline{V}$ as above, so that $|\partial^c u_k| = F_{k\#}(\mathscr{L}^n \lfloor_{\overline{V}})$. Moreover, $v_k \to v_\infty$ in $L^\infty(V)$, where v_∞ is the c^*-dual to u_∞. The uniform semiconvexity of v_k (i.e. convexity of $v_k(y) + \frac{1}{2}\|c\|_{C^2}|y|^2$) ensures pointwise convergence of $Dv_k \to Dv_\infty$ $\mathscr{L}^n$-a.e. on $\overline{V}$. From $D_y c(F_k(\tilde{y}), \tilde{y}) = -Dv_k(\tilde{y})$ we deduce $F_k \to F_\infty$ $\mathscr{L}^n$-a.e. on $\overline{V}$. This is enough to conclude $|\partial^c u_k| \rightharpoonup |\partial^c u_k|$, by testing the convergence against continuous functions and applying Lebesgue's dominated convergence theorem.

(c) To prove the converse, suppose u_k is a sequence of c-convex functions which vanish at $\tilde{x}$ and $|\partial^c u_k| \rightharpoonup \mu_\infty$ weakly-$*$. Since the u_k have Lipschitz constants dominated by $\|c\|_{C^1}$ and $\overline{U}$ is compact, any subsequence of the u_k admits a convergent further subsequence by the Ascoli-Arzelà Theorem. A priori, the limit u_∞ might depend on the subsequences, but (b) guarantees $|\partial^c u_\infty| = \mu_\infty$, after which [34, Proposition 4.1] identifies u_∞ uniquely in terms of $\mu^+ = \mu_\infty$ and $\mu^- = \mathscr{L}^n \lfloor_{\overline{V}}$, up to an additive constant; this arbitrary additive constant is fixed by the condition $u_\infty(\tilde{x}) = 0$. Thus the whole sequence u_k converges uniformly.

(e) Now assume a finite measure $\gamma \geq 0$ vanishes outside $\partial^c u$ and has marginal densities $f^\pm$. Then the second marginal $d\mu^- := f^- d\mathscr{L}^n$ of γ is absolutely continuous with respect to Lebesgue and γ vanishes outside the graph of $F : \overline{V} \longmapsto U$, whence $\gamma = (F \times id)_\# \mu^-$ by e.g. [1, Lemma 2.1]. (Here id denotes the identity map, restricted to the domain $\operatorname{dom} Dv$ of definition of F.) Recalling that $|\partial^c u| = F_\#(\mathscr{L}^n \lfloor_{\overline{V}})$ (see the proof of (a) above), for any Borel $X \subset U'$ we have

$$\lambda |\partial^c u|(X) = \lambda \mathscr{L}^n(F^{-1}(X)) \leq \int_{F^{-1}(X)} f^-(y) d\mathscr{L}^n(y) = \int_X f^+(x) d\mathscr{L}^n(x) \leq \Lambda \mathscr{L}^n(X)$$

whenever $\lambda \leq f^-$ and $f^+ \leq \Lambda$. We can also reverse the last four inequalities and interchange λ with Λ to establish claim (e) of the lemma.

(d) The last point remaining follows from (e) by taking $\gamma = (F \times id)_\# \mathscr{L}^n$. Indeed an upper bound λ on $|\partial^c u| = F_\# \mathscr{L}^n$ throughout $\overline{U}$ and lower bound 1 on $\mathscr{L}^n$ translate into a lower bound $1/\lambda$ on $|\partial^{c^*} u^{c^*}|$, since the reflection γ^* defined by $\gamma^*(Y \times X) := \gamma(X \times Y)$ for each $X \times Y \subset U \times V$ vanishes outside $\partial^{c^*} u^{c^*}$ and has second marginal absolutely continuous with respect to Lebesgue by the hypothesis $|\partial^c u| \leq \lambda$. $\square$

Remark 3.2 (Monge-Ampère type equation)**.** Differentiating (3.5) formally with respect to $\tilde{x}$ and recalling $|\det DG(\tilde{x})| = f^+(\tilde{x})/f^-(G(\tilde{x}))$ yields the Monge-Ampère type equation

$$\frac{\det[D^2 u(\tilde{x}) + D^2_{xx} c(\tilde{x}, G(\tilde{x}))]}{|\det D^2_{xy} c(\tilde{x}, G(\tilde{x}))|} = \frac{f^+(\tilde{x})}{f^-(G(\tilde{x}))} \tag{3.7}$$

on U, where $G(\tilde{x})$ is given as a function of $\tilde{x}$ and $Du(\tilde{x})$ by (3.5). Degenerate ellipticity follows from the fact that $y = G(x)$ produces equality in $u(x) + u^{c^*}(y) + c(x,y) \geq 0$. A condition under which c-convex weak-$*$ solutions are known to exist is given by

$$\int_{\overline{U}} f^+(x) d\mathscr{L}^n(x) = \int_{\overline{V}} f^-(y) d\mathscr{L}^n(y).$$

The boundary condition $\partial^c u(\overline{U}) \subset \overline{V}$ which then guarantees Du to be uniquely determined f^+-a.e. is built into our definition of c-convexity. In fact, [34, Proposition 4.1] shows u to be uniquely determined up to additive constant if either $f^+ > 0$ or $f^- > 0$ $\mathscr{L}^n$-a.e. on its connected domain, U or V.

A key result we shall exploit several times is a maximum principle first deduced from Trudinger and Wang's work [43] by Loeper; see [34, Theorem 3.2]. A simple and direct proof, and also an extension can be found in [28, Theorem 4.10], where the principle was also called 'double-mountain above sliding-mountain' (**DASM**). Other proofs and extensions appear in [44] [42] [46] [36] [18]:

Theorem 3.3 (Loeper's maximum principle '**DASM**'). *Assume* **(B0)**–**(B2)** *and* **(A3w)** *and fix $x, \tilde{x} \in \overline{U}$. If $t \in [0,1] \longmapsto -D_x c(\tilde{x}, y(t))$ is a line segment then $f(t) := -c(x, y(t)) + c(\tilde{x}, y(t)) \leq \max\{f(0), f(1)\}$ for all $t \in [0,1]$.*

It is through this theorem and the next that hypothesis **(A3w)** and the non-negative cross-curvature hypothesis **(B3)** enter crucially. Among the many corollaries Loeper deduced from this result, we shall need two. Proved in [34, Theorem 3.1 and Proposition 4.4] (alternately [28, Theorem 3.1] and [27, A.10]), they include the c-convexity of the so-called *contact set* (meaning the c^*-subdifferential at a point), and a local to global principle.

Corollary 3.4. *Assume* **(B0)**–**(B2)** *and* **(A3w)** *and fix $(\tilde{x}, \tilde{y}) \in \overline{U} \times \overline{V}$. If u is c-convex then $\partial^c u(\tilde{x})$ is c^*-convex with respect to $\tilde{x} \in U$, i.e. $-D_x c(\tilde{x}, \partial^c u(\tilde{x}))$ forms a convex subset of $T^*_{\tilde{x}} U$. Furthermore, any local minimum of the map $x \in U \longmapsto u(x) + c(x, \tilde{y})$ is a global minimum.*

As shown in [29, Corollary 2.11], the strengthening **(B3)** of hypothesis **(A3w)** improves the conclusion of Loeper's maximum principle. This improvement asserts that the altitude $f(t, x)$ at each point of the evolving landscape then accelerates as a function of $t \in [0,1]$:

Theorem 3.5 (**Time-convex DASM**). *Assume* **(B0)**–**(B3)** *and fix $x, \tilde{x} \in \overline{U}$. If $t \in [0,1] \longmapsto -D_x c(\tilde{x}, y(t))$ is a line segment then $t \in [0,1] \longmapsto f(t) := -c(x, y(t)) + c(\tilde{x}, y(t))$ is convex.*

Remark 3.6. Since all assumptions **(B0)**–**(B3)** and **(A3w)** on the cost are symmetric in x and y, all the results above still hold when exchanging x with y.

4 Cost-exponential coordinates, null Lagrangians, and affine renormalization

In this section, we set up the notation for the rest of the paper. Recall that $c \in C^4(\overline{U} \times \overline{V})$ is a non-negatively cross-curved cost function satisfying **(B1)**–**(B3)** on a pair of bounded domains U and V which are strongly c-convex with respect to each other **(B2u)**.

Fix $\lambda, \Lambda > 0$ and an open domain $U^\lambda \subset U$, and let u be a c-convex solution of the c-Monge-Ampère equation

$$\begin{cases} \lambda \mathscr{L}^n \leq |\partial^c u| \leq \frac{1}{\lambda}\mathscr{L}^n & \text{in } U^\lambda \subset U, \\ |\partial^c u| \leq \Lambda \mathscr{L}^n & \text{in } \overline{U}. \end{cases} \tag{4.1}$$

We sometimes abbreviate (4.1) by writing $|\partial^c u| \in [\lambda, 1/\lambda]$. In the following sections, we will prove interior differentiability of u on U^λ, that is $u \in C^1(U^\lambda)$; see Theorem 9.2.

Throughout D_y will denote the derivative with respect to the variable y, and iterated subscripts as in D^2_{xy} denote iterated derivatives. We also use

$$\beta_c^\pm = \beta_c^\pm(U \times V) := \|(D^2_{xy}c)^{\pm 1}\|_{L^\infty(U\times V)} \tag{4.2}$$

$$\gamma_c^\pm = \gamma_c^\pm(U \times V) := \|\det(D^2_{xy}c)^{\pm 1}\|_{L^\infty(U\times V)} \tag{4.3}$$

to denote the bi-Lipschitz constants $\beta_c^\pm$ of the coordinate changes (4.4) and the Jacobian bounds $\gamma_c^\pm$ for the same transformation. Notice $\gamma_c^+\gamma_c^- \geq 1$ for any cost satisfying **(B1)**, and equality holds whenever the cost function $c(x,y)$ is quadratic. So the parameter $\gamma_c^+\gamma_c^-$ crudely quantifies the departure from the quadratic case. The inequality $\beta_c^+\beta_c^- \geq 1$ is much more rigid, equality implying $D^2_{xy}c(x,y)$ is the identity matrix, and not merely constant.

4.1 Choosing coordinates which convexify c-convex functions

In the current subsection, we introduce an important transformation (mixing dependent and independent variables) for the cost $c(x,y)$ and potential $u(x)$, which plays a crucial role in the subsequent analysis. This change of variables and its most relevant properties are encapsulated in the following definition and theorem. *In the sequel, whenever we use the expression $\tilde{c}(q,\cdot)$ or $\tilde{u}(q)$ we refer to the modified cost function and convex potential defined here, unless otherwise stated.* Since properties **(B0)**–**(B3)**, **(A3w)** and **(B2u)** were shown to be tensorial in nature (i.e. coordinate independent) in [28] [34], the modified cost $\tilde{c}$ inherits these properties from the original cost c with one exception: (4.5) defines a C^3 diffeomorphism $q \in \overline{U}_{\tilde{y}} \longmapsto x(q) \in \overline{U}$, so the cost $\tilde{c} \in C^3(\overline{U}_{\tilde{y}} \times \overline{V})$ may not be C^4 smooth. However, its definition reveals that we may still differentiate $\tilde{c}$ four times as long as no more than three of the four derivatives fall on the variable q, and it leads to the same geometrical structure (pseudo-Riemannian curvatures, including (2.1)) as the original cost c since the metric tensor and symplectic form defined in [28] involve only mixed derivatives $D^2_{qy}\tilde{c}$, and therefore remain C^2 functions of the coordinates $(q,y) \in \overline{U}_{\tilde{y}} \times \overline{V}$.

Definition 4.1 (Cost-exponential coordinates and apparent properties)**.** *Given $c \in C^4\big(\overline{U} \times \overline{V}\big)$ strongly twisted* **(B0)**–**(B1)***, we refer to the coordinates $(q,p) \in \overline{U}_{\tilde{y}} \times \overline{V}_{\tilde{x}}$ defined by*

$$q = q(x) = -D_y c(x, \tilde{y}), \qquad p = p(y) = -D_x c(\tilde{x}, y), \tag{4.4}$$

as the cost exponential coordinates *from $\tilde{y} \in \overline{V}$ and $\tilde{x} \in \overline{U}$ respectively. We denote the inverse diffeomorphisms by $x : \overline{U}_{\tilde{y}} \subset T^*_{\tilde{y}}V \longmapsto \overline{U}$ and $y : \overline{V}_{\tilde{x}} \subset T^*_{\tilde{x}}U \longmapsto \overline{V}$; they satisfy*

$$q = -D_y c(x(q), \tilde{y}), \qquad p = -D_x c(\tilde{x}, y(p)). \tag{4.5}$$

The cost $\tilde{c}(q,y) = c(x(q),y) - c(x(q),\tilde{y})$ *is called the* modified cost at $\tilde{y}$. *A subset of* $\overline{U}$ *or function thereon is said to* appear from $\tilde{y}$ *to have property A, if it has property A when expressed in the coordinates* $q \in \overline{U}_{\tilde{y}}$.

Remark 4.2. Identifying the cotangent vector $0 \oplus q$ with the tangent vector $Q \oplus 0$ to $U \times V$ using the pseudo-metric of Kim and McCann [28] shows $x(q)$ to be the projection to U of the pseudo-Riemannian exponential map $\exp_{(\tilde{x},\tilde{y})} Q \oplus 0$; similarly $y(p)$ is the projection to V of $\exp_{(\tilde{x},\tilde{y})} 0 \oplus P$. Also, $x(q) =: c^*\text{-}\exp_{\tilde{y}} q$ and $y(p) =: c\text{-}\exp_{\tilde{x}} p$ in the notation of Loeper [34].

Our first contribution is the following theorem. For a non-negatively cross-curved cost **(B3)**, it shows that any $\tilde{c}$-convex potential appears convex from $\tilde{y} \in \overline{V}$. Even if the cost function is weakly regular **(A3w)**, the level sets of the $\tilde{c}$-convex potential appear convex from $\tilde{y}$, as was discovered independently from us by Liu [32], and exploited by Liu with Trudinger and Wang [33]. Note that although the difference between the cost $c(x,y)$ and the modified cost $\tilde{c}(q,y)$ depends on $\tilde{y}$, they differ by a null Lagrangian $c(x,\tilde{y})$ which — being independent of $y \in V$ — does not affect the question of which maps G attain the infimum (1.1). Having a function with convex level sets is a useful starting point, since it enables us to apply Caffarelli's affine renormalization of convex sets approach and a full range of techniques from Gutierrez [23] to address the regularity of c-convex potentials.

Theorem 4.3 (Modified c-convex functions appear convex). *Let* $c \in C^4\big(\overline{U} \times \overline{V}\big)$ *satisfying* **(B0)**–**(B2)** *be weakly regular* **(A3w)**. *If* $u = u^{c^*c}$ *is c-convex on* $\overline{U}$, *then* $\tilde{u}(q) = u(x(q)) + c(x(q),\tilde{y})$ *has convex level sets, as a function of the cost exponential coordinates* $q \in \overline{U}_{\tilde{y}}$ *from* $\tilde{y} \in \overline{V}$. *If, in addition, c is non-negatively cross-curved* **(B3)** *then* $\tilde{u}$ *is convex on* $\overline{U}_{\tilde{y}}$. *In either case* $\tilde{u}$ *is minimized at* q_0 *if* $\tilde{y} \in \partial^c u(x(q_0))$. *Furthermore,* $\tilde{u}$ *is* $\tilde{c}$*-convex with respect to the modified cost* $\tilde{c}(q,y) := c(x(q),y) - c(x(q),\tilde{y})$ *on* $\overline{U}_{\tilde{y}} \times \overline{V}$, *and* $\partial^{\tilde{c}}\tilde{u}(q) = \partial^c u(x(q))$ *for all* $q \in \overline{U}_{\tilde{y}}$.

Proof. The final sentences of the theorem are elementary: c-convexity $u = u^{c^*c}$ asserts

$$u(x) = \sup_{y \in \overline{V}} -c(x,y) - u^{c^*}(y) \quad \text{and} \quad u^{c^*}(y) = \sup_{q \in \overline{U}_{\tilde{y}}} -c(x(q),y) - u(x(q)) = \tilde{u}^{\tilde{c}^*}(y)$$

from (3.2), hence

$$\begin{aligned} \tilde{u}(q) &= \sup_{y \in \overline{V}} -c(x(q),y) + c(x(q),\tilde{y}) - u^{c^*}(y) \\ &= \sup_{y \in \overline{V}} -\tilde{c}(q,y) - \tilde{u}^{\tilde{c}^*}(y), \end{aligned}$$

and $\partial^{\tilde{c}}\tilde{u}(q) = \partial^c u(x(q))$ since all three suprema above are attained at the same $y \in \overline{V}$. Taking $y = \tilde{y}$ reduces the inequality $\tilde{u}(q) + \tilde{u}^{\tilde{c}^*}(y) + \tilde{c}(q,y) \geq 0$ to $\tilde{u}(q) \geq -\tilde{u}^{\tilde{c}^*}(\tilde{y})$, with equality precisely if $\tilde{y} \in \partial^{\tilde{c}}\tilde{u}(q)$. It remains to address the convexity claims.

Since the supremum $\tilde{u}(q)$ of a family of convex functions is again convex, it suffices to establish the convexity of $q \in \overline{U}_{\tilde{y}} \longmapsto -\tilde{c}(q,y)$ for each $y \in \overline{V}$ under hypothesis **(B3)**. For a similar reason, it suffices to establish the level-set convexity of the same family of functions under hypothesis **(A3w)**.

First assume **(A3w)**. Since

$$D_y\tilde{c}(q,\tilde{y}) = D_yc(x(q),\tilde{y}) := -q \tag{4.6}$$

we see that $\tilde{c}$-segments in $\overline{U}_{\tilde{y}}$ with respect to $\tilde{y}$ coincide with ordinary line segments. Let $q(s) = (1-s)q_0+sq_1$ be any line segment in the convex set $\overline{U}_{\tilde{y}}$. Define $f(s,y) := -\tilde{c}(q(s),y) = -c(x(q(s)),y) + c(x(q(s)),\tilde{y})$. Loeper's maximum principal (Theorem 3.3 above, see also Remark 3.6) asserts $f(s,y) \le \max\{f(0,y), f(1,y)\}$, which implies convexity of each set $\{q \in \overline{U}_{\tilde{y}} \mid -\tilde{c}(q,y) \le const\}$. Under hypothesis **(B3)**, Theorem 3.5 goes on to assert convexity of $s \in [0,1] \longmapsto f(s,y)$ as desired. □

The effect of this change of gauge on Jacobian inequalities is summarized in a corollary:

Corollary 4.4 (Transformed $\tilde{c}$-Monge-Ampère inequalities). *Using the hypotheses and notation of Theorem 4.3, if $|\partial^c u| \in [\lambda,\Lambda] \subset [0,\infty]$ on $U' \subset \overline{U}$, then $|\partial^{\tilde{c}}\tilde{u}| \in [\lambda/\gamma_c^+, \Lambda\gamma_c^-]$ on $U'_{\tilde{y}} = -D_yc(U',\tilde{y})$, where $\gamma_c^\pm = \gamma_c^\pm(U'\times V)$ and $\beta_c^\pm = \beta_c^\pm(U'\times V)$ are defined in* (4.2)–(4.3). *Furthermore, $\gamma_{\tilde{c}}^\pm := \gamma_{\tilde{c}}^\pm(U'_{\tilde{y}}\times V) \le \gamma_c^+\gamma_c^-$ and $\beta_{\tilde{c}}^\pm := \beta_{\tilde{c}}^\pm(U'_{\tilde{y}}\times V) \le \beta_c^+\beta_c^-$.*

Proof. From the Jacobian bounds $|\det D_xq(x)| \in [1/\gamma_c^-, \gamma_c^+]$ on U', we find $\mathscr{L}^n(X)/\gamma_c^- \le \mathscr{L}^n(q(X)) \le \gamma_c^+\mathscr{L}^n(X)$ for each $X \subset U'$. On the other hand, Theorem 4.3 asserts $\partial^{\tilde{c}}\tilde{u}(q(X)) = \partial^c u(X)$, so the claim $|\partial^{\tilde{c}}\tilde{u}| \in [\lambda/\gamma_c^+, \Lambda\gamma_c^-]$ follows from the hypothesis $|\partial^c u| \in [\lambda,\Lambda]$, by definition (3.6) and the fact that $q : \overline{U} \longrightarrow \overline{U}_{\tilde{y}}$ from (4.4) is a diffeomorphism; see **(B1)**. The bounds $\gamma_{\tilde{c}}^\pm \le \gamma_c^+\gamma_c^-$ and $\beta_{\tilde{c}}^\pm \le \beta_c^+\beta_c^-$ follow from $D^2_{qy}\tilde{c}(q,y) = D^2_{xy}c(x(q),y)D_qx(q)$ and $D_qx(q) = -D^2_{xy}c(x(q),\tilde{y})^{-1}$. □

4.2 Affine renormalization

The renormalization of a function $\tilde{u}$ by an affine transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ will be useful in Section 7 to prove our Alexandrov type estimates. Let us therefore record the following observations. Define

$$\tilde{u}^*(q) = |\det L|^{-2/n}\tilde{u}(Lq). \tag{4.7}$$

Here $\det L$ denotes the Jacobian determinant of L, i.e. the determinant of the linear part of L.

Lemma 4.5 (Affine invariance of $\tilde{c}$-Monge-Ampère measure). *Assuming* **(B0)**–**(B1)***, given a $\tilde{c}$-convex function $\tilde{u} : U_{\tilde{y}} \longmapsto \mathbf{R}$ and affine bijection $L : \mathbf{R}^n \longmapsto \mathbf{R}^n$, define the renormalized potential $\tilde{u}^*$ by* (4.7) *and renormalized cost*

$$\tilde{c}_*(q,y) = |\det L|^{-2/n}\tilde{c}(Lq, L^*y) \tag{4.8}$$

using the adjoint L^* to the linear part of L. Then, for all Borel $Z \subset \overline{U}_{\tilde{y}}$,

$$|\partial\tilde{u}^*|(L^{-1}Z) = |\det L|^{-1}|\partial\tilde{u}|(Z), \tag{4.9}$$

$$|\partial^{\tilde{c}_*}\tilde{u}^*|(L^{-1}Z) = |\det L|^{-1}|\partial^{\tilde{c}}\tilde{u}|(Z). \tag{4.10}$$

Proof. From (3.4) we see $\bar p \in \partial\tilde u(\bar q)$ if and only if $|\det L|^{-2/n}L^*\bar p \in \partial\tilde u^*(L^{-1}\bar q)$, thus (4.9) follows from $\partial\tilde u^*(L^{-1}Z) = |\det L|^{-2/n}L^*\big(\partial\tilde u(Z)\big)$. Similarly, since (3.2) yields $(\tilde u^*)^{\tilde c^*_*}(y) = |\det L|^{-2/n}\tilde u^{\tilde c^*}(L^*y)$, we see $\bar y \in \partial^{\tilde c}\tilde u(\bar q)$ is equivalent to $|\det L|^{-2/n}L^*\bar y \in \partial^{\tilde c_*}\tilde u^*(L^{-1}\bar q)$ from (3.3) (and Theorem 4.3), whence $\partial^{\tilde c_*}\tilde u^*(L^{-1}Z) = |\det L|^{-2/n}L^*\big(\partial^{\tilde c}\tilde u^*(Z)\big)$ to establish (4.10). □

As a corollary to this lemma, we recover the affine invariance not only of the Monge-Ampère equation satisfied by $\tilde u(q)$ — but also of the $\tilde c$-Monge-Ampère equation it satisfies — under coordinate changes on V (which induce linear transformations L on $T^*_{\tilde y}V$ and L^* on $T_{\tilde y}V$): for $q \in U_{\tilde y}$,

$$\frac{d|\partial\tilde u^*|}{d\mathscr{L}^n}(L^{-1}q) = \frac{d|\partial\tilde u|}{d\mathscr{L}^n}(q) \quad \text{and} \quad \frac{d|\partial^{\tilde c_*}\tilde u^*|}{d\mathscr{L}^n}(L^{-1}q) = \frac{d|\partial^{\tilde c}\tilde u|}{d\mathscr{L}^n}(q).$$

5 Strongly c-convex interiors and boundaries not mixed by $\partial^c u$

The subsequent sections of this paper are largely devoted to ruling out exposed points in $U_{\tilde y}$ of sets on which ordinary convexity of the $\tilde c$-convex potential from Theorem 4.3 fails to be strict. This current section rules out exposed points on the boundary of $U_{\tilde y}$. We do this by proving an important topological property of the (multi-valued) mapping $\partial^c u \subset \overline{U} \times \overline{V}$. Namely, we show that the subdifferential $\partial^c u$ maps interior points of $\operatorname{spt}|\partial^c u| \subset \overline{U}$ only to interior points of V, under hypothesis (4.1), and conversely that $\partial^c u$ maps boundary points of U only to boundary points of V. This theorem may be of independent interest, and was required by Figalli and Loeper to conclude their continuity result concerning maps of the plane which optimize **(A3w)** costs [17].

This section does not use the cross-curvature condition **(B3)** (nor **A3w**) on the cost function $c \in C^4(\overline{U} \times \overline{V})$, but relies crucially on the *strong* c-convexity **(B2u)** of its domains U and V (but importantly, not on $\operatorname{spt}|\partial^c u|$). No analog for Theorem 5.1 was needed by Caffarelli to establish $C^{1,\alpha}$ regularity of convex potentials $u(x)$ whose gradients optimize the classical cost $c(x,y) = -\langle x, y\rangle$ [8], since in that case he was able to take advantage of the fact that the cost function is smooth on the whole of $\mathbf{R}^n$ to chase potentially singular behaviour to infinity. (One general approach to showing regularity of solutions for degenerate elliptic partial differential equations is to exploit the threshold-hyperbolic nature of the solution to try to follow either its singularities or its degeneracies to the boundary, where they can hopefully be shown to be in contradiction with boundary conditions; the *degenerate* nature of the ellipticity precludes the possibility of *purely local* regularizing effects.)

Theorem 5.1 (Strongly c-convex interiors and boundaries not mixed by $\partial^c u$)**.** *Let c satisfy* **(B0)**–**(B1)** *and $u = u^{c^*c}$ be a c-convex function (which implies $\partial^c u(\overline{U}) = \overline{V}$), and $\lambda > 0$.*

(a) If $|\partial^c u| \geq \lambda$ on $X \subset \overline{U}$ and V is strongly c^*-convex with respect to X, then interior points of X cannot be mapped by $\partial^c u$ to boundary points of V: i.e. $(X \times \partial V) \cap \partial^c u \subset (\partial X \times \partial V)$.

(b) If $|\partial^c u| \le \Lambda$ on $\overline{U}$, and U is strongly c-convex with respect to V, then boundary points of U cannot be mapped by $\partial^c u$ into interior points of V: i.e. $\partial U \times V$ is disjoint from $\partial^c u$.

Proof. Note that when X is open the conclusion of (a) implies $\partial^c u$ is disjoint from $X \times \partial V$. We therefore remark that it suffices to prove (a), since (b) follows from (a) exchanging the role x and y and observing that $|\partial^c u| \le \Lambda$ implies $|\partial^{c^*} u^{c^*}| \ge 1/\Lambda$ as in Lemma 3.1(d).

Let us prove (a). Fix any point $\tilde{x}$ in the interior of X, and $\tilde{y} \in \partial^c u(\tilde{x})$. Assume by contradiction that $\tilde{y} \in \partial V$. At $(\tilde{x}, \tilde{y})$ we use **(B0)–(B1)** to define cost-exponential coordinates $(p, q) \longmapsto (x(q), y(p))$ by

$$\begin{array}{rcl} p = & -D_x c(\tilde{x}, y(p)) + D_x c(\tilde{x}, \tilde{y}) & \in T^*_{\tilde{x}}(U) \\ q = & D^2_{xy} c(\tilde{x}, \tilde{y})^{-1}(D_y c(x(q), \tilde{y}) - D_y c(\tilde{x}, \tilde{y})) & \in T_{\tilde{x}}(U) \end{array}$$

and define a modified cost and potential by subtracting null Lagrangian terms:

$$\begin{array}{rcl} \tilde{c}(q, p) & := & c(x(q), y(p)) - c(x(p), \tilde{y}) - c(\tilde{x}, y(p)) \\ \tilde{u}(q) & := & u(x(q)) + c(x(q), \tilde{y}). \end{array}$$

Similarly to Corollary 4.4, $|\partial^{\tilde{c}} \tilde{u}| \ge \tilde{\lambda} := \lambda/(\gamma_c^+ \gamma_c^-)$, where $\gamma_c^\pm$ denote the Jacobian bounds (4.3) for the coordinate change. Note $(\tilde{x}, \tilde{y}) = (x(\mathbf{0}), y(\mathbf{0}))$ corresponds to $(p, q) = (\mathbf{0}, \mathbf{0})$. Since c-segments with respect to $\tilde{y}$ correspond to line segments in $U_{\tilde{y}} := -D_y c(U, \tilde{y})$ we see $D_p \tilde{c}(q, \mathbf{0})$ depends linearly on q, whence $D^3_{qqp} \tilde{c}(q, \mathbf{0}) = 0$; similarly c^*-segments with respect to $\tilde{x}$ become line segments in the p variables, $D_q \tilde{c}(\mathbf{0}, p)$ depends linearly on p, $D^3_{ppq} c(\mathbf{0}, p) = 0$, and the extra factor $D^2_{xy} c(\tilde{x}, \tilde{y})^{-1}$ in our definition of $x(q)$ makes $-D^2_{pq} \tilde{c}(\mathbf{0}, \mathbf{0})$ the identity matrix (whence $q = -D_p \tilde{c}(\mathbf{0}, q)$ and $p = -D_q \tilde{c}(p, \mathbf{0})$ for all q in $U_{\tilde{y}} = x^{-1}(U)$ and p in $V_{\tilde{x}} := y^{-1}(V)$). Although the change of variables $(q, p) \longmapsto (x(p), y(q))$ is only a C^3 diffeomorphism, we can still take four derivatives of the modified cost provided at least one of the four derivatives is with respect to q and another is with respect to p. We denote $X_{\tilde{y}} := x^{-1}(X)$ and choose orthogonal coordinates on U which make $-\hat{e}_n$ the outer unit normal to $V_{\tilde{x}} \subset T^*_{\tilde{x}} U$ at $\tilde{p} = \mathbf{0}$. Note that $V_{\tilde{x}}$ is strongly convex by hypothesis (a).

In these variables, consider a small cone of height ε and angle θ around the $-\hat{e}_n$ axis:

$$E_{\theta,\varepsilon} := \left\{ q \in \mathbf{R}^n \mid \left| -\hat{e}_n - \frac{q}{|q|} \right| \le \theta, |q| \le \varepsilon \right\}$$

Observe that, if θ, ε are small enough, then $E_{\theta,\varepsilon} \subset X_{\tilde{y}}$, and its measure is of order $\varepsilon^n \theta^{n-1}$. Consider now a slight enlargement

$$E'_{\theta, C_0\varepsilon} := \left\{ p = (P, p_n) \in \mathbf{R}^n \mid p_n \le \theta |p| + C_0 \varepsilon |p|^2 \right\},$$

of the polar dual cone, where ε will be chosen sufficiently small depending on the large parameter C_0 forced on us later.

The strong convexity ensures $V_{\tilde{x}}$ is contained in a ball $B_R(R\hat{e}_n)$ of some radius $R > 1$ contained in the half-space $p_n \ge 0$ with boundary sphere passing through the origin. As long

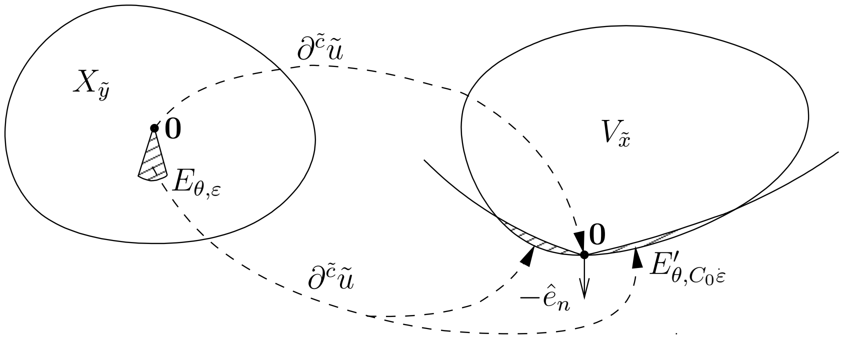

Figure 1: If $\partial^{\tilde c}\tilde u$ sends an interior point onto a boundary point, then by $\tilde c$-monotonicity of $\partial^{\tilde c}\tilde u$ the small cone $E_{\theta,\varepsilon}$ has to be sent onto $E'_{\theta,C_0\varepsilon}\cap V_{\tilde x}$. Since for $\varepsilon>0$ small but fixed $\mathscr{L}^n(E_{\theta,\varepsilon})\sim\theta^{n-1}$, while $\mathscr{L}^n(E'_{\theta,C_0\varepsilon}\cap V_{\tilde x})\lesssim\theta^{n+1}$ (by the uniform convexity of $\tilde V_{\tilde x}$), we get a contradiction as $\theta\to 0$.

as $C_0\varepsilon<(6R)^{-1}$ we claim $E'_{\theta,C_0\varepsilon}$ intersects this ball — a fortiori $V_{\tilde x}$ — in a set whose volume tends to zero like θ^{n+1} as $\theta\to 0$. Indeed, from the inequality

$$p_n\le\theta\sqrt{|P|^2+p_n^2}+\frac{1}{6}|P|+\frac{1}{3}p_n$$

satisfied by any $(P,p_n)\in E'_{\theta,C_0\varepsilon}\cap B_R(R\hat e_n)$ we deduce $p_n^2\le|P|^2(1+9\theta^2)/(2-9\theta^2)$, i.e. $p_n<|P|$ if θ is small enough. Combined with the further inequalities

$$\frac{|P|^2}{2R}\le p_n\le\theta\sqrt{|P|^2+p_n^2}+C_0\varepsilon|P|^2+C_0\varepsilon p_n^2$$

(the first inequality follows by the strong convexity of $V_{\tilde x}$), this yields $|P|\le 6\theta\sqrt{2}$ and $p_n\le O(\theta^2)$ as $\theta\to 0$. Thus $\mathscr{L}^n(E'_{\theta,C_\varepsilon}\cap V_{\tilde x})\le C\theta^{n+1}$ for a dimension dependent constant C, provided $C_0\varepsilon<(6R)^{-1}$.

The contradiction now will come from the fact that, thanks to the $\tilde c$-cyclical monotonicity of $\partial^{\tilde c}\tilde u$, if we first choose C_0 big and then we take ε sufficiently small, the image of all $q\in E_{\theta,\varepsilon}$ by $\partial^{\tilde c}\tilde u$ has to be contained in $E'_{\theta,C_0\varepsilon}$ for θ small enough. Since $\partial^{\tilde c}\tilde u\Big(\overline{X_{\tilde y}}\Big)\subset\overline{V_{\tilde x}}$ this will imply

$$\varepsilon^n\theta^{n-1}\sim\tilde\lambda\mathscr{L}^n(E_{\theta,\varepsilon})\le|\partial^{\tilde c}\tilde u|(E_{\theta,\varepsilon})\le\mathscr{L}^n(V_{\tilde x}\cap E'_{\theta,C_0\varepsilon})\le C\theta^{n+1},$$

which gives a contradiction as $\theta\to 0$, for $\varepsilon>0$ small but fixed.

Thus all we need to prove is that, if C_0 is big enough, then $\partial^{\tilde c}\tilde u(E_{\theta,\varepsilon})\subset E'_{\theta,C_0\varepsilon}$ for any ε sufficiently small. Let $q\in E_{\theta,\varepsilon}$ and $p\in\partial^{\tilde c}\tilde u(q)$. Combining

$$\int_0^1 ds\int_0^1 dt\,D^2_{qp}\tilde c(sq,tp)[q,p]=\tilde c(q,p)+\tilde c(\mathbf{0},\mathbf{0})-\tilde c(q,\mathbf{0})-c(\mathbf{0},p)\le 0$$

(where the last inequality is a consequence of $\tilde{c}$-monotonicity of $\partial^{\tilde{c}}\tilde{u}$; see for instance [46, Definitions 5.1 and 5.7]) with

$$\begin{aligned}
D^2_{qp}\tilde{c}(sq,tp) =& D^2_{qp}\tilde{c}(\mathbf{0},tp) + \int_0^s ds' D^3_{qqp}\tilde{c}(s'q,tp)[q] \\
=& D^2_{qp}\tilde{c}(\mathbf{0},\mathbf{0}) + \int_0^t dt' D^3_{qpp}\tilde{c}(\mathbf{0},t'p)[p] \\
&+ \int_0^s ds' D^3_{qqp}\tilde{c}(s'q,\mathbf{0})[q] + \int_0^s ds' \int_0^t dt' D^4_{qqpp}\tilde{c}(s'q,t'p)[q,p]
\end{aligned}$$

yields

$$\begin{aligned}
-\langle q,p\rangle &\leq -\int_0^1 ds \int_0^1 dt \int_0^s ds' \int_0^t dt' D^4_{qqpp}\tilde{c}(s'q,t'p)[q,q,p,p] \\
&\leq C_0|q|^2|p|^2
\end{aligned}$$

since $D^3_{qpp}\tilde{c}(\mathbf{0},t'p)$ and $D^3_{qqp}\tilde{c}(s'q,\mathbf{0})$ vanish in our chosen coordinates and $-D^2_{pq}\tilde{c}(\mathbf{0},\mathbf{0})$ is the identity matrix. Due to the tensorial nature of the cross-curvature (2.1), C_0 depends on $\|c\|_{C^4(U\times V)}$ and the bi-Lipschitz constants $\beta^{\pm}_c$ from (4.2).

From the above inequality and the definition of $E_{\theta,\varepsilon}$ we deduce

$$p_n = \langle p, \hat{e}_n + \frac{q}{|q|}\rangle - \langle p, \frac{q}{|q|}\rangle \leq \theta|p| + C_0\varepsilon|p|^2$$

so $p \in E'_{\theta,C_0\varepsilon}$ as desired. □

6 The Monge-Ampère measure dominates the $\tilde{c}$-Monge-Ampère measure

In this section we shall prove that — up to constants — the ordinary Monge-Ampère measure $|\partial\tilde{u}|$ dominates the $\tilde{c}$-Monge-Ampère measure $|\partial^{\tilde{c}}\tilde{u}|$, when defined in the coordinates introduced in Theorem 4.3. Let us begin with a lemma which motivates our proposition heuristically. The conclusions of the lemma extend easily from smooth to non-smooth functions by an approximation argument combining Lemma 3.1(b)–(c) with results of Trudinger and Wang [44]. However this approach would require the domains U and V to be smooth, so in Proposition 6.2 we prefer to construct an explicit approximation which proves the statement we need, requires no additional smoothness hypotheses, and is logically independent of both Lemma 6.1 and [44].

Lemma 6.1. *Assume* **(B0)**–**(B3)** *and let* $\tilde{u} : \overline{U}_{\tilde{y}} \longmapsto \mathbf{R}$ *be a convex $\tilde{c}$-convex function as in Theorem 4.3. If* $\tilde{u} \in C^2(U'_{\tilde{y}})$ *for some open set* $U'_{\tilde{y}} \subset U_{\tilde{y}}$*, then* $|\partial^{\tilde{c}}\tilde{u}| \leq \gamma^-_{\tilde{c}}|\partial\tilde{u}|$ *on* $U'_{\tilde{y}}$*, where* $\gamma^{\pm}_{\tilde{c}} = \gamma^{\pm}_{\tilde{c}}(U'_{\tilde{y}} \times V)$ *are defined as in (4.3).*

Proof. In addition to the convexity of $\tilde{u}(q)$, for any $y \in \overline{V}$ Theorem 4.3 asserts the convexity of the $\tilde{c}$-convex function $q \in \overline{U}_{\tilde{y}} \longmapsto -\tilde{c}(q,y)$. Thus

$$\det(D^2_{qq}\tilde{u}(\tilde{q}) + D^2_{qq}\tilde{c}(\tilde{q},y)) \leq \det D^2_{qq}\tilde{u}(\tilde{q})$$

by the concavity of $S \longmapsto \det^{1/n}(S)$ on symmetric non-negative definite matrices. On the other hand, for $\tilde{c}$-convex $\tilde{u} \in C^2(U'_{\tilde{y}})$, the measure $|\partial^{\tilde{c}}\tilde{u}|$ is absolutely continuous, with Lebesgue density given by the left hand side of (3.7). Thus at any $\tilde{q} \in U'_{\tilde{y}}$,

$$\frac{d|\partial^{\tilde{c}}\tilde{u}|}{d\mathscr{L}^n}(\tilde{q}) = \frac{\det(D^2_{qq}\tilde{u}(\tilde{q}) + D^2_{qq}c(\tilde{q},\tilde{G}(\tilde{q})))}{|\det D^2_{qy}\tilde{c}(y,\tilde{G}(\tilde{q}))|} \leq \gamma^-_{\tilde{c}} \det D^2_{qq}\tilde{u}(\tilde{q}) \tag{6.1}$$

as desired. □

We now prove the proposition that we actually need subsequently.

Proposition 6.2 (Monge-Ampère measure dominates $\tilde{c}$-Monge-Ampère measure)**.** *Assume* **(B0)**–**(B3)***, and let* $\tilde{u} : \overline{U}_{\tilde{y}} \longmapsto \mathbf{R}$ *be a convex $\tilde{c}$-convex function from Theorem 4.3. Then* $|\partial^{\tilde{c}}\tilde{u}| \leq \gamma^-_{\tilde{c}}|\partial\tilde{u}|$ *on* $U'_{\tilde{y}} \subset U_{\tilde{y}}$*, where* $\gamma^{\pm}_{\tilde{c}} = \gamma^{\pm}_{\tilde{c}}(U'_{\tilde{y}} \times V)$ *are defined as in (4.3).*

Proof. It suffices to prove $|\partial^{\tilde{c}}\tilde{u}|\big(B_r(\bar{q})\big) \leq \gamma^-_{\tilde{c}}|\partial\tilde{u}|\big(\overline{B_r(\bar{q})}\big)$ for each ball whose closure is contained in $U'_{\tilde{y}}$. Given such a ball, let $h(q) := |q-\bar{q}|$, and let $\rho_\varepsilon(q) = \varepsilon^{-n}\rho(q/\varepsilon) \geq 0$ be a smooth mollifier vanishing outside $B_\varepsilon(\mathbf{0})$ and carrying unit mass. For $\varepsilon > 0$ sufficiently small, we can define the smooth convex function

$$\tilde{u}_{\varepsilon,\delta} = (\tilde{u} + \delta h) * \rho_\varepsilon$$

on $\overline{B_r(\bar{q})}$. Since $\tilde{u}$ and h are locally Lipschitz, letting R denote a bound for $\mathrm{Lip}(\tilde{u}) + \mathrm{Lip}(h)$ inside $B_r(\bar{q})$ yields

$$\|\tilde{u}_{\varepsilon,\delta} - (\tilde{u} + \delta h)\|_{L^\infty(B_r(\bar{q}))} \leq \varepsilon R$$

for all $\delta < 1$.

Claim: Fix $0 < t < 1$ and $0 < \delta < 1$. For $\varepsilon > 0$ sufficiently small, we claim that to each $y_0 \in \partial^{\tilde{c}}\tilde{u}(B_{tr}(\bar{q}))$ corresponds some $q_{\varepsilon,\delta} \in B_r(\bar{q})$ such that $(q_{\varepsilon,\delta}, y_0) \in \partial^{\tilde{c}}\tilde{u}_{\varepsilon,\delta}$.

Indeed, for $y_0 \in \partial^{\tilde{c}}\tilde{u}(q_0)$ with $|q_0 - \bar{q}| < tr$, observe

$$\tilde{u}(q) \geq -c(q,y_0) - \tilde{u}^{\tilde{c}^*}(y_0) \qquad \forall\, q \in \overline{U}_{\tilde{y}},$$

with equality at q_0. Moreover $h(q_0) \leq tr$. Therefore

$$\begin{aligned}\tilde{u}_{\varepsilon,\delta}(q_0) - (-c(q_0,y_0) - \tilde{u}^{\tilde{c}^*}(y_0)) &\leq \varepsilon R + \tilde{u}(q_0) + \delta h(q_0) - (-c(q_0,y_0) - \tilde{u}^{\tilde{c}^*}(y_0)) \\ &= \varepsilon R + \delta h(q_0) \leq \varepsilon R + tr\delta.\end{aligned}$$

On the other hand, if $q \in \partial B_r(\bar{q})$, then $h(q) = r$, and so

$$\begin{aligned}\tilde{u}_{\varepsilon,\delta}(q) - (-c(q,y_0) - \tilde{u}^{\tilde{c}^*}(y_0)) &\geq -\varepsilon R + \tilde{u}(q) + \delta h(q) - (-c(q,y_0) - \tilde{u}^{\tilde{c}^*}(y_0)) \\ &\geq -\varepsilon R + \delta h(q) = -\varepsilon R + r\delta.\end{aligned}$$

Thus, if for fixed δ small we choose ε small enough so that

$$r\delta - \varepsilon R > tr\delta + \varepsilon R,$$

we deduce that if we lower the graph of the function $-c(q, y_0) - \tilde{u}^{\tilde{c}^*}(y_0)$ to the lowest level at which it intersects the graph of $\tilde{u}_{\varepsilon,\delta}$, then the point of intersection must lie over $B_r(\bar{q})$. This proves the claim.

Having established the claim, let $E \subset B_r(\bar{q})$ denote the (Borel) set of all $q_{\varepsilon,\delta}$ which arise from $\partial\tilde{u}_{\varepsilon,\delta}(B_{tr}(\bar{q}))$ in this way. Since $\tilde{u}_{\varepsilon,\delta}$ is smooth, the condition $y_0 \in \partial^{\tilde{c}}\tilde{u}_{\varepsilon,\delta}(q_{\varepsilon,\delta})$ implies

$$D_q\tilde{u}_{\varepsilon,\delta}(q_{\varepsilon,\delta}) = -D_q\tilde{c}(q_{\varepsilon,\delta}, y_0), \tag{6.2}$$

as well as

$$D_{qq}\tilde{u}_{\varepsilon,\delta}(q_{\varepsilon,\delta}) \geq -D_{qq}\tilde{c}(q_{\varepsilon,\delta}, y_0). \tag{6.3}$$

By (6.2) and **(B1)** we can define a smooth map $G_{\varepsilon,\delta}(q_0)$ throughout E using the relation

$$D_q\tilde{u}_{\varepsilon,\delta}(q_0) = -D_q\tilde{c}(q_0, G_{\varepsilon,\delta}(q_0)),$$

and find that $\partial^{\tilde{c}}\tilde{u}_{\varepsilon,\delta}(q_{\varepsilon,\delta}) = \{G_{\varepsilon,\delta}(q_{\varepsilon,\delta})\}$ is a singleton. In this way we obtain

$$\begin{aligned}
|\partial^c\tilde{u}|(B_{tr}(\bar{q})) \leq |\partial^c\tilde{u}_{\varepsilon,\delta}|(E) &= \int_E |\det D_qG_{\varepsilon,\delta}|(q)\, dq \\
&= \int_E \frac{\det(D_{qq}\tilde{u}_{\varepsilon,\delta}(q) + D_{qq}\tilde{c}(q, G_{\varepsilon,\delta}(q)))}{|\det D^2_{qy}\tilde{c}(q, G_{\varepsilon,\delta}(q))|}\, dq,
\end{aligned}$$

where in the last equality we used (6.3) to deduce that $D_{qq}\tilde{u}_{\varepsilon,\delta}(q) + D_{qq}c(q, G_{\varepsilon,\delta}(q))$ is non-negative definite. Hence the inequality

$$\frac{\det(D_{qq}\tilde{u}_{\varepsilon,\delta}(q) + D_{qq}\tilde{c}(q, G_{\varepsilon,\delta}(q)))}{|\det D^2_{qy}\tilde{c}(q, G_{\varepsilon,\delta}(q))|} \leq \gamma^-_{\tilde{c}} \det D_{qq}\tilde{u}_{\varepsilon,\delta}(q)$$

holds (similarly to (6.1) above), and so

$$|\partial^c\tilde{u}|(B_{tr}(\bar{q})) \leq \gamma^-_{\tilde{c}} \int_E \det(D_{qq}\tilde{u}_{\varepsilon,\delta}(q))\, dq \leq \gamma^-_{\tilde{c}}\, |\partial\tilde{u}_{\varepsilon,\delta}|(B_r(\bar{q})).$$

Letting first $\varepsilon \to 0$ and then $\delta \to 0$, we finally deduce

$$|\partial^c\tilde{u}|(B_{tr}(\bar{q})) \leq \limsup_{\varepsilon,\delta\to 0} \gamma^-_{\tilde{c}}\, |\partial\tilde{u}_{\varepsilon,\delta}|(B_r(\bar{q})) \leq \gamma^-_{\tilde{c}}\, |\partial\tilde{u}|\big(\overline{B_r(\bar{q})}\big).$$

Here, to see the last inequality one may, for instance, use Lemma 3.1(b) with $c(x, y) = -\langle x, y\rangle$. Arbitrariness of $0 < t < 1$ yields the desired result.

$\square$

7 Alexandrov type estimates and affine renormalization

In this section we prove the key estimates for c-convex potential functions which will eventually lead to the continuity and injectivity of optimal maps. Namely, we extend Alexandrov type estimates commonly used in the analysis of Monge-Ampère equations (thus for the cost $c(x, y) = -\langle x, y\rangle$), to general non-negatively curved cost functions. This is established in Lemma 7.2 (plus Proposition 6.2) and Lemma 7.9. These estimates are used to compare the infimum of c-convex function on a section with the size of the section, which are the key ingredients in the proof of our main results; see Propositions 7.3 and 7.10. Lemma 7.9 represents the most nontrivial and technical result we obtain in this section.

We recall a basic lemma for convex sets due to Fritz John [24], which will play an essential role in the rest of the paper.

Lemma 7.1 (John's lemma)**.** *For a compact convex set $S \subset \mathbf{R}^n$, there exists an affine transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ such that $\overline{B_1} \subset L^{-1}(S) \subset \overline{B_n}$.*

We now estimate the infimum of $\tilde{u}$ in terms of the Monge-Ampère measure in a section. The following lemma is a standard fact for convex functions. With Lemma 7.1 in mind, we state it for normalized functions $\tilde{u}^*$ and sections Z^*. However, the estimate (7.1) is invariant under the affine renormalization (4.7); according to (4.9), it holds with or without stars.

Lemma 7.2 (Upper bound on Dirichlet solutions to Monge-Ampère inequalities)**.** *Let $\tilde{u}^* : \mathbf{R}^n \longmapsto \mathbf{R} \cup \{+\infty\}$ be a convex function whose section $Z^* := \{\tilde{u}^* \leq 0\}$ satisfies $B_1 \subset Z^* \subset \overline{B}_n$. Assume that $\tilde{u}^* = 0$ on ∂Z^*. Then, for all $t \in (0, 1)$,*

$$|\partial\tilde{u}^*|\,(tZ^*) \leq \frac{C(n)}{(1-t)^n}\frac{|\inf_{Z^*}\tilde{u}^*|^n}{\mathscr{L}^n(Z^*)}, \tag{7.1}$$

where tZ^* denotes the dilation of Z^* by a factor t with respect to the origin.

Although the proof of this result is classical (see for instance [23]), for sake of completeness we prefer to give all the details.

Proof. We can assume that $\tilde{u}^*|_{Z^*} \not\equiv 0$, otherwise the estimate is trivial. It is not difficult to prove that

$$|p^*| \leq \frac{|\inf_{Z^*}\tilde{u}^*|}{(1-t)} \qquad \forall\, p^* \in \partial\tilde{u}^*(tZ^*). \tag{7.2}$$

Indeed, if $q^* \in tZ^*$ and $p^* \in \partial\tilde{u}^*(q^*)$, then

$$\langle p^*, q - q^*\rangle \leq \tilde{u}(q) - \tilde{u}(q^*) = |\tilde{u}(q^*)| \qquad \forall\, q \in \partial Z^*,$$

and taking the supremum in the left hand side among all $q \in \partial Z^*$ (7.2) follows. Thus, since $\mathscr{L}^n(Z^*) \leq C(n)$, we conclude

$$|\partial\tilde{u}^*|(tZ^*) \leq \frac{C(n)}{(1-t)^n}\frac{|\inf_{Z^*}\tilde{u}^*|^n}{\mathscr{L}^n(Z^*)}.$$

□

Combining the above lemmas, we obtain:

Proposition 7.3. *Assume* **(B0)**–**(B3)** *and define* $\gamma^{\pm}_{\tilde{c}} = \gamma^{\pm}_{\tilde{c}}(Z \times V)$ *as in (4.3). Any convex* $\tilde{c}$*-convex function* $\tilde{u} : \overline{U}_{\tilde{y}} \longmapsto \mathbf{R}$ *from Theorem 4.3 which satisfies* $|\partial^c \tilde{u}| \in [\lambda, 1/\lambda]$ *in a section of the form* $Z := \{q \in \overline{U}_{\tilde{y}} |\ \tilde{u}(q) \leq 0\}$, *and* $\tilde{u} = 0$ *on* ∂Z, *also satisfies*

$$\mathscr{L}^n(Z)^2 \leq C(n) \frac{\gamma^-_{\tilde{c}}}{\lambda} |\inf_Z \tilde{u}|^n. \tag{7.3}$$

Proof. First use the affine map L as given in Lemma 7.1 to renormalize $\tilde{u}$ into $\tilde{u}^*$ using (4.7). This does not change the bound $|\partial \tilde{u}^*| \in [\lambda, 1/\lambda]$, but allows us to apply Lemma 7.2 with $t = 1/2$. Its conclusion (7.1) has been expressed in a form which holds with or without the stars, in view of (4.9). Proposition 6.2 now yields the desired inequality (7.3). □

7.1 $\tilde{c}$-cones over convex sets

We now progress toward the Alexandrov type estimate in Lemma 7.9. In this subsection we construct and study the $\tilde{c}$-cone associated to the section of a $\tilde{c}$-convex function. This $\tilde{c}$-cone plays an essential role in our proof of Lemma 7.9.

Definition 7.4 ($\tilde{c}$-cone)**.** *Assume* **(B0)**–**(B2)** *and* **(A3w)**, *and let* $\tilde{u} : \overline{U}_{\tilde{y}} \longmapsto \mathbf{R}$ *be the* $\tilde{c}$*-convex function with convex level sets from Theorem 4.3. Let* Z *denote the section* $\{\tilde{u} \leq 0\}$, *fix* $\tilde{q} \in \operatorname{int} Z$, *and assume* $\tilde{u} = 0$ *on* ∂Z. *The* $\tilde{c}$-cone $h^{\tilde{c}} : U_{\tilde{y}} \longmapsto \mathbf{R}$ generated by $\tilde{q}$ and Z with height $-\tilde{u}(\tilde{q}) > 0$ *is given by*

$$h^{\tilde{c}}(q) := \sup_{y \in \overline{V}} \{-\tilde{c}(q, y) + \tilde{c}(\tilde{q}, y) + \tilde{u}(\tilde{q}) \mid -\tilde{c}(q, y) + \tilde{c}(\tilde{q}, y) + \tilde{u}(\tilde{q}) \leq 0 \text{ on } \partial Z\}. \tag{7.4}$$

Notice the $\tilde{c}$-cone $h^{\tilde{c}}$ depends only on the convex set $Z \subset \overline{U}_{\tilde{y}}$, $\tilde{q} \in \operatorname{int} Z$, and the value $\tilde{u}(\tilde{q})$, but is otherwise independent of $\tilde{u}$. Recalling that $\tilde{c}(q, \tilde{y}) \equiv 0$ on $U_{\tilde{y}}$, we record several key properties of the $\tilde{c}$-cone:

Lemma 7.5 (Basic properties of $\tilde{c}$-cones)**.** *Adopting the notation and hypotheses of Definition 7.4, let* $h^{\tilde{c}} : \overline{U}_{\tilde{q}} \longmapsto \mathbf{R}$ *be the* $\tilde{c}$*-cone generated by* $\tilde{q}$ *and* Z *with height* $-\tilde{u}(\tilde{q})$. *Then*

(a) $h^{\tilde{c}}$ *has convex level sets; it is a convex function if* **(B3)** *holds;*

(b) $h^{\tilde{c}}(q) \geq h^{\tilde{c}}(\tilde{q}) = \tilde{u}(\tilde{q})$ *for all* $q \in Z$;

(c) $h^{\tilde{c}} = 0$ *on* ∂Z;

(d) $\partial^{\tilde{c}} h^{\tilde{c}}(\tilde{q}) \subset \partial^{\tilde{c}} \tilde{u}(Z)$.

Proof. Property (a) is a consequence of the level-set convexity of $q \longmapsto -\tilde{c}(q, y)$ proved in Theorem 4.3, or its convexity assuming **(B3)**. Moreover, since $-\tilde{c}(q, \tilde{y}) + \tilde{c}(\tilde{q}, \tilde{y}) + \tilde{u}(\tilde{q}) = \tilde{u}(\tilde{q})$ for all $q \in U_{\tilde{y}}$, (b) follows. For each pair $z \in \partial Z$ and $y_z \in \partial^{\tilde{c}} \tilde{u}(z)$, consider the supporting mountain $m_z(q) = -\tilde{c}(q, y_z) + \tilde{c}(z, y_z)$, i.e. $m_z(z) = 0 = \tilde{u}(z)$ and $m_z \leq \tilde{u}$. Consider the

$\tilde{c}$-segment $\sigma(t)$ connecting $\sigma(0)=\tilde{y}$ and $\sigma(1)=y_z$ in V with respect to z. Since $-\tilde{c}(q,\tilde{y})\equiv 0$, by continuity there exists some $t\in]0,1]$ for which $\bar{m}_z(q):=-\tilde{c}(q,\sigma(t))+\tilde{c}(z,\sigma(t))$ satisfies $\bar{m}_z(\tilde{q})=\tilde{u}(\tilde{q})$. From Loeper's maximum principle (Theorem 3.3 above), we have

$$\bar{m}_z \leq \max[m_z, -\tilde{c}(\cdot,\tilde{y})] = \max[m_z, 0],$$

and therefore, from $m_z \leq \tilde{u}$,

$$\bar{m}_z \leq 0 \text{ on } Z.$$

By the construction, $\bar{m}_z$ is of the form

$$-\tilde{c}(\cdot,y)+\tilde{c}(\tilde{q},y)+\tilde{u}(\tilde{q}),$$

and vanishes at z. This proves (c). Finally (d) follows from (c) and the fact that $h^{\tilde{c}}(\tilde{q})=\tilde{u}(\tilde{q})$. Indeed, it suffices to move down the supporting mountain of $h^{\tilde{c}}$ at $\tilde{q}$ until the last moment at which it touches the graph of $\tilde{u}$ on Z from below. The conclusion then follows from Loeper's local to global principle, Corollary 3.4 above. □

The following estimate shows that the Monge-Ampère measure, and the relative location of the vertex within the section which generates it, control the height of any well-localized $\tilde{c}$-cone. Afficionados of the Monge-Ampère theory may be less surprised by this estimate once it is recognized that the localization in coordinates ensures the cost is approximately affine, at least in one of its two variables. Still, it is vital that the approximation be controlled! Together with Lemma 7.5(d), this proposition plays a key role in the proof of our Alexandrov type estimate (Lemma 7.9).

Proposition 7.6 (Lower bound on the Monge-Ampère measure of a small $\tilde{c}$-cone)**.** *Assume* **(B0)**–**(B3)** *and define $\tilde{c}\in C^3\big(\overline{U}_{\tilde{y}}\times\overline{V}\big)$ as in Definition 4.1. Let $Z\subset\overline{U}_{\tilde{y}}$ be a closed convex set and $h^{\tilde{c}}$ the $\tilde{c}$-cone generated by $\tilde{q}\in\operatorname{int}Z$ of height $-h^{\tilde{c}}(\tilde{q})>0$ over Z. Let Π^+,Π^- be two parallel hyperplanes contained in $T^*_{\tilde{y}}V\setminus Z$ and touching ∂Z from two opposite sides. Then there exists $\varepsilon_c>0$ small, depending only on the cost (and given by Lemma 7.7), and a constant $C(n)>0$ depending only on dimension, such that if $\operatorname{diam}(Z)\leq\varepsilon_c/C(n)$ then*

$$|h^{\tilde{c}}(\tilde{q})|^n \leq C(n)\frac{\min\{\operatorname{dist}(\tilde{q},\Pi^+),\operatorname{dist}(\tilde{q},\Pi^-)\}}{\ell_{\Pi^+}}|\partial h^{\tilde{c}}|(\{\tilde{q}\})\mathscr{L}^n(Z), \tag{7.5}$$

where ℓ_{Π^+} denotes the maximal length among all the segments obtained by intersecting Z with a line orthogonal to Π^+.

To prove this, we first observe a basic estimate on the cost function c.

Lemma 7.7. *Assume* **(B0)**–**(B2)**. *For $\tilde{c}\in C^3\big(\overline{U}_{\tilde{y}}\times\overline{V}\big)$ from Definition 4.1 and each $y\in\overline{V}$ and $q,\tilde{q}\in\overline{U}_{\tilde{y}}$,*

$$|-D_q\tilde{c}(q,y)+D_q\tilde{c}(\tilde{q},y)| \leq \frac{1}{\varepsilon_c}|q-\tilde{q}|\,|D_q\tilde{c}(\tilde{q},y)| \tag{7.6}$$

where ε_c is given by $\varepsilon_c^{-1}=2(\beta_c^+)^4(\beta_c^-)^6\|D^3_{xxy}c\|_{L^\infty(U\times V)}$ in the notation (4.2).

Proof. For fixed $\tilde{q} \in \overline{U}_{\tilde{y}}$ introduce the $\tilde{c}$-exponential coordinates $p(y) = -D_q\tilde{c}(\tilde{q}, y)$. The bi-Lipschitz constants (4.2) of this coordinate change are estimated by $\beta^{\pm}_{\tilde{c}} \leq \beta^+_c\beta^-_c$ as in Corollary 4.4. Thus

$$\begin{aligned}\operatorname{dist}(y, \tilde{y}) &\leq \beta^-_{\tilde{c}}|-D_q\tilde{c}(\tilde{q}, y) + D_q\tilde{c}(\tilde{q}, \tilde{y})| \\ &= \beta^+_c\beta^-_c|D_q\tilde{c}(\tilde{q}, y)|.\end{aligned}$$

where $\tilde{c}(q, \tilde{y}) \equiv 0$ from Definition 4.1 has been used. Similarly, noting the convexity **(B2)** of $V_{\tilde{q}} := p(V)$,

$$\begin{aligned}|-D_q\tilde{c}(\tilde{q}, y) + D_q\tilde{c}(q, y)| &= |-D_q\tilde{c}(\tilde{q}, y) + D_q\tilde{c}(q, y) + D_q\tilde{c}(\tilde{q}, \tilde{y}) - D_q\tilde{c}(q, \tilde{y})| \\ &\leq \|D^2_{qq}D_p\tilde{c}\|_{L^\infty(U_{\tilde{y}}\times\tilde{V}_{\tilde{q}})}|\tilde{q} - q||p(y) - p(\tilde{y})| \\ &\leq \|D^2_{qq}D_y\tilde{c}\|_{L^\infty(U_{\tilde{y}}\times V)}(\beta^-_c\beta^+_c)^2|\tilde{q} - q| \operatorname{dist}(y, \tilde{y})\end{aligned}$$

The result follows since $|D^2_{qq}D_y\tilde{c}| \leq ((\beta^-_c)^2 + \beta^+_c(\beta^-_c)^3))|D^2_{xx}D_yc| \leq 2\beta^+_c(\beta^-_c)^3|D^2_{xx}D_yc|$. (The last inequality follows from $\beta^+_c\beta^-_c \geq 1$.) □

Proof of Proposition 7.6. We fix $\tilde{q} \in Z$. Let Π^i, $i = 1, \cdots n$, (with Π^1 equal either Π^+ or Π^-) be hyperplanes contained in $\mathbf{R}^n \setminus Z$, touching ∂Z, and such that $\{\Pi^+, \Pi^2, \ldots, \Pi^n\}$ are all mutually orthogonal (so that also $\{\Pi^-, \Pi^2, \ldots, \Pi^n\}$ are mutually orthogonal). Moreover we choose $\{\Pi^2, \ldots, \Pi^n\}$ in such a way that, if $\pi^1(Z)$ denotes the projection of Z on Π^1 and $\mathscr{H}^{n-1}(\pi^1(Z))$ denotes its $(n-1)$-dimensional Hausdorff measure, then

$$C(n)\mathscr{H}^{n-1}(\pi^1(Z)) \geq \prod_{i=2}^{n} \operatorname{dist}(\tilde{q}, \Pi^i), \tag{7.7}$$

for some universal constant $C(n)$. Indeed, as $\pi^1(Z)$ is convex, by Lemma 7.1 we can find an ellipsoid E such that $E \subset \pi^1(Z) \subset (n-1)E$, and for instance we can choose $\{\Pi^2, \ldots, \Pi^n\}$ among the hyperplanes orthogonal to the axes of the ellipsoid (for each axis we have two possible hyperplanes, and we can always choose the furthest one so that (7.7) holds).

Each hyperplane Π^i touches Z from outside, say at $q^i \in T^*_{\tilde{y}}V$. Let $p_i \in T_{\tilde{y}}V$ be the outward (from Z) unit vector at q^i orthogonal to Π^i. Then $s_ip_i \in \partial h^{\tilde{c}}(q^i)$ for some $s_i > 0$, and by Corollary 3.4 there exists $y_i \in \partial^{\tilde{c}}h^{\tilde{c}}(q^i)$ such that

$$-D_q\tilde{c}(q^i, y_i) = s_ip_i.$$

Define $y_i(t)$ as

$$-D_q\tilde{c}(q^i, y_i(t)) = t\, s_ip_i,$$

i.e. $y_i(t)$ is the $\tilde{c}$-segment from $\tilde{y}$ to y_i with respect to q^i. As in the proof of Lemma 7.5 (c), the intermediate value theorem yields $0 < t_i \leq 1$ such that

$$-\tilde{c}(\cdot, y_i(t_i)) + \tilde{c}(\tilde{q}, y_i(t_i)) + h^{\tilde{c}}(\tilde{q}) \leq 0 \text{ on } Z$$

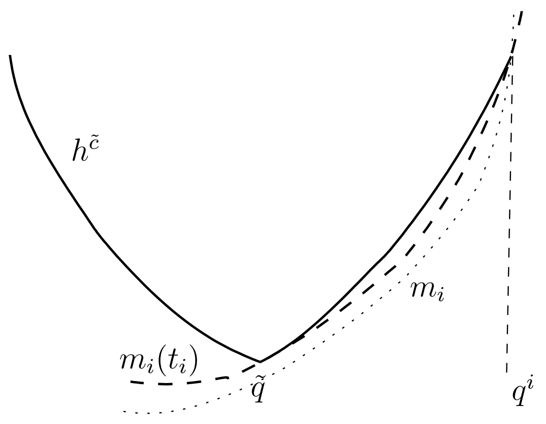

Figure 2: The dotted line represents the graph of $m_i := -\tilde{c}(\cdot, y_i) + \tilde{c}(\tilde{q}, y_i) + h^{\tilde{c}}(\tilde{q})$, while the dashed one represents the graph of $m_i(t_i) := -\tilde{c}(\cdot, y_i(t_i)) + \tilde{c}(\tilde{q}, y_i(t_i)) + h^{\tilde{c}}(\tilde{q})$. The idea is that, whenever we have m_i a supporting function for $h^{\tilde{c}}$ at a point $q^i \in \partial Z$, we can let y vary continuously along the $\tilde{c}$-segment from $\tilde{y}$ to y_i with respect to q^i, to obtain a supporting function $m_i(t_i)$ which touches $h^{\tilde{c}}$ also at $\tilde{y}$.

with equality at q^i. Thus, by the definition of $h^{\tilde{c}}$, $y_i(t_i) \in \partial^{\tilde{c}} h^{\tilde{c}}(\tilde{q}) \cap \partial^{\tilde{c}} h^{\tilde{c}}(q^i)$,

$$-D_q\tilde{c}(\tilde{q}, y_i(t_i)) \in \partial h^{\tilde{c}}(\tilde{q}) \quad \text{and} \quad t_i s_i p_i = -D_q\tilde{c}(q^i, y_i(t_i)) \in \partial h^{\tilde{c}}(q^i).$$

Therefore by the convexity of $h^{\tilde{c}}$ shown in Lemma 7.5(a), the affine function P^i with slope $-D_q\tilde{c}(q^i, y_i(t_i))$ and with $P^i(\Pi^i) \equiv 0$ satisfies $P^i(\tilde{q}) \le h^{\tilde{c}}(\tilde{q})$. This shows

$$|-D_q\tilde{c}(q^i, y_i(t_i))| \ge \frac{|h^{\tilde{c}}(\tilde{q})|}{\operatorname{dist}(\tilde{q}, \Pi^i)}. \tag{7.8}$$

Also, by (7.6)

$$\begin{aligned} |-D_q\tilde{c}(\tilde{q}, y_i(t_i)) + D_q\tilde{c}(q^i, y_i(t_i))| &\le \frac{1}{\varepsilon_c}|\tilde{q} - q^i|\,|-D_q\tilde{c}(\tilde{q}, y_i(t_i)| \\ &\le \frac{1}{\varepsilon_c}\operatorname{diam} Z\,|-D_q\tilde{c}(\tilde{q}, y_i(t_i)|. \end{aligned}$$

Therefore if $\operatorname{diam} Z \le \delta_n \varepsilon_c$ with $\delta_n > 0$ small, each vector $-D_q\tilde{c}(\tilde{q}, y_i(t_i))$ is close to $-D_q\tilde{c}(q^i, y_i(t_i))$, say

$$|-D_q\tilde{c}(\tilde{q}, y_i(t_i)) + D_q\tilde{c}(q^i, y_i(t_i))| \le \delta_n|-D_q\tilde{c}(\tilde{q}, y_i(t_i))|.$$

Since the vectors $\{-D_q\tilde{c}(q^i, y_i(t_i))\}$ are mutually orthogonal, the above estimate implies that for δ_n small enough the convex hull of $\{-D_q\tilde{c}(\tilde{q}, y_i(t_i))\}$ has measure of order $\prod_{i=1}^n |-D_q\tilde{c}(q^i, y_i(t_i))|$. Thus, by the lower bound (7.8) and the convexity of $\partial h^{\tilde{c}}(\tilde{q})$, we obtain

$$\mathscr{L}^n(\partial h^{\tilde{c}}(\tilde{q})) \ge C(n)\frac{|h^{\tilde{c}}(\tilde{q})|^n}{\prod_{i=1}^n \operatorname{dist}(\tilde{q}, \Pi^i)}.$$

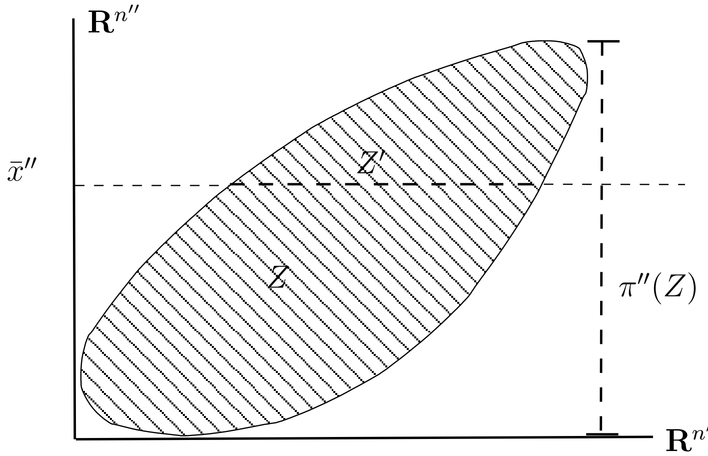

Figure 3: The volume of any convex set always controls the product (measure of one slice) · (measure of the projection orthogonal to the slice).

Since Π^1 was either Π^+ or Π^-, we have proved that

$$|h^{\tilde{c}}(\tilde{q})|^n \leq C(n) \min\{\mathrm{dist}(\tilde{q}, \Pi^+), \mathrm{dist}(\tilde{q}, \Pi^-)\} \prod_{i=2}^{n} \mathrm{dist}(\tilde{q}, \Pi^i) |\partial h^{\tilde{c}}|(\{\tilde{q}\}).$$

To conclude the proof, we apply Lemma 7.8 below with Z' given by the segment obtained intersecting Z with a line orthogonal to Π^+. Combining that lemma with (7.7), we obtain

$$C(n)|Z| \geq \ell_{\Pi^+} \prod_{i=2}^{n} \mathrm{dist}(\tilde{q}, \Pi^i),$$

and last two inequalities prove the proposition (taking $C(n) \geq 1/\delta_n$ larger if necessary). □

Lemma 7.8 (Estimating a convex volume using one slice and an orthogonal projection)**.** *Let Z be a convex set in $\mathbf{R}^n = \mathbf{R}^{n'} \times \mathbf{R}^{n''}$. Let π', π'' denote the projections to the components $\mathbf{R}^{n'}$, $\mathbf{R}^{n''}$, respectively. Let Z' be a slice orthogonal to the second component, that is*

$$Z' = (\pi'')^{-1}(\bar{x}'') \cap Z \qquad \textit{for some } \bar{x}'' \in \pi''(Z).$$

Then there exists a constant $C(n)$, depending only on $n = n' + n''$, such that

$$C(n)\mathscr{L}^n(Z) \geq \mathscr{H}^{n'}(Z')\mathscr{H}^{n''}(\pi''(Z)),$$

where $\mathscr{H}^d$ denotes the d-dimensional Hausdorff measure.

Proof. Let $L : \mathbf{R}^{n''} \to \mathbf{R}^{n''}$ be an affine map with determinant 1 given by Lemma 7.1 such that $B_r \subset L(\pi''(Z)) \subset B_{n''r}$ for some $r > 0$. Then, if we extend L to the whole $\mathbf{R}^n$ as $\tilde{L}(x', x'') = (x', Lx'')$, we have $\mathscr{L}^n(L(Z)) = \mathscr{L}^n(Z)$, $\mathscr{H}^{n'}(\tilde{L}(Z')) = \mathscr{H}^{n'}(Z')$, and

$$\mathscr{H}^{n''}(\pi''(\tilde{L}(Z))) = \mathscr{H}^{n''}(L(\pi''(Z))) = \mathscr{H}^{n''}(\pi''(Z)).$$

Hence, we can assume from the beginning that $B_r \subset \pi''(Z) \subset B_{n''r}$. Let us now consider the point $\bar{x}''$, and we fix an orthonormal basis $\{\hat{e}_1, \ldots, \hat{e}_{n''}\}$ in $\mathbf{R}^{n''}$ such that $\bar{x}'' = c\hat{e}_1$ for some $c \leq 0$. Since $\{r\hat{e}_1, \ldots, r\hat{e}_{n''}\} \subset \pi''(Z)$, there exist points $\{x_1, \ldots, x_{n''}\} \subset Z$ such that $\pi''(x_i) = r\hat{e}_i$. Let C' denote the convex hull of Z' with x_1, and let V' denote the $(n'+1)$-dimensional strip obtained taking the convex hull of $\mathbf{R}^{n'} \times \{\bar{x}''\}$ with x_1. Observe that $C' \subset V'$, and so

$$\mathscr{H}^{n'+1}(C') = \frac{1}{n'+1} \operatorname{dist}(x_1, \mathbf{R}^{n'} \times \{\bar{x}''\}) \mathscr{H}^{n'}(Z') \geq \frac{r}{n'+1} \mathscr{H}^{n'}(Z'). \tag{7.9}$$

We now remark that, since $\pi''(x_i) = r\hat{e}_i$ and $\hat{e}_i \perp V'$ for $i = 2, \ldots, n''$, we have $\operatorname{dist}(x_i, V') = r$ for all $i = 2, \ldots, n''$. Moreover, if $y_i \in V'$ denotes the closest point to x_i, then the segments joining x_i to y_i parallels $\hat{e}_i$, hence these segments are all mutually orthogonal, and they are all orthogonal to V' too. From this fact it is easy to see that, if we define the convex hull

$$C := \operatorname{co}(x_2, \ldots, x_{n''}, C'),$$

then, since $|x_i - y_i| = r$ for $i = 2, \ldots, n''$, by (7.9) and the inclusion $\pi''(Z) \subset B_{n''r} \subset \mathbf{R}^{n''}$ we get

$$\mathscr{L}^n(C) = \frac{(n'+1)!}{n!} \mathscr{H}^{n'+1}(C') r^{n''-1} \geq \frac{n'!}{n!} \mathscr{H}^{n'}(Z') r^{n''} \geq C(n) \mathscr{H}^{n'}(Z') \mathscr{H}^{n''}(\pi''(Z)).$$

This concludes the proof, as $C \subset Z$. □

7.2 An Alexandrov type estimate

The next Alexandrov type lemma holds for localized sections Z of $\tilde{c}$-convex functions.

Lemma 7.9 (Alexandrov type estimate and lower barrier)**.** *Assume* **(B0)**–**(B3)** *and let $\tilde{u} : \overline{U}_{\tilde{y}} \longmapsto \mathbf{R}$ be a convex $\tilde{c}$-convex function from Theorem 4.3. Let Z denote the section $\{\tilde{u} \leq 0\}$, assume $\tilde{u} = 0$ on ∂Z, and fix $\tilde{q} \in \operatorname{int} Z$. Let Π^+, Π^- be two parallel hyperplanes contained in $\mathbf{R}^n \setminus Z$ and touching ∂Z from two opposite sides. Then there exists $\varepsilon_c'(n) > 0$ small, depending only on dimension and the cost function (with $\varepsilon_c'(n) = \varepsilon_c / C(n)$ given by Proposition 7.6) such that if* $\operatorname{diam}(Z) \leq \varepsilon_c'(n)$ *then*

$$|\tilde{u}(\tilde{q})|^n \leq C(n) \gamma_{\tilde{c}}^{+}(Z \times V) \frac{\min\{\operatorname{dist}(\tilde{q}, \Pi^+), \operatorname{dist}(\tilde{q}, \Pi^-)\}}{\ell_{\Pi^+}} |\partial^{\tilde{c}} \tilde{u}|(Z) \mathscr{L}^n(Z),$$

where ℓ_{Π^+} denotes the maximal length among all the segments obtained by intersecting Z with a line orthogonal to Π^+, and $\gamma_{\tilde{c}}^{\pm} = \gamma_{\tilde{c}}^{+}(Z \times V)$ is defined as in (4.3).

Proof. Fix $\tilde{q} \in Z$. Observe that $\tilde{u} = 0$ on ∂Z and consider the $\tilde{c}$-cone $h^{\tilde{c}}$ generated by $\tilde{q}$ and Z of height $-h^{\tilde{c}}(\tilde{q}) = -\tilde{u}(\tilde{q})$ as in (7.4). From Lemma 7.5(d) we have

$$|\partial^{\tilde{c}} h^{\tilde{c}}|(\{\tilde{q}\}) \leq |\partial^{\tilde{c}} \tilde{u}|(Z),$$

and from Loeper's local to global principle, Corollary 3.4 above,

$$\partial h^{\tilde{c}}(\tilde{q}) = -D_q \tilde{c}(\tilde{q}, \partial^{\tilde{c}} h^{\tilde{c}}(\tilde{q})).$$

Therefore

$$|\partial h^{\tilde{c}}|(\{\tilde{q}\}) \leq \| \det D^2_{qy} \tilde{c} \|_{C^0(\{\tilde{q}\} \times V)} |\partial^c h^c|(\{\tilde{q}\}).$$

The lower bound on $|\partial h^{\tilde{c}}|(\{\tilde{q}\})$ comes from (7.5). This finishes the proof. □

7.3 Estimating solutions to the $\tilde{c}$-Monge-Ampère inequality $|\partial^{\tilde{c}} \tilde{u}| \in [\lambda, 1/\lambda]$

Combining the results of Proposition 7.3 and Lemma 7.9 yields:

Proposition 7.10 (Bounding local Dirichlet solutions to $\tilde{c}$-Monge-Ampère inequalities)**.** *Assume* **(B0)**–**(B3)** *and let $\tilde{u} : \overline{U}_{\tilde{y}} \longmapsto \mathbf{R}$ be a convex $\tilde{c}$-convex function from Theorem 4.3. There exists $\varepsilon'_c(n) > 0$ small, depending only on dimension and the cost function (and given by Lemma 7.9), and constants $C(n)$, $C_i(n) > 0$, $i = 1, 2$, depending only on dimension, such that the following holds: Letting Z denote the section $\{\tilde{u} \leq 0\}$, assume $|\partial^{\tilde{c}} \tilde{u}| \in [\lambda, 1/\lambda]$ in Z and $\tilde{u} = 0$ on ∂Z. Let $\Pi^+ \neq \Pi^-$ be parallel hyperplanes contained in $T^*_{\tilde{y}} V \setminus Z$ and supporting Z from two opposite sides. If $\mathrm{diam}(Z) \leq \varepsilon'_c(n)$ then*

$$C_1(n) \frac{\lambda}{\gamma^-_{\tilde{c}}} \leq \frac{|\inf_Z \tilde{u}|^n}{\mathscr{L}^n(Z)^2} \leq \; C_2(n) \frac{\gamma^+_{\tilde{c}}}{\lambda} \tag{7.10}$$

and

$$\frac{|\tilde{u}(q)|^n}{\mathscr{L}^n(Z)^2} \leq C(n) \frac{\gamma^+_{\tilde{c}}}{\lambda} \frac{\min\{\mathrm{dist}(q, \Pi^+), \mathrm{dist}(q, \Pi^-)\}}{\ell_{\Pi^+}} \qquad \forall\, q \in \mathrm{int}\, Z, \tag{7.11}$$

where ℓ_{Π^+} denotes the maximal length among all the segments obtained by intersecting Z with a line orthogonal to Π^+, and $\gamma^{\pm}_{\tilde{c}} = \gamma^{\pm}_{\tilde{c}}(Z \times V)$ is defined as in (4.3).

Proof. Equation (7.11) follows from Lemma 7.9 and the assumption $|\partial^{\tilde{c}} \tilde{u}| \leq 1/\lambda$. Now, by Lemma 7.1, we deduce that there exists an ellipsoid E such that $E \subset Z \subset nE$, where nE denotes the dilation of E by a factor n with respect to its barycenter $\bar{q}$. Taking Π^+ and Π^- orthogonal to one of the longest axes of E and $q = \bar{q}$ in (7.11) yields

$$|\tilde{u}(\bar{q})|^n \leq C(n) \frac{\gamma^+_{\tilde{c}}}{\lambda} \frac{n}{2} \mathscr{L}^n(Z)^2.$$

On the other hand, convexity of $\tilde{u}$ along the segment which crosses Z and passes through both $\bar{q}$ and the point $\tilde{q}$ where $\inf_Z \tilde{u}$ is attained implies

$$|\inf_Z \tilde{u}| \leq n |\tilde{u}(\bar{q})|,$$

since the barycenter of E divides the segment into a ratio at most $n : 1$. Combining these two estimates with (7.3) we obtain (7.10), to complete the proof. □

Remark 7.11 (Stability of bounds under affine renormalization)**.** Noting $\gamma^{\pm}_{\tilde{c}} \le \gamma^+_c\gamma^-_c$ from Corollary 4.4, we observe that the estimate (7.10) is stable under affine renormalization: let L be an affine transformation, and recall the renormalization

$$\tilde{u}^*(q) := |\det L|^{-2/n}\tilde{u}(Lq).$$

Then $L^{-1}(Z)$ is a section for $\tilde{u}^*$ and

$$|\inf_{L^{-1}(Z)} \tilde{u}^*|^n = |\det L|^{-2}|\inf_Z \tilde{u}|^n \sim |\det L|^{-2}\mathscr{L}^n(Z)^2 = \mathscr{L}^n(L^{-1}(Z))^2.$$

On the other hand, estimate (7.11) is not stable under affine renormalization (a line orthogonal to Π^+ is not an affinely invariant concept). For this reason, both in the proof of the c-strict convexity (Section 8) and in the proof of differentiability $u \in C^1$ (Section 9) we apply our Alexandrov estimates directly to the original sections, without renormalizing them. Using this strategy, our estimates turn out to be strong enough to adapt to our situation the strict convexity and interior continuity theory of Caffarelli [5] [8]. We perform this in the remainder of the manuscript.

8 The contact set is either a single point or crosses the domain

In this section and the final one, we complete the crucial step of proving the strict c-convexity of the c-convex potentials $u : \overline{U} \longmapsto \mathbf{R}$ arising in optimal transport, meaning $\partial^c u(x)$ should be disjoint from $\partial^c u(\tilde{x})$ whenever $x, \tilde{x} \in U^\lambda$ are distinct. This is accomplished in Theorem 9.1. In this section, we show that, if the contact set does not consist of a single point, then it extends to the boundary of U. Our method relies on the non-negative cross-curvature **(B3)** of the cost c.

From now on we adopt the following notation: $a \sim b$ means that there exist two positive constants C_1 and C_2, depending on n and $\gamma^+_c\gamma^-_c/\lambda$ only, such that $C_1 a \le b \le C_2 a$. Analogously we will say that $a \lesssim b$ (resp. $a \gtrsim b$) if there exists a positive constant C, depending on n and $\gamma^+_c\gamma^-_c/\lambda$ only, such that $a \le Cb$ (resp. $Ca \ge b$).

Recall that a point x of a convex set $S \subset \mathbf{R}^n$ is *exposed* if there is a hyperplane supporting S exclusively at x. Although the *contact set* $S := \partial^{c^*}u^{c^*}(\tilde{y})$ may not be convex, it appears convex from $\tilde{y}$ by Corollary 3.4, meaning its image $q(S) \subset U_{\tilde{y}}$ in the coordinates (4.4) is convex. The following theorem shows this convex set is either a singleton, or contains a segment which stretches across the domain. We prove it by showing the solution geometry near certain exposed points of $q(S)$ inside $U_{\tilde{y}}$ would be inconsistent with the bounds established in the previous section.

Theorem 8.1 (The contact set is either a single point or crosses the domain)**.** *Assume* **(B0)**–**(B3)** *and let u be a c-convex solution of* (4.1) *with $U^\lambda \subset U$ open. Fix $\tilde{x} \in U^\lambda$ and $\tilde{y} \in \partial^c u(\tilde{x})$, and define the contact set $S := \{x \in \overline{U} \mid u(x) = u(\tilde{x}) - c(x,\tilde{y}) + c(\tilde{x},\tilde{y})\}$. Assume that $S \neq \{\tilde{x}\}$, i.e. it is not a singleton. Then S intersects ∂U.*

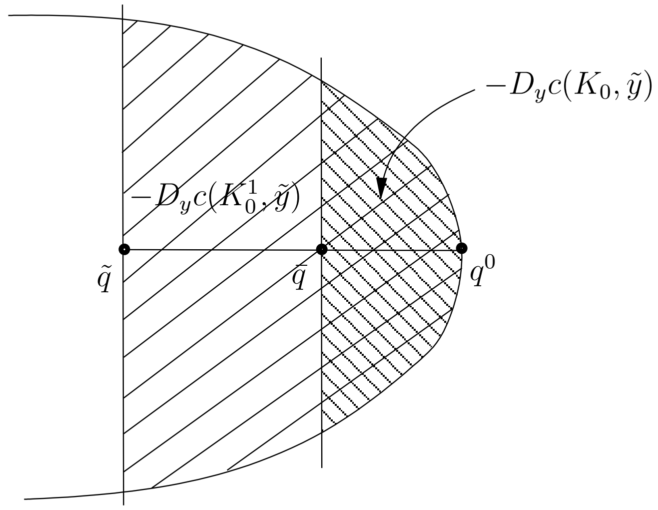

Figure 4: If the contact set $S_{\tilde{y}}$ has an exposed point q_0, we can cut two portions of $S_{\tilde{y}}$ with two hyperplanes orthogonal to $\tilde{q} - q^0$. The diameter of $-D_y c(K_0, \tilde{y})$ needs to be sufficiently small to apply the Alexandrov estimate Lemma 7.9, while $-D_y c(K_0^1, \tilde{y})$ has to intersect $U_{\tilde{y}}^{\lambda}$ is a set of positive measure to make use of Lemma 7.2 in the case q^0 is not an interior point of spt $|\partial^{\tilde{c}} u|$.

Proof. As in Definition 4.1, we transform $(x, u) \longmapsto (q, \tilde{u})$ with respect to $\tilde{y}$, i.e. we consider the transformation $q \in \overline{U}_{\tilde{y}} \longmapsto x(q) \in \overline{U}$, defined on $\overline{U}_{\tilde{y}} := -D_y c(\overline{U}, \tilde{y}) \subset T^*_{\tilde{y}} V$ by the relation

$$-D_y c(x(q), \tilde{y}) = q,$$

and the modified cost function $\tilde{c}(q, y) := c(x(q), y) - c(x(q), \tilde{y})$ on $\overline{U}_{\tilde{y}} \times \overline{V}$, for which the $\tilde{c}$-convex potential function $q \in \overline{U}_{\tilde{y}} \longmapsto \tilde{u}(q) := u(x(q)) + c(x(q), \tilde{y})$ is convex. We observe that $\tilde{c}(q, \tilde{y}) \equiv 0$ for all q, and moreover the set $S = \partial^{c^*} u^{c^*}(\tilde{y})$ appears convex from $\tilde{y}$, meaning $S_{\tilde{y}} := -D_y c(S, \tilde{y}) \subset \overline{U}_{\tilde{y}}$ is convex, by the Corollary 3.4 to Loeper's maximum principle.

Our proof is reminiscent of Caffarelli's for the cost $\tilde{c}(q, y) = -\langle q, y \rangle$ [8, Lemma 3]. Observe $\tilde{q} := -D_y c(\tilde{x}, \tilde{y})$ lies in the interior of the set $U_{\tilde{y}}^{\lambda} := -D_y c(U^{\lambda}, \tilde{y})$ where $|\partial^{\tilde{c}} \tilde{u}| \in [\lambda/\gamma_c^+, \gamma_c^-/\lambda]$, according to Corollary 4.4. Choose the point $q^0 \in S_{\tilde{y}} \subset \overline{U}_{\tilde{y}}$ furthest from $\tilde{q}$; it is an exposed point of $S_{\tilde{y}}$. We claim either $q^0 = \tilde{q}$ or $q^0 \in \partial U_{\tilde{y}}$. To derive a contradiction, suppose the preceding claim fails, meaning $q^0 \in U_{\tilde{y}} \setminus \{\tilde{q}\}$.

For a suitable choice of Cartesian coordinates on V we may, without loss of generality, take $q^0 - \tilde{q}$ parallel to the positive y^1 axis. Denote by $\hat{e}_i$ the associated orthogonal basis for $T_{\tilde{y}} V$, and set $b^0 := \langle q^0, \hat{e}_1 \rangle$ and $\tilde{b} := \langle \tilde{q}, \hat{e}_1 \rangle$, so the halfspace $q_1 = \langle q, \hat{e}_1 \rangle \geq b^0$ of $T^*_{\tilde{y}} V \simeq \mathbf{R}^n$ intersects $S_{\tilde{y}}$ only at q^0. Use the fact that q^0 is an exposed point of $S_{\tilde{y}}$ to cut a corner K_0 off the contact set S by choosing $\bar{s} > 0$ small enough that $\bar{b} = (1 - \bar{s}) b^0 + \bar{s} \tilde{b}$ satisfies:

(i) $-D_y c(K_0, \tilde{y}) := S_{\tilde{y}} \cap \{q \in \overline{U}_{\tilde{y}} \mid q_1 \geq \bar{b}\}$ is a compact convex set in the interior of $U_{\tilde{y}}$;

(ii) $\operatorname{diam}(-D_y c(K_0, \tilde{y})) \leq \varepsilon_c'/2$, where ε_c' is from Lemma 7.9.

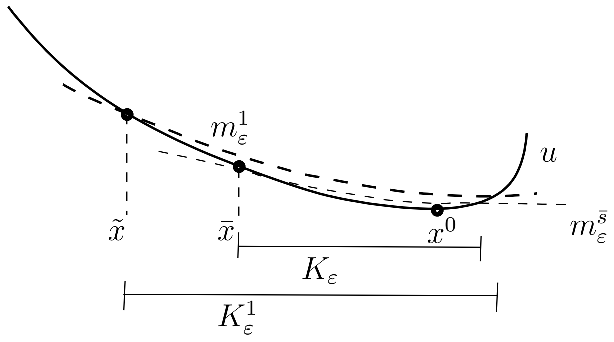

Figure 5: We cut the graph of u with the two functions $m_\varepsilon^{\bar{s}}$ and m_ε^1 to obtain two sets $K_\varepsilon \approx K_0$ and $K_\varepsilon^1 \approx K_0^1$ inside which we can apply our Alexandrov estimates to get a contradiction (both Lemma 7.2 and Lemma 7.9 to K_ε, but only Lemma 7.2 to K_ε^1). The idea is that the value of $u - m_\varepsilon^{\bar{s}}$ at x_0 is comparable to its minimum inside K_ε, but this is forbidden by our Alexandrov estimates since x_0 is too close to the boundary of K_0^ε. However, to make the argument work we need also to take advantage of the section K_ε^1, in order to "capture" some positive mass of the c-Monge-Ampère measure.

Defining $q^s := (1-s)q^0 + s\tilde{q}$, $x^s := x(q^s)$ the corresponding c-segment with respect to $\tilde{y}$, and $\bar{q} = q^{\bar{s}}$, note that $S_{\tilde{y}} \cap \{q_1 = \bar{b}\}$ contains $\bar{q}$, and K_0 contains $\bar{x} := x^{\bar{s}}$ and x^0. Since the corner K_0 needs not intersect the support of $|\partial^c u|$ (especially, when q^0 is not an interior point of $\operatorname{spt}|\partial^c u|$), we shall need to cut a larger corner K_0^1 as well, defined by $-D_y c(K_0^1, \tilde{y}) := S_{\tilde{y}} \cap \{q \in \overline{U}_{\tilde{y}} \mid q_1 \geq \tilde{b}\}$, which intersects U^λ at $\tilde{x}$. By tilting the supporting function slightly, we shall now define sections $K_\varepsilon \subset K_\varepsilon^1$ of u whose interiors include the extreme point x^0 and whose boundaries pass through $\bar{x}$ and $\tilde{x}$ respectively, but which converge to K_0 and K_0^1 respectively as $\varepsilon \to 0$.

Indeed, set $y_\varepsilon := \tilde{y} + \varepsilon \hat{e}_1$ and observe

$$\begin{aligned} m_\varepsilon^s(x) &:= -c(x, y_\varepsilon) + c(x, \tilde{y}) + c(x^s, y_\varepsilon) - c(x^s, \tilde{y}) \\ &= \varepsilon\langle -D_y c(x, \tilde{y}) + D_y c(x^s, \tilde{y}), \hat{e}_1\rangle + o(\varepsilon) \\ &= \varepsilon(\langle -D_y c(x, \tilde{y}), \hat{e}_1\rangle - (1-s)b^0 - s\tilde{b}) + o(\varepsilon) \end{aligned} \tag{8.1}$$

Taking $s \in \{\bar{s}, 1\}$ in this formula and $\varepsilon > 0$ shows the sections defined by

$$\begin{aligned} K_\varepsilon &:= \{x \mid u(x) \leq u(\bar{x}) - c(x, y_\varepsilon) + c(\bar{x}, y_\varepsilon)\}, \\ K_\varepsilon^1 &:= \{x \mid u(x) \leq u(\tilde{x}) - c(x, y_\varepsilon) + c(\tilde{x}, y_\varepsilon)\}, \end{aligned}$$

both include a neighbourhood of x_0 but converge to K_0 and K_0^1 respectively as $\varepsilon \to 0$.

We remark that there exist a priori no coordinates in which all set K_ε are convex. However for each fixed $\varepsilon > 0$, we can change coordinates so that both K_ε and K_ε^1 become convex: use y_ε to make the transformations

$$\begin{aligned} q &:= -D_y c(x_\varepsilon(q), y_\varepsilon), \\ \tilde{c}_\varepsilon(q, y) &:= c(x_\varepsilon(q), y) - c(x_\varepsilon(q), y_\varepsilon), \end{aligned}$$

so that the functions

$$\tilde u_\varepsilon(q) := u(x_\varepsilon(q)) + c(x_\varepsilon(q), y_\varepsilon) - u(\bar x) - c(\bar x, y_\varepsilon),$$
$$\tilde u^1_\varepsilon(q) := u(x_\varepsilon(q)) + c(x_\varepsilon(q), y_\varepsilon) - u(\tilde x) - c(\tilde x, y_\varepsilon).$$

are convex on $U_{y_\varepsilon} := D_y c(U, y_\varepsilon)$. Observe that, in these coordinates, K_ε and K^1_ε become convex:

$$\tilde K_\varepsilon := -D_y c(K_\varepsilon, y_\varepsilon) = \{q \in \overline{U}_{y_\varepsilon} \mid \tilde u_\varepsilon(q) \le 0\},$$
$$\tilde K^1_\varepsilon := -D_y c(K^1_\varepsilon, y_\varepsilon) = \{q \in \overline{U}_{y_\varepsilon} \mid \tilde u^1_\varepsilon(q) \le 0\},$$

and either $\tilde K_\varepsilon \subset \tilde K^1_\varepsilon$ or $\tilde K^1_\varepsilon \subset \tilde K_\varepsilon$ since $\tilde u_\varepsilon(q) - \tilde u^1_\varepsilon(q) = const$. For $\varepsilon > 0$ small, the inclusion must be the first of the two since the limits satisfy $\tilde K_0 \subset \tilde K^1_0$ and $\tilde q \in \tilde K^1_0 \setminus \tilde K_0$.

In the new coordinates, our original point $\tilde x \in U^\lambda$, the exposed point x^0, and the c-convex combination $\bar x$ with respect to $\tilde y$, correspond to

$$\tilde q_\varepsilon := -D_y c(\tilde x, y_\varepsilon), \quad q^0_\varepsilon := -D_y c(x^0, y_\varepsilon), \quad \bar q_\varepsilon := -D_y c(\bar x, y_\varepsilon).$$

Thanks to (ii), we have $\mathrm{diam}(\tilde K_\varepsilon) \le \varepsilon'_c$ for ε sufficiently small, so that the estimate of Lemma 7.9 applies. In these coordinates (chosen for each ε) we consider the parallel hyperplanes $\Pi^+_\varepsilon \ne \Pi^-_\varepsilon$ which support $\tilde K_\varepsilon \subset \overline{U}_{y_\varepsilon}$ from opposite sides and which are orthogonal to the segment joining q^0_ε with $\bar q_\varepsilon$. Since $\lim_{\varepsilon\to 0} q^0_\varepsilon - \bar q_\varepsilon = q^0 - \bar q$ paralles the $\hat e_1$ axis the limiting hyperplanes $\Pi^\pm_0 = \lim_{\varepsilon\to 0}\Pi^\pm_\varepsilon$ must coincide with $\Pi^+_0 = \{q \in T_{\tilde y}V \mid q_1 = b^0\}$ and $\Pi^-_0 = \{q \in T_{\tilde y}V \mid q_1 = \bar b\}$. Thus $q^0 \in \Pi^+_0$ and

$$\frac{\mathrm{dist}(q^0_\varepsilon, \Pi^+_\varepsilon)}{|q^0_\varepsilon - \bar q_\varepsilon|} \to 0 \ \text{ as } \varepsilon \to 0.$$

Observing that $|q^0_\varepsilon - \bar q_\varepsilon|$ is shorter than segment obtained intersecting $\tilde K_\varepsilon$ with the line orthogonal to Π^+_ε and passing through $q^0_\varepsilon \in \mathrm{int}\, \tilde K_\varepsilon$, Lemma 7.9 combines with $K_\varepsilon \subset K^1_\varepsilon$ and $|\partial^{\tilde c_\varepsilon} \tilde u_\varepsilon|(K_\varepsilon) \le \Lambda \gamma^-_c \mathscr{L}^n(K_\varepsilon)$ from (4.1) and Corollary 4.4 to yield

$$\frac{|\tilde u_\varepsilon(q^0_\varepsilon)|^n}{\Lambda \gamma^-_c \mathscr{L}^n(\tilde K^1_\varepsilon)^2} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{8.2}$$

On the other hand, $\bar x \in S$ implies $\tilde u_\varepsilon(q^0_\varepsilon) = -m^{\bar s}_\varepsilon(x^0)$, and $\tilde x \in S$ implies $\tilde u^1_\varepsilon(q^0_\varepsilon) = -m^1_\varepsilon(x^0)$ similarly. Thus (8.1) yields

$$\frac{\tilde u_\varepsilon(q^0_\varepsilon)}{\tilde u^1_\varepsilon(q^0_\varepsilon)} = \frac{\varepsilon(b^0 - \bar b) + o(\varepsilon)}{\varepsilon(b^0 - \tilde b) + o(\varepsilon)} \to \bar s \quad \text{as} \quad \varepsilon \to 0. \tag{8.3}$$

Our contradiction with (8.2)–(8.3) will be established by bounding the ratio $|\tilde u^1(q^0_\varepsilon)|^n / \mathscr{L}^n(K^1_\varepsilon)^2$ away from zero.

Recall that

$$b^0 = \langle -D_y c(x^0, \tilde y), \hat e_1\rangle \ = \ \max\{q_1 \mid q \in -D_y c(K_0, \tilde y)\}) \ > \ \tilde b$$

and $u(x) - u(\tilde{x}) \geq -c(x, \tilde{y}) + c(\tilde{x}, \tilde{y})$ with equality at x^0. From the convergence of K^1_ε to K^1_0 and the asymptotic behaviour (8.1) of $m^1_\varepsilon(x)$ we get

$$
\begin{aligned}
\frac{\tilde{u}^1_\varepsilon(q^0_\varepsilon)}{\inf_{\tilde{K}^1_\varepsilon} \tilde{u}^1_\varepsilon} &= \frac{-u(x^0) - c(x^0, y_\varepsilon) + u(\tilde{x}) + c(\tilde{x}, y_\varepsilon)}{\sup_{q \in \tilde{K}^1_\varepsilon}[-u(x(q)) - c(x(q), y_\varepsilon) + u(\tilde{x}) + c(\tilde{x}, y_\varepsilon)]} \\
&\geq \frac{-c(x^0, y_\varepsilon) + c(\tilde{x}, y_\varepsilon) + c(x^0, \tilde{y}) - c(\tilde{x}, \tilde{y})}{\sup_{x \in K^1_\varepsilon}[-c(x, y_\varepsilon) + c(\tilde{x}, y_\varepsilon) + c(x, \tilde{y}) - c(\tilde{x}, \tilde{y})]} \\
&\geq \frac{\varepsilon(\langle -D_y c(x^0, \tilde{y}), e_1\rangle - \tilde{b}) + o(\varepsilon)}{\varepsilon(\max\{q_1 \mid q \in -D_y c(K^1_\varepsilon, \tilde{y})\} - \tilde{b}) + o(\varepsilon)} \\
&\geq \frac{1}{2}
\end{aligned}
\tag{8.4}
$$

for ε sufficiently small. This shows $\tilde{u}^1(q^0_\varepsilon)$ is close to the minimum value of $\tilde{u}^1_\varepsilon$. We would like to appeal to Lemma 7.2 to conclude the proof, but are unable to do so since we only have bounds $|\partial^c u| \in [\lambda, 1/\lambda]$ on the potentially small intersection of U^λ with K^1_ε. However, this intersection occupies a stable fraction of K^1_ε as $\varepsilon \to 0$, which we shall prove as in [8, Lemma 3].

Since K^1_ε converges to K^1_0 for sufficiently small ε, observe that K^1_ε is uniformly bounded. Therefore the affine transformation $(L^1_\varepsilon)^{-1}$ that sends $\tilde{K}^1_\varepsilon$ to $B_1 \subset \tilde{K}^{1,*}_\varepsilon \subset \overline{B}_n$ as in Lemma 7.1 is an expansion, i.e. $|(L^1_\varepsilon)^{-1}q - (L^1_\varepsilon)^{-1}q'| \geq C_0|q - q'|$, with a constant $C_0 > 0$ independent of ε. Since $\tilde{x}$ is an interior point of U^λ, $B_{2\beta^-_c \delta/C_0}(\tilde{x}) \subset U^\lambda$ for sufficiently small $\delta > 0$, hence $B_{2\delta/C_0}(\tilde{q}_\varepsilon) \subset U^\lambda_{y_\varepsilon}$ with β^-_c from (4.2). Defining $U^\lambda_{y_\varepsilon} := -D_y c(U^\lambda, y_\varepsilon)$, we have

$$U^{\lambda,*}_{y_\varepsilon} := (L^1_\varepsilon)^{-1}(U^\lambda_{y_\varepsilon}) \supset B_{2\delta}(\tilde{q}^*_\varepsilon).$$

Reducing δ if necessary to ensure $\delta < 1$, define (to apply Lemma 7.2 later)

$$\tilde{K}^{1,*}_{\varepsilon,\delta} := (1 - \delta)\tilde{K}^{1,*}_\varepsilon.$$

(As in Lemma 7.2, $(1-\delta)\tilde{K}^{1,*}_\varepsilon$ denotes the dilation of $\tilde{K}^{1,*}_\varepsilon$ of a factor $(1-\delta)$ with respect to the origin.) Since $\tilde{K}^{1,*}_\varepsilon$ is convex, it contains the convex hull of $B_1 \cup \{\tilde{q}^*_\varepsilon\}$, and so

$$\mathscr{L}^n(B_{2\delta}(\tilde{q}^*_\varepsilon) \cap \tilde{K}^{1,*}_{\varepsilon,\delta}) \geq C\delta^n.$$

for some constant $C = C(n) > 0$ depending on dimension only. Letting $\tilde{K}^1_{\varepsilon,\delta} := L^1_\varepsilon(\tilde{K}^{1,*}_{\varepsilon,\delta})$ this implies

$$\mathscr{L}^n(U^\lambda_{y_\varepsilon} \cap \tilde{K}^1_{\varepsilon,\delta}) \geq C|\det L^1_\varepsilon|\delta^n \sim \mathscr{L}^n(\tilde{K}^1_\varepsilon)\delta^n.$$

Recalling that $\gtrsim$ and $\lesssim$ denote inequalities which hold up to multiplicative constants depending on n, λ and $\gamma^+_c\gamma^+_{\bar{c}}/\lambda$, Proposition 6.2 combines with this estimate to yield

$$|\partial\tilde{u}^1_\varepsilon|(\tilde{K}^1_{\varepsilon,\delta}) \gtrsim |\partial^{\tilde{c}_\varepsilon}\tilde{u}^1_\varepsilon|(\tilde{K}^1_{\varepsilon,\delta}) \gtrsim \mathscr{L}^n(K^1_\varepsilon)\delta^n,$$

where (4.1) and Corollary 4.4 have been used. Finally, since the conclusion of Lemma 7.2 holds with or without stars in light of (4.7)–(4.9), taking $t = (1-\delta)$ in (7.1) yields

$$\frac{|\inf_{\tilde K^1_\varepsilon} \tilde u^1_\varepsilon|^n}{\mathscr{L}^n(\tilde K^1_\varepsilon)^2} \gtrsim \delta^{2n}.$$

Since $\delta > 0$ is independent of ε this contradicts (8.2)–(8.3) to complete the proof. $\square$

Remark 8.2. As can be easily seen from the proof, one can actually show that if $U^\lambda = U$ and S is not a singleton, then $S_{\tilde y}$ has no exposed points in the interior of $U_{\tilde y}$. Indeed, if by contradiction there exists q^0 an exposed point of $S_{\tilde y}$ belonging to the interior of $U_{\tilde y}$, we can choose a point $\tilde q \in S_{\tilde y}$ in the interior of $U_{\tilde y} = U^\lambda_{\tilde y}$ such that the segment $q^0 - \tilde q$ is orthogonal to a hyperplane supporting $S_{\tilde y}$ at q^0. Then it can immediately checked that the above proof (which could even be simplified in this particular case) shows that such a point q^0 cannot exist.

9 Continuity and injectivity of optimal maps

The first theorem below combines results of Sections 5 and 8 to deduce strict c-convexity of the c-potential for an optimal map, if its target is strongly c-convex. This strict c-convexity — which is equivalent to injectivity of the map — will then be combined with an adaptation of Caffarelli's argument [5, Corollary 1] to obtain interior continuity of the map — or equivalently C^1-regularity of its c-potential function — for non-negatively cross-curved costs, yielding the concluding theorem of the paper.

Theorem 9.1 (Injectivity of optimal maps to a strongly c-convex target)**.** *Let c satisfy* **(B0)**–**(B3)** *and* **(B2u)***. If u is a c-convex solution of* (4.1) *on $U^\lambda \subset U$ open, then u is strictly c-convex on U^λ, meaning $\partial^c u(x)$ and $\partial^c u(\tilde x)$ are disjoint whenever $x, \tilde x \in U^\lambda$ are distinct.*

Proof. Suppose by contradiction that $\tilde y \in \partial^c u(x) \cap \partial^c u(\tilde x)$ for two distinct points $x, \tilde x \in U^\lambda$, and set $S = \partial^{c^*} u^{c^*}(\tilde y)$. According to Theorem 8.1, the set S intersects the boundary of U at a point $\bar x \in \partial U \cap \partial^{c^*} u^{c^*}(\tilde y)$. Since (4.1) asserts $\lambda \le |\partial^c u|$ on U^λ and $|\partial^c u| \le \Lambda$ on $\overline{U}$, Theorem 5.1(a) yields $\tilde y \in V$ (since $x, \tilde x \in U^\lambda$), and hence $\bar x \in U$ by Theorem 5.1(b). This contradicts $\bar x \in \partial U$ and proves the theorem. $\square$

Theorem 9.2 (Continuity of optimal maps to strongly c-convex targets)**.** *Let c satisfy* **(B0)**–**(B3)** *and* **(B2u)***. If u is a c-convex solution of* (4.1) *on $U^\lambda \subset U$ open, then u is continuously differentiable inside U^λ.*

Proof. Recalling that c-convexity implies semiconvexity, all we need to show is that the c-subdifferential $\partial^c u(\tilde x)$ of u at every point $\tilde x \in U_\lambda$ is a singleton.

Assume by contradiction that is not. As $\partial^c u(\tilde x)$ is compact, one can find a point y_0 in the set $\partial^c u(\tilde x)$ such that $-D_x c(\tilde x, y_0) \in \partial u(\tilde x)$ is an exposed point of the compact convex set $\partial u(\tilde x)$. Similarly to Definition 4.1, we transform $(x, u) \longmapsto (q, \tilde u)$ with respect to y_0, i.e.

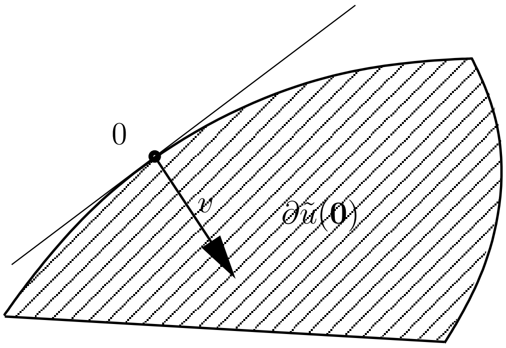

Figure 6: $v \in \partial\tilde{u}(\mathbf{0})$ and the hyperplane orthogonal to v is supporting $\partial\tilde{u}(\mathbf{0})$ at 0.

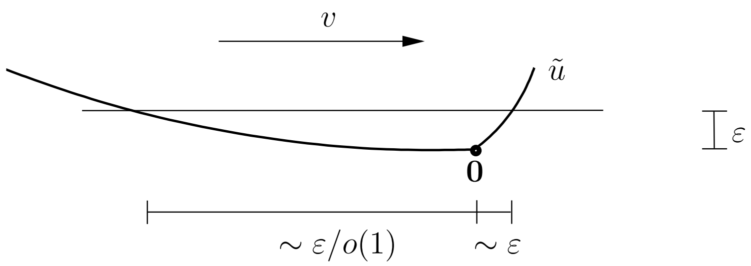

Figure 7: Since the hyperplane orthogonal to v is supporting $\partial\tilde{u}(\mathbf{0})$ at 0, we have $\tilde{u}(-tv) = o(t)$ for $t \geq 0$. Moreover, $\tilde{u}$ grows at least linearly in the direction of v.

we consider the transformation $q \in \overline{U}_{y_0} \longmapsto x(q) \in \overline{U}$, defined on $\overline{U}_{y_0} = -D_y c(\overline{U}, y_0) + D_y c(\tilde{x}, y_0) \subset T^*_{y_0} V$ by the relation

$$-D_y c(x(q), y_0) + D_y c(\tilde{x}, y_0) = q,$$

and the modified cost function $\tilde{c}(q, y) := c(x(q), y) - c(x(q), y_0)$ on $\overline{U}_{y_0} \times \overline{V}$, for which the $\tilde{c}$-convex potential function $q \in \overline{U}_{y_0} \longmapsto \tilde{u}(q) := u(x(q)) - u(\tilde{x}) + c(x(q), y_0) - c(\tilde{x}, y_0)$ is convex. We observe that $\tilde{c}(q, y_0) \equiv 0$ for all q, the point $\tilde{x}$ is sent to $\mathbf{0}$, $\tilde{u} \geq \tilde{u}(\mathbf{0}) = 0$, and $\tilde{u}$ is strictly convex thanks to Theorem 9.1. Moreover, since $-D_x c(\tilde{x}, y_0) \in \partial u(\tilde{x})$ was an exposed point of $\partial u(\tilde{x})$, $0 = -D_q \tilde{c}(\mathbf{0}, y_0)$ is an exposed point of $\partial\tilde{u}(\mathbf{0})$. Hence, we can find a vector $v \in \partial\tilde{u}(\mathbf{0}) \setminus \{0\}$ such that the hyperplane orthogonal to v is a supporting hyperplane for $\partial\tilde{u}(\mathbf{0})$ at 0. Thanks to the convexity of $\tilde{u}$, this implies that

$$\tilde{u}(-tv) = o(t) \quad \text{for } t \geq 0, \qquad \tilde{u}(q) \geq \langle v, q \rangle + \tilde{u}(\mathbf{0}) \quad \text{for all } q \in U_{y_0}. \tag{9.1}$$

Let us now consider the section $K_\varepsilon := \{\tilde{u} \leq \varepsilon\}$. Since $\tilde{u}(\mathbf{0}) = 0$, $\tilde{u} \geq 0$ and $\tilde{u}$ is strictly convex, $K_\varepsilon \to \{\mathbf{0}\}$ as $\varepsilon \to 0$. Thus by (9.1) it is easily seen that for ε sufficiently small the following hold:

$$K_\varepsilon \subset \{q \mid \langle q, v \rangle \leq \varepsilon\}, \qquad -\alpha(\varepsilon) v \in K_\varepsilon,$$

where $\alpha(\varepsilon) > 0$ is a positive constant depending on ε and such that $\alpha(\varepsilon)/\varepsilon \to +\infty$ as $\varepsilon \to 0$. Since $\mathbf{0}$ is the minimum point of $\tilde{u}$, this immediately implies that one between our Alexandrov estimates (7.10) or (7.11) must be violated by $\tilde{u}$ inside K_ε for ε sufficiently small, which is the desired contradiction. $\square$

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