Convex solutions to the power-of-mean curvature flow, conformally invariant inequalities and regularity results in some applications of optimal transportation

by

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Abstract

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In this thesis we study three different problems: convex ancient solutions to the power-of-mean curvature flow; Sharp inequalities; regularity results in some applications of optimal transportation.

The second chapter is devoted to the power-of-mean curvature flow; We prove some estimates for convex ancient solutions (the existence time for the solution starts from $-\infty$) to the power-of-mean curvature flow, when the power is strictly greater than $\frac{1}{2}$. As an application, we prove that in two dimension, the blow-down of an entire convex translating solution, namely $u_h = \frac{1}{h}u(h^{\frac{1}{1+\alpha}}x)$, locally uniformly converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$ as $h \to \infty$. The second application is that for generalized curve shortening flow (convex curve evolving in its normal direction with speed equal to a power of its curvature), if the convex compact ancient solution sweeps the whole space $\mathbb{R}^2$, it must be a shrinking circle. Otherwise the solution must be defined in a strip region. In the first section of the third chapter, we prove a one-parameter family of sharp conformally invariant integral inequalities for functions on the n-dimensional unit ball. As a limiting case, we obtain an inequality that generalizes Carleman's inequality for harmonic functions in the plane to poly-harmonic functions in higher dimensions. The second section represents joint work with Tobias Weth and Rupert Frank; the main result is that, one can always put a sharp remainder term on the righthand side of the sharp fractional sobolev inequality.

In the first section of the final chapter, under some suitable condition, we prove that the solution to the principal-agent problem must be C^1. The proof is based on a perturbation argument. The second section represents joint work with Emanuel Indrei; the main result is that, under (A3S) condition on the cost and c-convexity condition on the domains, the free boundary in the optimal partial transport problem is $C^{1,\alpha}$.

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Contents

Chapter 1

Introduction

1.1 The power-of-mean curvature flow

Classifying ancient convex solution to mean curvature flow is very important in studying the singularities of mean curvature flow. Translating solutions arise as a special case of ancient solution when one uses a proper procedure to blow up the mean convex flow near type II singular points, and general ancient solutions arise at general sigularities. Some important progress was made by Wang [69], and Daskalopoulos, Hamilton and Sesum [29]. In [69] Wang proved that an entire convex translating solution to mean curvature flow must be rotationally symmetric which was a conjecture formulated explicitly by White in [68]. Wang also constructed some entire convex translating solution with level set neither spherical nor cylindrical in dimension greater than or equal to 3. In the same paper, Wang also proved that if a convex ancient solution to the curve shortening flow sweeps the whole space $\mathbb{R}^2$, it must be a shrinking circle, otherwise the convex ancient solution must be defined in a strip region and he indeed constructed such solutions by some compactness argument. Daskalopoulos, Hamilton and Sesum [29] showed that apart from the shrinking circle, the so called Angenent oval(a convex ancient solution of the curve shortening flow discovered by Angenent that decomposes into two translating solutions of the flow) is the only other embedded convex compact ancient solution of the curve shortening flow. That means the corresponding curve shortening solution defined in a strip region constructed by Wang is exactly the "Angenent oval".

The power-of-mean curvature flow, in which a hypersurface evolves in its normal direction with speed equal to a power α of its mean curvature H, was studied by Andrews [1], [2], [3], Schulze [61], Chou and Zhu [28] and Sheng and Wu [64] . Schulze [61] called it H^α-flow. In the following, we will also call the one dimensional power-of-curvature flow the generalized curve shortening flow. Similar to the mean curvature flow, when one blows up the flow near the type II singularity appropriately, a convex translating solution will arise, see [64] for details. It will be very interesting if one could classify the ancient convex solutions. In the second chapter of this thesis, we use the method developed by Wang [69] to study the geometric asymptotic behavior of ancient convex solutions to H^α-flow. The general equation for H^α-flow is $\frac{\partial F}{\partial t} = -H^\alpha \vec{v}$, where $F : M \times [0, T) \to \mathbb{R}^{n+1}$ is a time-dependent embedding of the evolving hypersurface, $\vec{v}$ is the unit normal vector to the hypersurface $F(M, t)$ in $\mathbb{R}^{n+1}$ and H is its mean curvature. If the evolving hypersurface can be represented as a graph of a function $u(x, t)$ over some domain in $\mathbb{R}^n$, then we can project the evolution equation to the $(n+1)$th coordinate direction of $\mathbb{R}^{n+1}$ and the equation becomes

$$u_t = \sqrt{1 + |Du|^2} \left(\text{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) \right)^\alpha .$$

Then a translating solution to the H^α-flow will satisfy the equation

$$\sqrt{1 + |Du|^2} (\text{div}(\frac{Du}{\sqrt{1 + |Du|^2}}))^\alpha = 1,$$

which is equivalent to the following equation (1.3) when $\sigma = 1$,

$$L_\sigma(u) = (\sqrt{\sigma + |Du|^2})^{\frac{1}{\alpha}} \mathrm{div}(\frac{Du}{\sqrt{\sigma + |Du|^2}}) \tag{1.1}$$

$$= (\sigma + |Du|^2)^{\frac{1}{2\alpha} - \frac{1}{2}} \sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{\sigma + |Du|^2}) u_{ij} \tag{1.2}$$

$$= 1, \tag{1.3}$$

where $\sigma \in [0, 1]$, $\alpha \in (\frac{1}{2}, \infty]$ is a constant, $n = 2$ is the dimension of $\mathbb{R}^2$. If u is a convex solution of (1.3), then $u + t$, as a function of $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$, is a translating solution to the flow

$$u_t = \sqrt{\sigma + |Du|^2} (\mathrm{div}(\frac{Du}{\sqrt{\sigma + |Du|^2}}))^\alpha. \tag{1.4}$$

When $\sigma = 1$, equation (1.4) is the non-parametric power-of-mean curvature flow. When $\sigma = 0$,(1.3) is the level set flow. That is, if u is a solution of (1.3) with $\sigma = 0$, the level set $\{u = -t\}$, where $-\infty < t < -\inf u$, evolves by the power-of-mean curvature.

In the following we will assume $\sigma \in [0, 1]$, $\alpha \in (\frac{1}{2}, \infty]$ and the dimension $n = 2$, although some of the estimates do hold in high dimension. The main results of the second chapter are the following theorems.

Theorem 1. *Let u be an entire convex solution of* (1.3). *Let $u_h(x) = h^{-1} u(h^{\frac{1}{1+\alpha}} x)$. Then u_h locally uniformly converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$, as $h \to \infty$.*

Theorem 2. *Let u_σ be an entire convex solution of* (1.3). *Then $u_0(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}$ up to a translation of the coordinate system. When $\sigma \in (0, 1]$, if $|D^2 u(x)| = O(|x|^\beta)$ as $|x| \to \infty$, for any fixed constant β satisfying $\beta < 3\alpha - 2$, then u_σ is rotationally symmetric after a proper translation of the coordinate system.*

Corollary 1. *A convex compact ancient solution to the generalized curve shortening flow which sweeps the whole space $\mathbb{R}^2$ must be a shrinking circle.*

Remark 1.1.1. . *The condition* $\alpha > \frac{1}{2}$ *is necessary for our results. One can consider the translating solution* $v(x)$ *to* (1.3) *with* $\sigma = 1$ *in one dimension. In fact when* $\alpha \leq \frac{1}{2}$, *the translating solution* $v(x)$ *is a convex function defined on the entire real line ([28] page 28). Then one can construct a function* $u(x, y) = v(x) - y$ *defined on the entire plane, and* u *will satisfy* (1.3) *with* $\sigma = 0$ *and it is obviously not rotationally symmetric.*

We would also like to point out that this elementary construction can be used to give a slight simplification of Wang's proof for Theorem 2.1 in [69](corresponding to our Corollary 3 for $\alpha = 1$). Let v_σ be an entire convex solution to (1.3) in dimension n with $\sigma \in (0, 1]$. Then $u(x, y) = v_\sigma(x) - \sqrt{\sigma}y$ will be an entire convex solution to (1.3) in dimension $n + 1$ with $\sigma = 0$. Hence if one has proved the estimate in Corrollary 3 for $\sigma = 0$ in all dimensions, the estimates for $\sigma \in (0, 1]$ follows immediately from the above construction.

1.2 Conformally invariant inequalities and remainder terms in the fractional Sobolev Inequality

1.2.1 Carleman type conformally invariant inequalities

There is a well known inequality by Carleman [17]

$$\int_{B_2} e^{2u} dx \leq \frac{1}{4\pi} (\int_{\partial B_2} e^u d\theta)^2, \tag{1.5}$$

for all harmonic functions in the unit ball B_2 of $\mathbb{R}^2$. Equality occurs exactly for $u = c$ and $u = -2 \log |x - x_0| + c$, where c is a constant and $x_0 \in \mathbb{R}^2 - \overline{B_2}$.

Although Carleman proved (1.5) initially for harmonic functions, it follows from the maximum principle that inequality (1.5) holds for subharmonic functions. Beckenbach and Rado [8] used Carleman's inequality to prove the isoperimetric inequality on a surface with non-positive Gauss curvature: If on a surface with non-positive Gauss curvature an analytic curve C of length L encloses a simply-connected domain D of area A, then the inequality

$$L^2 \geq 4\pi A$$

holds. This is exactly the sharp isoperimetric inequality in the plane. Their proof is quite simple: In isothermal coordinates (x, y) for a simply-connected domain $\widetilde{D}$ which is slightly larger than D, then the metric on $\widetilde{D}$ can be written as $e^{2w}(dx^2+dy^2)$, for (x, y) in some bounded domain $\Omega \in \mathbb{R}^2$. Now, the coordinate image of D in Ω is a Jordan domain, so by the Riemann mapping theorem we can map it to B_2 conformally, which means D with the metric induced by the metric of the surface is isometric to $(B_2, e^{2u}g)$, where u is a subharmonic function (By the non-positive curvature condition). Beckenbach and Rado's result now follows directly from Carleman's inequality.

The 2008 paper by Hang, Wang and Yan [45], generalized this inequality to higher dimensions as follows. For any harmonic function u in the unit ball $B_n \subset \mathbb{R}^n$,

$$\|u\|_{L^{\frac{2n}{n-2}}(B_n)} \leq n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} \|u\|_{L^{\frac{2(n-1)}{n-2}}(\partial B_n)}, \tag{1.6}$$

where $n \geq 3$ and ω_n is the volume of B_n. Any constant is an optimizer and it is unique up to a conformal transformation (as will be explained before the proof of Theorem 4). This is a special case of our Theorem (3) ($a = 0$), and again because of the maximum principle, this inequality holds for subharmonic functions. Hang, Wang and Yan interpreted their inequality as the isoperimetric inequality for B_n with metric $\rho^{\frac{4}{n-2}}g$, where ρ is subharmonic (which means non-positive scalar curvature). By using the conformal map (1.10), the equivalent form of inequality (3.17) in the upper-half space is

$$\|Pf\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} \leq n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} \|f\|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}, \tag{1.7}$$

for all $f \in L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})$. Here $\mathbb{R}^{n-1}$ is the boundary of $\mathbb{R}^n_+$ and Pf is the harmonic extension of f to the upper halfspace. The optimizers are $f(Y) = \frac{c}{(\lambda^2+|Y-Y_0|^2)^{\frac{n-2}{2}}}$, for some constant c, positive constant λ and $Y_0 \in \mathbb{R}^{n-1}$.

In the first section of the third chapter of this thesis we prove an analogous result for a one-parameter family $\{P_a\}_{2-n<a<1}$ of Poisson-type kernels on B_n, which includes Hang, Wang, and Yan theorem thanks to the fact that $P_0 = P$, and which includes some new interesting cases, like k-harmonic functions in B_{2k}, corresponding to the choices $a = 2-2k$, $n = 2k$ (thus providing another direction of generalization of Carlemans inequality).

In the following B_n denotes the n-dimensional unit ball in Euclidean space, $\|u\|_{L^p(\Omega)}$ is the L^p norm of function u defined on domain Ω, $|B_n|$ is the volume of B_n and $c(n,a,p)$ is some constant which depends on n, a and p. The parameter a satisfies $2-n<a<1$. Before giving the main theorems, we will give an interesting corollary for the reason that it is easy to state and it is clearly a natural generalization of Carleman's inequality in four dimension.

Corollary 2. *For any u satisfying $\Delta^2 u \le 0$ on B_4 and $-\frac{\partial u}{\partial \gamma} \le 1$ on ∂B_4, where γ is the outer normal of ∂B_4, we have*

$$(\int_{B_4} e^{4u}dx)^{\frac{1}{4}} \le S(\int_{\partial B_4} e^{3u}d\xi)^{\frac{1}{3}}, \tag{1.8}$$

The sharp constant is assumed by the solution of $\Delta^2 u = 0$ in B_4 with boundary values $-\frac{\partial u}{\partial \gamma} = 1$ and $u = 0$ on ∂B_4.

For a function f defined on $\mathbb{R}^{n-1}$ (thought of as the boundary of the upper half-space $\mathbb{R}^n_+$), we define a poly-harmonic extension as follows: for $(X, x_n) \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, +\infty)$,

$$(P_a f)(X, x_n) = d_{n,a} \int_{\mathbb{R}^{n-1}} \frac{x_n^{1-a}}{((X-Y)^2 + x_n^2)^{\frac{n-a}{2}}} f(Y) d^{n-1}Y. \tag{1.9}$$

Here the choice of integrand guarantees independence of $P_a 1$ on (X, x_n), while $a < 1$ ensures $P_a 1 < \infty$, and the normalization constants $d_{n,a}$ are chosen so that $P_a 1 = 1$ (and can be expressed explicitly using Γ functions). Recalling that inversion in the unit sphere maps the halfspace $x_n > 1/2$ to the unit ball centered at $(\mathbf{0}, 1)$, we see the conformal map

$$\phi(X, x_n) = \frac{(X, x_n + \frac{1}{2})}{|(X, x_n + \frac{1}{2})|^2} - (\mathbf{0}, 1) \tag{1.10}$$

maps the upper halfspace $x_n > 0$ to the standard ball $\phi : \mathbb{R}^n_+ \longrightarrow B_n$. Conformality of this map makes it easy to compute its Jacobian $J(\phi) = |(X, x_n + \frac{1}{2})|^{-2n}$, and the Jacobian $J(\phi|_{\partial \mathbb{R}^n_+}) = |(X, \frac{1}{2})|^{-2(n-1)}$ of its boundary trace. Indeed, ϕ pulls back the Euclidean metric g on B_n to the conformally flat metric $\phi^* g = |(X, x_n + \frac{1}{2})|^{-4} \sum dx_i^2$ on $\mathbb{R}^n_+$. Then it is not hard to check the formula

$$f(X, x_n) = |(X, x_n + \tfrac{1}{2})|^{2-n-a} \tilde{f} \circ \phi(X, x_n) \tag{1.11}$$

and its restriction to $x_n = 0$ boundary trace define Banach space isometries from $\tilde{f} \in L^{\frac{2n}{n-2+a}}(B_n)$ to $f \in L^{\frac{2n}{n-2+a}}(\mathbb{R}^n_+)$ and from $L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)$ to $L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})$ respectively. We define the poly-harmonic extension $\tilde{P}_a \tilde{f}$ of $\tilde{f} \in L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)$ implicitly by using P_a after pulling back from the ball to the halfspace:

$$(\tilde{P}_a \tilde{f}) \circ \phi(X, x_n) = |(X, x_n + \tfrac{1}{2})|^{n+a-2} P_a \left(\frac{\tilde{f} \circ \phi(Y, \frac{1}{2})}{|(Y, \frac{1}{2})|^{n+a-2}} \right). \tag{1.12}$$

When $a = 0$, $\tilde{P}_a \tilde{f}$ again becomes the usual harmonic extension to the ball. Another case of special interest is $a = 2 - n$, in which case the conformal factors are suppressed so that $\tilde{P}_{2-n} 1 = 1$, and the isometric Banach spaces are both of L^∞ type. When $n = 2k$ the extended function turns out to be k harmonic on the $2k$ dimensional ball, i.e. $\Delta^k \tilde{P}_{2-2k} \tilde{f} = 0$.

Theorem 3. *For any* $f \in L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)$*,* $n \geq 2$*,* $a < 1$ *and* $n-2+a > 0$*, we have the sharp inequality*

$$\|\widetilde{P}_a f\|_{L^{\frac{2n}{n-2+a}}(B_n)} \leq S_{n,a}\|f\|_{L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)}, \tag{1.13}$$

where the sharp constant $S_{n,a}$ *depends only on* n *and* a*. The optimizers are unique up to a conformal transform and include the constant function* $f = 1$*.*

We now study the limiting information. Letting $f = 1 + \frac{n-2+a}{2}F$ and $a \to 2-n$, we get the following inequality

Theorem 4. *For any* F *such that* $e^F \in L^{n-1}(\partial B_n)$*,* $n > 2$*, we have*

$$\|e^{I_n + \widetilde{P}_{2-n}F}\|_{L^n(B_n)} \leq S_n\|e^F\|_{L^{n-1}(\partial B_n)}, \tag{1.14}$$

where

$$I_n = \left(\log(X^2 + (x_n + \frac{1}{2})^2) - d_{n,2-n}\int_{\mathbb{R}^{n-1}} \frac{x_n^{n-1}}{((X-Y)^2 + x_n^2)^{n-1}}\log(Y^2 + \frac{1}{4})d^{n-1}Y\right) \circ \phi^{-1}.$$

Up to a conformal transform any constant is an optimizer.

Remark 1.2.1. *We point out that the sharp inequality* (1.6) *combines with Brezis and Lieb's dual argument ([13] page10-11) to give the sharp version of inequality* (1.9) *in [13] when the domain is a ball:*

$$\|\nabla f\|_{L^2(B_n)} + C(n)\|f\|_{L^{\frac{2(n-1)}{n-2}}(\partial B_n)} \geq S_n\|f\|_{L^{\frac{2n}{n-2}}(B_n)},$$

where S_n *is sharp Sobolev constant and* $C(n)$ *can be determined by letting* $f = 1$ *when the inequality becomes equality. This sharp Sobolev inequality with trace term was also proved by Maggi and Villani in [58] by using methods from optimal transportation.*

Remark 1.2.2. *When $-1 < a < 1$, from Caffarelli and Silvestre [16] we know $u = P_a f$ is the unique solution to the boundary value problem*

$$\begin{aligned} \operatorname{div}(x_n^a \nabla u(X, x_n)) &= 0, (X, x_n) \in \mathbb{R}_+^n \\ u(X, 0) &= f, X \in \mathbb{R}^{n-1}. \end{aligned}$$

Then the fractional Laplacian can be defined by using an analogue of the Dirichlet to Neumann map $(-\Delta)^{\frac{1-a}{2}} f = -\lim_{x_n \to 0+} x_n^a u_y$. So, our equivalent form of inequality (1.13) on $\mathbb{R}_+^n$, namely $\|P_a f\|_{L^{\frac{2n}{n-2+a}}(\mathbb{R}_+^n)} \le \widetilde{S}_{n,a} \|f\|_{L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^n)}$, provides a sharp estimate for the $L^{\frac{2n}{n-2+a}}$ norm of solution of the above boundary value problem.

1.2.2 Remainder terms in the fractional Sobolev Inequality

In the joint work with Frank and Weth, we consider the fractional Sobolev inequality

$$\|u\|_{s/2}^2 \ge \mathcal{S} \left(\int_{\mathbb{R}^N} |u|^q dx \right)^{\frac{2}{q}} \qquad \text{for all } u \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N), \tag{1.15}$$

where $0 < s < N$, $q = \frac{2N}{N-s}$, and $\mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ is the space of all tempered distributions u such that

$$\hat{u} \in L^1_{loc}(\mathbb{R}^N) \qquad \text{and} \qquad \|u\|_{s/2}^2 := \int_{\mathbb{R}^N} |\xi|^s |\hat{u}|^2 d\xi < \infty.$$

Here, as usual, $\hat{u}$ denotes the (distributional) Fourier transform of u. The best Sobolev constant

$$\mathcal{S} = \mathcal{S}(N, s) = 2^s \pi^{\frac{s}{2}} \frac{\Gamma(\frac{N+s}{2})}{\Gamma(\frac{N-s}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{s/N}, \tag{1.16}$$

i.e., the largest possible constant in (1.15), has been computed first in the special case $s = 2$, $N = 3$ by Rosen [60] and then independently by Aubin [5] and Talenti [65] for $s = 2$ and all dimensions N. For general $s \in (0, N)$, the best constant has been given by Lieb [50] for an equivalent reformulation of inequality (1.15), the (diagonal) Hardy-

Littlewood-Sobolev inequality. In order to discuss this equivalence in some more detail, we note that

$$\|u\|_{s/2}^2 = \int_{\mathbb{R}^N} u(-\Delta)^{s/2} u dx \tag{1.17}$$

for every Schwartz function u, where the operator $(-\Delta)^{s/2}$ is defined by

$$\widehat{(-\Delta)^{s/2}} u(\xi) = |\xi|^s \widehat{u}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^N.$$

Moreover, $\mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ is also given as the completion of smooth functions with compact support under the norm $\|\cdot\|_{s/2}$. The (diagonal) Hardy-Littlewood-Sobolev inequality states that

$$\Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \, dx \, dy \Big| \leq \pi^{\lambda/2} \frac{\Gamma(\frac{N-\lambda}{2})}{\Gamma(N-\frac{\lambda}{2})} \Big(\frac{\Gamma(N)}{\Gamma(N/2)} \Big)^{1-\frac{\lambda}{N}} |f|_p |g|_p \tag{1.18}$$

for all $f, g \in L^p(\mathbb{R}^N)$, where $0 < \lambda < N$ and $p = \frac{2N}{2N-\lambda}$. Here and in the following, we let $|\cdot|_r$ denote the usual L^r-norm for $1 \leq r \leq \infty$. The equivalence of (1.15) and (1.18) follows – by a duality argument – from the fact that for every $f \in L^{\frac{q}{q-1}}(\mathbb{R}^N)$ there exists a unique solution $(-\Delta)^{-s/2} f \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ of the equation $(-\Delta)^{s/2} u = f$ given by convolution with the Riesz potential, i.e., by

$$[(-\Delta)^{-s/2} f](x) = 2^{-s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N-s}{2})}{\Gamma(s/2)} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-s}} f(y) \, dy \qquad \text{for a.e. } x \in \mathbb{R}^N. \tag{1.19}$$

In [50], Lieb identified the extremal functions for (1.18), and his results imply that equality holds in (1.15) for nontrivial u if and only if u is contained in an $N+2$-dimensional submanifold $\mathcal{M}$ of $\mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ given as the set of functions which, up to translation, dilation and multiplication by a nonzero constant, coincide with

$$U \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N), \qquad U(x) = (1+|x|^2)^{-\frac{N-s}{2}}. \tag{1.20}$$

For the special case $s = 2$, i.e., the first order Sobolev inequality, Brezis and Lieb [13] asked the question whether a remainder term – proportional to the quadratic distance of the function u to the manifold $\mathcal{M}$ – can be added to the right hand side of (1.15). This question was answered affirmatively in the case $s = 2$ by Bianchi and Egnell [10], and their result was extended later to the case $s = 4$ in [56] and to the case of an arbitrary even positive integer $s < N$ in [7]. In the second section of the third chapter we prove the corresponding remainder term inequality for all (real) values $s \in (0, N)$. Our main result is the following.

Theorem 5. *Let*

$$\mathcal{M} := \left\{ c\,U\big(\frac{\cdot - x_0}{\varepsilon}\big) \;:\; c \in \mathbb{R} \setminus \{0\}, x_0 \in \mathbb{R}^N, \varepsilon > 0 \right\} \subset \overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N), \tag{1.21}$$

where U is defined in (1.20). Then there exists a positive constant α depending only on the dimension N and $s \in (0, N)$ such that

$$d^2(u, \mathcal{M}) \geq \int_{\mathbb{R}^N} u(-\Delta)^{s/2}(u)dx - \mathcal{S}\left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{2}{q}} \geq \alpha\, d^2(u, \mathcal{M}) \tag{1.22}$$

for all $u \in \overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N)$, where $d(u, \mathcal{M}) = \min\{\|u - \varphi\|_{s/2} \;:\; \varphi \in \mathcal{M}\}$.

As a corollary of Theorem 5, we also derive a remainder term inequality for the space $\overset{\circ}{H}{}^{\frac{s}{2}}(\Omega) \subset \overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N)$ of all functions $u \in \overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N)$ which vanish in $\mathbb{R}^N \setminus \Omega$. In the case where Ω is bounded and has a continuous boundary, $\overset{\circ}{H}{}^{\frac{s}{2}}(\Omega)$ coincides with the completion of $\mathcal{C}_0^\infty(\Omega) \subset \overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{s/2}$, whereas in general it may be a slightly larger space (see e.g. [43, Theorem 1.4.2.2]). We also recall that, for $1 < r < \infty$, the weak L^r-norm of a measurable function u on Ω is given by

$$|u|_{w,r,\Omega} = \sup_{\substack{A \subset \Omega \\ |A| > 0}} |A|^{\frac{1}{r} - 1} \int_A |u|\, dx,$$

see e.g. [46].

Theorem 6. *Let, as before,* $q = \frac{2N}{N-s}$. *Then there exists a constant* $C > 0$ *depending only on* N *and* $s \in (0, N)$ *such that for every domain* $\Omega \subset \mathbb{R}^N$ *with* $|\Omega| < \infty$ *and every* $u \in \mathring{H}^{\frac{s}{2}}(\Omega)$ *we have*

$$\|u\|_{s/2}^2 - \mathcal{S} \left(\int_\Omega |u|^q dx \right)^{\frac{2}{q}} \geq C|\Omega|^{-\frac{2}{q}} |u|_{w,q/2,\Omega}^2 \tag{1.23}$$

For fixed bounded domains $\Omega \subset \mathbb{R}^N$, the existence of a weak $L^{q/2}$-remainder term is due to Brezis and Lieb [13] in the case $s = 2$ and to Gazzola and Grunau [40] in the case of an arbitrary even positive integer $s < N$. Bianchi and Egnell [10] gave an alternative proof in the case $s = 2$ using the corresponding special case of inequality (1.22). We will follow similar ideas in our proof of Theorem 6, using Theorem 5 in full generality. We note that some additional care is needed to get a remainder term which only depends on $|\Omega|$ and not on Ω itself.

1.3 Regularity results in some applications of optimal transportation

Recall that in the optimal transportation problem, one is given a source domain Ω with density $f(x)dx$ and a target domain Λ with density $g(x)dx$, where f and g are nonnegative Borel functions and satisfy the mass balance condition ($\int_\Omega f = \int_\Lambda g$). Then, given a cost function $c(x, y)$, a central problem is to find an optimal transference plan, namely a Borel map ϕ which minimizes the total cost $\int_\Omega c(x, \varphi(x))dx$ among all maps $\varphi : \Omega \to \Lambda$ pushing $f(x)dx$ forward to $g(x)dx$. Through the efforts of many authors, a rather satisfactory theory has been developed for the existence, uniqueness and regularity of the optimal transference plan (see [67] and references therein). However, in the two applications below, one can not apply the existing regularity theory of classical optimal transportation directly to them. The main reason is that both cases involve transporting densities which are a priori unknown. Moreover, examples show these unknown densities won't generally satisfy the hypotheses demanded by all of the existing theories for smoothness of optimal maps.

1.3.1 The principal-agent problem

In economics, the principal-agent problem arises when the two parties have different interests and asymmetric information. Knowing the distribution of different types of agents and their preference, the principal needs to make some decision that maximizes her total profit or minimizes her total loss. A typical example is that a monopolist wants to market automobiles ($y \in Y$) to a population of potential buyers ($x \in X$), with some known information as following: preference function $b(x, y)$ that measures the the preference of a buyer x for the car y, the density of different types of buyers in the population $d\mu(x)$ and the cost $c(y)$ for manufacturing a car of type y. In order to maximize her profit, the principal needs to decide what kind of cars to manufacture and the price of each type of car. Recently Figalli, Kim and McCann [36] identified the conditions that ensure the existence and uniqueness of the strategy that the principal can use for maximizing her profit. They also discussed various interesting phenomena that the optimal strategy may display.

For some important special preference functions, the problem was studied by Wilson [70], Armstrong [4], and Rochet and Choné [27]. Later, for general preference function Carlier [20] reformulated it as a minimization problem over the space of b-convex functions (see Definition 1), however for general $b(x, y)$ the space of admissible functions is generally not convex, which is the main reason that apart from existence, he was unable to deduce many key properties of the optimal strategy, such as its uniqueness. By adapting a strengthened version of the so called MTW condition in optimal transportation together with the conditions bi-twist and bi-convexity which had also been used in the regularity theory of optimal transportation, Figalli, Kim and McCann successfully established various convexity properties in the principal-agent problem, and those convexity properties enable them to prove the uniqueness and some other important properties of the solution. For a more complete discussion of the problem we refer the reader to [36] and references therein.

In the first section of the last chapter of this thesis we study the regularity of the solution to the principal-agent problem. For the special case when $b(x, y) = x \cdot y$ and $c(y) = |y|^2$, the C^1 regularity of the minimizer was proved by Carlier and Lachand-Robert [21]. Later Caffarelli and Lions [14] gave a very beautiful proof of $C^{1,1}$ regularity. For the general preference function, under some suitable condition we will show that the minimizer is C^1. The proof is based on a perturbation argument, which is usually more difficult in the case of general $b(x, y)$. Since for the bilinear case $b(x, y) = x \cdot y$, the space of admissible functions is a subset of convex functions, one can cut the graph of the minimizer by a hyperplane and replace the part below the hyperplane with the flat one. But for general $b(x, y)$ when one uses the natural choice of $b(\cdot, y) + \lambda$ (λ is a constant) instead of a hyperplane to perturb the minimizer, the shape of the domain where the function is perturbed is hard to control except for some special y.

Before giving the main result, we will list the definitions and conditions which will be used later. The following **(B0)**-**(B3)** conditions were introduced into the principal-agent problem by Figalli, Kim and McCann [36].

Let $\overline{X}$ be the closure of a set $\mathbf{X} \subset \mathbb{R}^n$. For each fixed $(x_0, y_0) \in \overline{X} \times \overline{Y}$ we assume:

(**B0**) $b \in C^4(\overline{\mathbf{X}} \times \overline{\mathbf{Y}})$, where $\mathbf{X} \subset \mathbb{R}^n$ and $\mathbf{Y} \subset \mathbb{R}^n$ are open and bounded;

(**B1**) (bi-twist) both $x \in \overline{\mathbf{X}} \mapsto D_y b(x, y_0)$ and $y \in \overline{Y} \mapsto D_x b(x_0, y)$ are diffeomorphisms onto their ranges;

(**B2**) (bi-convexity) both $\mathbf{X}_{y_0} := D_y b(\mathbf{X}, y_0)$ and $\mathbf{Y}_{x_0} := D_x b(x_0, \mathbf{Y})$ are convex subsets of $\mathbb{R}^n$.

(**B3**) (non-negative cross-curvature)

$$\frac{\partial^4}{\partial s^2 \partial t^2}\Big|_{(s,t)=(0,0)} b(x(s), y(t)) \geq 0 \tag{1.24}$$

whenever either of the two curves $s \in [-1,1] \mapsto D_y b(x(s), y(0))$ and $t \in [-1,1] \mapsto D_x b(x(0), y(t))$ forms an affinely parameterized line segment ($\in X_{y(0)}$, or $\in Y_{x_0}$, respectively).

Now in order to formulate the principal-agent problem as a minimization problem over some space of admissible functions, we need the definition of b-convexity.

Definition 1. A function $u : \mathbf{X} \mapsto \mathbb{R}$ is called b-convex if $u = (u^{b^*})^b$, where

$$v^b(x) = \sup_{y \in \overline{\mathbf{Y}}} \{b(x,y) - v(y)\}, \text{and } u^{b^*}(y) = \sup_{x \in \overline{\mathbf{X}}} \{b(x,y) - u(x)\}. \tag{1.25}$$

By the above definition and **(B0)**, it is easy to see that a b-convex function is semi-convex, which implies that it is differentiable almost everywhere. In the following we will use DomDu to denote the set where u is differentiable. Then by (**B1**), we can define the so called b-exponential map.

Definition 2. For each $q \in \overline{Y}_x$ we define $y_b(x,q)$ as the unique solution to

$$D_x b(x, y_b(x,q)) = q, \tag{1.26}$$

where the uniqueness is guaranteed by **(B1)**.

If the principal selects a price menu given by the function $v(y)$, each agent x will try to choose the product that maximize the quantity $b(x,y) - v(y)$ among all $y \in \overline{\mathbf{Y}}$. Then one can define a new function $u(x) = \sup_{y \in \overline{\mathbf{Y}}} \{b(x,y) - v(y)\}$, which is b-convex by definition 1. By **(B1)**, we have $u(x) = b(x, y_b(x, Du(x))) - v(y_b(x, Du(x)))$ for all $x \in \text{Dom}Du$. As discussed in section 4 of [36], the principal-agent problem is equivalent to the following minimization problem.

$$\min_{u \in \mathcal{U}_0} L(u), \tag{1.27}$$

where $\mathcal{U}_0 := \{u | u(x) \text{ is } b\text{-convex and } u \geq u_0(x) = b(x, y_0) - c(y_0)\}$, and

$$L(u) = \int_{\mathbf{X}} [c(y_b(x, Du(x)) - b(x, y_b(x, Du(x))) + u] d\mu. \tag{1.28}$$

Recall that $c(y_b(x, Du(x))$ is the cost for the principal to manufacture the car $y_b(x, Du(x))$ and $b(x, y_b(x, Du(x)) - u(x)$ is the price of the car $y_b(x, Du(x))$. Therefore, the quantity $L(u)$ exactly measures the total loss of the principal. Note that the point y_0 in the definition of $\mathcal{U}_0$ is the so called null product (or outside option), which the principal is compelled to offer to all agents at zero profit. So $u_0(x)$ is a quantity below which the agent x will reject the principal's offer.

In the following, we will assume the density of different types of agents is given by $d\mu = f(x)dx$, where $f \in C^0(\overline{\mathbf{X}}) \cap W^{1,\infty}(\overline{\mathbf{X}})$ is a positive function. Below is one of the main theorems in [36].

Theorem 7. *[36] If b satisfies **(B0)-(B3)**, and if $c(y)$ is strictly b^*-convex, i.e., if $DomDc^b = \overline{\mathbf{X}}$, then there exists a unique solution to the above minimization problem.*

To state our regularity result, we need the following condition on $c(y)$.

Condition 1. $c \in C^{1,1}(\overline{\mathbf{X}})$, and $c(y_b(x, q)) - b(x, y_b(x, q))$ is strongly convex on the variable $q \in Y_x$ for $x \in \overline{\mathbf{X}}$ uniformly, namely there exists a fixed $\delta > 0$, such that $c(y_b(x, q)) - b(x, y_b(x, q)) - \delta|q|^2$ is a convex function with respect to q.

Remark 1.3.1. *By Proposition 4.4 in [36], if b satisfies **(B0)-(B3)**, and if $c(y)$ is (strictly) b^*-convex, we have that $c(y_b(x, q)) - b(x, y_b(x, q))$ is a (strictly) convex function with respect to q.*

Theorem 8. *If b satisfies **(B0)-(B3)**, and if c satisfies Condition 1, then the unique solution to the above minimization problem is in $C^1(X)$.*

1.3.2 Regularity of the free boundary in the optimal partial transport problem for general cost functions

The optimal partial transport problem is a natural extension of the classical optimal transport problem. One has the source density $f\chi_\Omega$ and the target density $g\chi_\Omega$, where f and g are two nonnegative functions. Then given a mass m satisfying $0 < m \leq \min\{\|f\|_{L_1}, \|g\|_{L_1}\}$, one wants to find an optimal transference plan between f and g with mass m. By a transference plan we mean a nonnegative, finite Borel measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ with its first and second marginal controlled by f and g respectively, namely for any Borel set $A \subset \mathbb{R}^n$ we have:

$$\gamma(A \times \mathbb{R}^n) \leq \int_A f(x)dx,\ \gamma(\mathbb{R}^n \times A) \leq \int_A g(y)dy.$$

An optimal transference plan is a minimizer of the following functional

$$\gamma \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)d\gamma,$$

where c is a nonnegative cost function.

The existence and uniqueness of the optimal transference plan have been addressed by Caffarelli and McCann under some suitable condition on the cost. Later Figalli extended it to more general situation with a different method. The regularity of the free boundary in the optimal partial transport is also highly interesting. For quadratic cost with the supports of the densities convex and separated by a hyperplane, Caffarelli and McCann proved that the free boundary is $C^{1,\alpha}$ away from some bad points. Figalli extends the C^1 regularity to the situation that allows the densities to have overlap, and Indrei improved

Figalli's result to $C^{1,\alpha}$. Moreover, Indrei investigated the size of the bad points of the free boundary, he proved some estimates for the Hausdorff measure of singular points. In the second section of the last chapter of this thesis we will prove the following result for the regularity of free boundary in optimal partial transport for costs satisfying **(B0)-(B2)** and the precursor **(A3)** of Ma, Trudinger and Wang [57] which inspired **(B3)**.

(A3) (Ma-Trudinger-Wang condition)

$$\frac{\partial^4}{\partial s^2 \partial t^2}\Big|_{(s,t)=(0,0)} b(x(s), y(t)) > 0 \tag{1.29}$$

whenever either of the two curves $s \in [-1,1] \mapsto D_y b(x(s), y(0))$ and $t \in [-1,1] \mapsto D_x b(x(0), y(t))$ forms an affinely parameterized line segment ($\in X_{y(0)}$, or $\in Y_{x_0}$, respectively) and

$$\frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} b(x(s), y(t)) = 0.$$

Below we establish the following result obtained in joint work with Indrei.

Theorem 9. *Let $f = f\chi_\Omega \in L^p(\mathbb{R}^n)$ be a nonnegative function with $p \in (\frac{n+1}{2}, \infty]$, and $g = g\chi_\Lambda$ a positive function bounded away from zero, Moreover, assume that Ω and Λ are bounded, Λ is relatively c-convex with respect to a neighborhood of $\Omega \cup \Lambda$, and $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$. Let $c \in \mathcal{F}_0$ and $m \in (0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$. Then the free boundary $\partial U_m \cap \Omega$ is locally a $C^{1,\alpha}$ graph with $\alpha = \frac{2p-n-1}{2p(2n-1)-n+1}$.*

Chapter 2

Convex solutions to the power-of-mean curvature flow

In this chapter we study the convex solutions to the power-of-mean curvature flow. It is divided into four sections. The first section is devoted to the proof of power growth estimate of the solutions. The second section contains the proof of Theorem 1 and the first part of Theorem 2. The third section establishes Corollary 1 and the last section completes the proof of Theorem 2.

2.1 Power growth estimate

In this section, we prove a key estimate, which says that any entire convex solution u to the equation (1.3) must satisfy

$$u(x) \leq C(1 + |x|^{1+\alpha}),$$

where the constant C depends only on the upper bound of $u(0)$ and $|Du(0)|$. When $\alpha = 1$, the estimate was proved by Wang [69]. To apply Wang's method, the main difficulty is that now the speed function is nonlinear in the curvature, we overcome this difficulty by further exploiting some elementary convexity properties.

For any constant $h > 0$, we denote

$$\begin{aligned} \Gamma_h &= \{x \in \mathbb{R}^n : u(x) = h\}, \\ \Omega_h &= \{x \in \mathbb{R}^n : u(x) < h\}, \end{aligned}$$

so that Γ_h is the boundary of Ω_h. Let κ be the curvature of the level curve Γ_h. We have

$$L_\sigma(u) = (\sigma + u_\gamma^2)^{\frac{1}{2\alpha} - \frac{1}{2}}(\kappa u_\gamma + \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}) \tag{2.1}$$

$$\geq \kappa u_\gamma^{\frac{1}{\alpha}} = L_0(u), \tag{2.2}$$

where γ is the unit outward normal to Ω_h, and $u_{\gamma\gamma} = \gamma_i \gamma_j u_{ij}$.

Lemma 1. *Let u be a complete convex solution of* (1.3). *Suppose $u(0) = 0$ and the infimum $\inf\{|x| : x \in \Gamma_1\}$ is attained at $x_0 = (0, -\delta) \in \Gamma_1$, for some $\delta > 0$ sufficiently small. Let D_1 be the projection of Γ_1 on the axis $\{x_2 = 0\}$. Then D_1 contains the interval $(-R, R)$, and when $\alpha \leq 1$, R satisfies*

$$R \geq C_1(-\log \delta - C_2)^{\frac{\alpha}{\alpha+1}}, \tag{2.3}$$

where $C_1, C_2 > 0$ are independent of δ; when $\alpha > 1$, $R \geq C$ for some positive constant C.

Note that the above lemma seems too weak when $\alpha > 1$, in Remark 2.1.1, we will show how to strengthen it for the purpose of the proof of Corollary 3.

Proof. First, we prove the lemma when $\frac{1}{2} < \alpha \leq 1$. Suppose near x_0, Γ_1 is given by $x_2 = g(x_1)$. Then g is a convex function, $g(0) = -\delta$, and $g'(0) = 0$. Let $b > 0$ be a constant such that $g'(b) = 1$. To prove (2.3) it suffices to prove

$$b \geq C_1(-\log\delta - C_2)^{\frac{\alpha}{\alpha+1}}. \tag{2.4}$$

For any $y = (y_1, y_2) \in \Gamma_1$, where $y_1 \in [0, b]$, let $\xi = \frac{y}{|y|}$, by convexity of u we have

$$u_\xi(y) \geq \frac{u(y) - u(0)}{|y|} = \frac{1}{|y|}.$$

Let θ denote the angle between ξ and the tangential vector $\frac{1}{\sqrt{1+g'^2}}(1, g')$ of Γ_1 at y. Then

$$\cos\theta = \frac{\xi_1 + \xi_2 g'(y_1)}{\sqrt{1+g'^2}},$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\xi_1 g' - \xi_2}{\sqrt{1+g'^2}}.$$

Hence

$$u_\gamma(y) \geq \frac{\sqrt{1+g'^2}}{y_1 g' - y_2}, \tag{2.5}$$

where γ is the unit normal of the sub-level set Ω_1 . Since $L_0 u \leq 1$, we obtain,

$$\frac{g''}{(1+g'^2)^{\frac{3}{2}}} \frac{(1+g'^2)^{\frac{1}{2\alpha}}}{(y_1 g' - y^2)^{\frac{1}{\alpha}}} \leq \kappa u_\gamma^{\frac{1}{\alpha}} \leq 1, \tag{2.6}$$

where κ is the curvature of the level curve Γ_1. Hence

$$g''(y_1) \leq (1+g'^2)^{\frac{3}{2}-\frac{1}{2\alpha}}(y_1 g' - y_2)^{\frac{1}{\alpha}} \tag{2.7}$$

$$\leq 10 y_1^{\frac{1}{\alpha}} g' + 10\delta \tag{2.8}$$

where $y_2 = g(y_1)$ and $g'(y_1) \leq 1$ for $y_1 \in (0, b)$. The inequality from (2.7) to (2.8) is trivial when $y_2 \geq 0$. When $y_2 \leq 0$, since $|y_2| \leq \delta$, we have either $y_1 g' \leq \delta$ or $y_1 g' > \delta$, for the former $(y_1 g' - y_2)^{\frac{1}{\alpha}} \leq (2\delta)^{\frac{1}{\alpha}} \leq 4\delta$, for the latter $(y_1 g' - y_2)^{\frac{1}{\alpha}} \leq (2y_1 g')^{\frac{1}{\alpha}} \leq 4y_1^{\frac{1}{\alpha}} g'$, since $g'(y_1) \leq 1$. We consider the equation

$$\rho''(t) = 10t^{\frac{1}{\alpha}}\rho' + 10\delta \tag{2.9}$$

with initial conditions $\rho(0) = -\delta$ and $\rho'(0) = 0$. Then for $t \in (0, b)$ we have

$$\rho'(t) = 10\delta e^{\frac{10\alpha}{\alpha+1}t^{\frac{\alpha+1}{\alpha}}} \int_0^t e^{-\frac{10\alpha}{\alpha+1}s^{\frac{\alpha+1}{\alpha}}} ds. \tag{2.10}$$

Since $\int_0^\infty e^{-\frac{10\alpha}{\alpha+1}s^{\frac{\alpha+1}{\alpha}}} ds$ is bounded above by some constant C, we have

$$\begin{aligned} 1 = \rho'(b) &= 10\delta e^{\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}} \int_0^b e^{-\frac{10\alpha}{\alpha+1}s^{\frac{\alpha+1}{\alpha}}} ds. && (2.11) \\ &\leq C_1 \delta e^{\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}}, && (2.12) \end{aligned}$$

from where (2.4) follows.

When $\alpha > 1$, the situation is different. First, We introduce a number a such that $g'(a) = \frac{1}{2}$. Then, we can follow the above proof until (2.7). For (2.8) the inequality becomes

$$g''(y_1) \leq 10y_1^{\frac{1}{\alpha}} g' + 10\delta^{\frac{1}{\alpha}},$$

for $y_1 \in [a, b]$. Now (2.12) becomes

$$e^{-\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}}\rho'(b) - e^{-\frac{10\alpha}{\alpha+1}a^{\frac{\alpha+1}{\alpha}}}\rho'(a) \leq C_1\delta,$$

after rearranging the terms we have

$$1 \leq C_1\delta e^{\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}} + \frac{1}{2}e^{\frac{10\alpha}{\alpha+1}(b^{\frac{\alpha+1}{\alpha}} - a^{\frac{\alpha+1}{\alpha}})},$$

then it is easy to see that when δ is small, $b \geq C$, for some fixed constant C.

Remark 2.1.1. *When $\alpha \leq 1$, It follows from Lemma 1 that when δ is sufficiently small, by convexity and in view of Figure 1, we see that Ω_1 contains the shadowed region. Then it is easy to check that Ω_1 contains an ellipse*

$$E = \{(x_1, x_2) | \frac{x_1^2}{(\frac{R}{6})^2} + \frac{(x_2 - \frac{7\delta^* - 5\delta}{12})^2}{(\frac{\delta^* + \delta}{4})^2} = 1\}, \tag{2.13}$$

where δ^* is a positive constant such that $u(0, \delta^*) = 1$ and R is defined in the Lemma 1.

When $\alpha > 1$, if δ^* is very large, in the part $\{x | u(x) \leq 1, x_1 \geq 0\}$, by convexity we can find an ellipse which has short axis bounded from below and long axis very large, and if we let the ellipse evolve under the generalized curve shortening flow, it will take time more than 1 to converge to a round point. When δ^* is less than some fixed constant, we need to consider two cases. Case 1, when the set $\{u \leq 1\}$ is not compact. In this case when we project $\{u(x) = 1\}$ to the axis $\{x_2 = 0\}$, and denote the leftmost(rightmost) point as $(-l, 0)$($(r, 0)$), then either l or r is very large, which guarantees that one can still find an ellipse inside $\{x | u(x) \leq 1, x_1 \leq 0\}$ (or $\{x | u(x) \leq 1, x_1 \geq 0\}$) with the similar property as before. Case 2, when $\{u \leq 1\}$ is compact. For this case, we will always assume 0 is the minimum point of u, and $u(0) = 0$. We claim that when δ is very small, for the purpose of the proof of Corollary 3, we can assume one of l and r is very large. Indeed, if the claim is not true, we have a sequence of functions u_i satisfying that $\{u_i \leq 1\}$ has width bounded by some constant independent of i, and the distance dist$(0, \{u_i \leq 1\}) \to 0$, as $i \to \infty$. And in view of the following proof of corollary 3, we can assume u_i satisfies equation (1.3) with $\sigma_i \to 0$ (see the first paragraph in the proof of Corollary 3). Then by passing to a subsequence, we can assume $\{u_i \leq 1\}$ converges to a convex curve C_0 in hausdorff distance, and let C_0 evolves under the generalized curve shortening flow, it will converge to a point on itself, which is clearly impossible. Once l or r is very large, we can find the ellipse as in the case 1.

Remark 2.1.2. *One can also establish similar lemma in higher dimensions, which says*

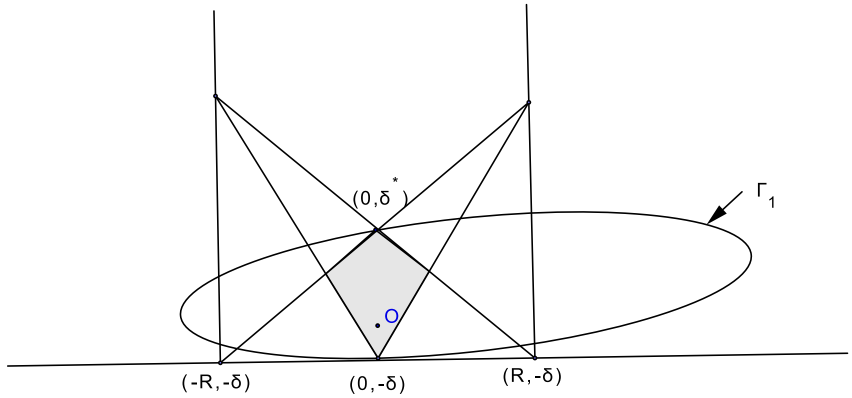

Figure 2.1: Γ_1 contains the shadow part.

that D_1 (convex set with dimension greater than 1) contains a ball centered at the origin with radius $R \geq C_n(-\log\delta - C)^{\frac{\alpha}{\alpha+1}}$, where C_n is a constant depending only on n and C is a positive constant independent of δ. The proof can be reduced to the two dimensional case. For the details of how to reduce the situation to lower dimensional case we refer the reader to the proof of Lemma 2.6 in [69].

Lemma 2. *Let u be a complete convex solution of* (1.3)*. Suppose $u(0) = 0$, δ and δ^* are defined as in Lemma 1 and Remark 2.1.1. Then if δ and δ^* are sufficiently small, u is defined in a strip region.*

The proof of Lemma 2 is based on a careful study of the shape of the level curve of u, we will give an important corollary first.

Corollary 3. *Let u be an entire convex solution of* (1.3) *in $\mathbb{R}^2$, then*

$$u(x) \leq C(1 + |x|^{1+\alpha}), \tag{2.14}$$

where the constant C depends only on the upper bound for $u(0)$ and $|Du(0)|$.

Proof. By adding a constant to u we may suppose $u(0) = 0$. It suffices to prove that $\text{dist}(0, \Gamma_h) \geq Ch^{\frac{1}{1+\alpha}}$ for all large h. By the rescaling $u_h(x) = \frac{1}{h}u(h^{\frac{1}{1+\alpha}}x)$ we need only to prove $\text{dist}(0, \Gamma_{1,u_h}) \geq C$. Note that $|Du_h(0)| = \frac{1}{h^{\frac{\alpha}{1+\alpha}}}|Du(0)| \to 0$, as $h \to \infty$. Hence by convexity $\inf_{B_r(0)} u_h$ goes to 0 uniformly for fixed radius r. Note also that u_h satisfies equation (1.3) with $\sigma \to 0$ as $h \to \infty$.

If the estimate

$$\text{dist}(0, \Gamma_{1,u_h}) \geq C, \text{ for all large } h$$

fails, we can find a sequence $h_k \to \infty$ such that $\delta_k = \inf\{|x| : x \in \Gamma_{1,u_{h_k}}\} \to 0$. Now, we take δ_k^* as in Remark 2.1.1 with respect to u_{h_k}. δ_k^* has a positive lower bound δ^*, otherwise by Lemma 2 u_{h_k} can not be an entire solution for large k.

If $\delta_k^* \leq 1000$ for all large k, since the ellipse E_k defined for u_{h_k} as in Remark 2.1.1 is contained in $\Omega_{1,u_{h_k}}$ and the distance between the center O_k of E_k and the origin is bounded above by 1000, by the previous discussion we know $u_{h_k}(O_k)$ is bounded bellow by -1 when k is large. Let $E_k(t)$ be the solution to the generalized curve shortening flow starting from time $t = -1$, with initial condition $E_k(-1) = E_k$. (1) When $\sigma = 0$, $\partial\Omega_{-t,u_{h_k}}$ evolves under the generalized curve shortening flow, we have the inclusion $E_k(t) \subset \partial\Omega_{-t,u_{h_k}}$ for all $t > -1$. Hence $\inf_{B_{1000}(0)} u_{h_k}$ is smaller than 1 minus the time needed for E_k to shrink to O_k. However, by the size of E_k, the time needed for it to shrink to a point goes to infinity as k goes to infinity, which is contradictory to the discussion at the beginning of the proof that u_{h_k} converges to 0 uniformly in the ball $B_{1000}(0)$ as h_k goes to infinity. (2)When $\sigma \in (0, 1]$, we can take v_k as the solution of $L_{\sigma_k} v = 1$ in E_k with $v = 1$ on ∂E_k, where $\sigma_k = h_k^{-\frac{2\alpha}{1+\alpha}}$. Passing to a subsequence and adjusting the size of E_k if necessary, we can assume E_k converge to some ellipse E with the length of its long axis very large, the length of its short axis bigger than some fixed positive number and the distance from its center to the origin is less than 1000. Then v_k converges to a solution of the generalized curve shortening flow, and a contradiction can be made as for the case $\sigma = 0$

Otherwise, by the definition of b in the proof of Lemma 1 and the convexity of Ω_{1,u_h} we can find a disc B_k with center $O = (0, 50)$ and radius 20 inside $\Omega_{1,u_{h_k}}$, obviously it will take time more than 2 for B_k to shrink to O. We can take $B_k(t)$ as a solution to the generalized curve shortening flow starting from time $t = -1$ with $B_k(-1) = B_k$, then a similar contradiction will be made as before.

Remark 2.1.3. *The estimate in Corollary 3 is also true for higher dimensions, one can prove it by reducing the problem to the two dimensional case similar to the corresponding part in [69].*

Proof of Lemma 2. By a rotation of coordinates we assume the axial directions of E in Remark 2.1.1 coincide with those of the coordinate system. Let $\mathcal{M}_u$ be the graph of u, which consists of two parts, $\mathcal{M}_u = \mathcal{M}^+ \cup \mathcal{M}^-$, where $\mathcal{M}^+ = \{(x, u(x)) \in \mathbb{R}^3 : \partial_{x_2} u \geq 0\}$ and $\mathcal{M}^- = \{(x, u(x)) \in \mathbb{R}^3 : \partial_{x_2} u \leq 0\}$. Then $\mathcal{M}^\pm$ can be represented as graphs of functions $g^\pm$ in the form $x_2 = g^\pm(x_1, x_3), (x_1, x_2) \in D$ and D is the projection of $\mathcal{M}_u$ on the plane $\{x_2 = 0\}$. The functions g^+ and g^- are respectively concave and convex, and we have $x_3 = u(x_1, g^\pm(x_1, x_3))$. Denote

$$g = g^+ - g^-. \tag{2.15}$$

Then g is a positive, concave function in D, vanishing on ∂D. For any $h > 0$ we also denote $g_h(x_1) = g(x_1, h)$, $g_h^\pm(x_1) = g(x_1, h)$, and $D_h = \{x_1 \in \mathbb{R}^1 : (x_1, h) \in D\}$. Then g_h is a positive, concave function in D_h, vanishing on ∂D_h, and $D_h = (-\underline{a}_h, \overline{a}_h)$ is an interval containing the origin. We denote $b_h = g_h(0)$. We will consider the case $\sigma = 0$ first.

Claim 1: suppose h large, $g_1(0) = \delta^* + \delta$ small, $b_h \leq 4$ and $\underline{a}_h, \overline{a}_h \geq b_h$. Then $\overline{a}_h \geq \frac{1}{1000} \frac{h}{b_h^\alpha}$ for $\alpha \leq 1$ and $\overline{a}_h \geq \frac{1}{1000} \frac{h^{\frac{1}{2\alpha-1}}}{b_h^{\frac{1}{2\alpha-1}}}$ for $\alpha > 1$.

Proof. Without loss of generality, we assume $\overline{a}_h \leq \underline{a}_h$. Denote $U_h = \Omega_h \cap \{x_1 > 0\}$. By the convexity of U_h and the assumption $\underline{a}_h, \overline{a}_h \geq b_h$, we have $\underline{a}_s, \overline{a}_s \geq \frac{1}{2} b_h$ for all $s \in (\frac{1}{2}h, h)$. Hence by the concavity of g we have $|\frac{d}{dx_1} g_s(0)| \leq 2$ for $s \in (\frac{1}{2}h, h)$, which means the arc-length of the image of $\Gamma_s \cap \{x_1 > 0\}$ under Gauss map is bigger then $\frac{\pi}{6}$. Notice that Ω_1 contains E, which was defined in Remark 2.1.1. When δ and δ^* are very

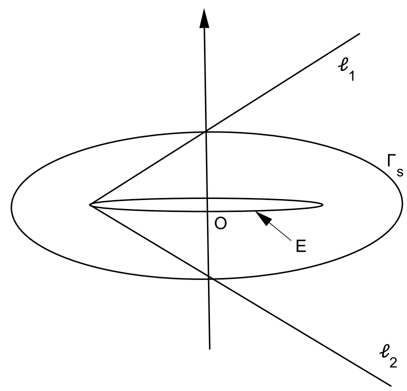

Figure 2.2: $\Gamma_s \cap \{x_1 > 0\}$ is trapped between two lines

small, E is very thin and long. The centre of E is very close to the origin, in fact for our purpose we can just pretend E is centered at the origin. By convexity of Ω_h and in view of Figure 2, we see that $\Gamma_s \cap \{x_1 > 0\}$ is trapped between two lines ℓ_1 and ℓ_2, and the slopes of ℓ_1 and ℓ_2 are very close to 0 when E is very long and thin. Then it is clear that the largest distance from the points on $\Gamma_s \cap \{x_1 > 0\}$ to the origin can not be bigger than $10\overline{a}_h$. By convexity of u, we have $u_\gamma(x) \geq \frac{h}{20\overline{a}_h}$, for $x \in \Gamma_s \cap \{x_1 > 0\}$. Since $\Gamma_s \cap \{x_1 > 0\}$ evolves under the generalized curve shortening flow, when $\alpha \leq 1$ we have the following estimate

$$\frac{d}{ds}(|U_s|) = \int_{\Gamma_s \cap \{x_1 > 0\}} \kappa^\alpha d\xi \tag{2.16}$$

$$= \int_{\Gamma_s \cap \{x_1 > 0\}} u_\gamma^{\frac{1}{\alpha}-1} \kappa d\xi \tag{2.17}$$

$$\geq \frac{1}{50}(\frac{h}{\overline{a}_h})^{\frac{1}{\alpha}-1}\frac{\pi}{6}, \tag{2.18}$$

from (2.16) to (2.17) we used the equation $\kappa u_\gamma^{\frac{1}{\alpha}} = 1$. The claim follows by the simple fact $\frac{3}{2}b_h\overline{a}_h \geq |U_h| \geq \frac{1}{50}(\frac{h}{\overline{a}_h})^{\frac{1}{\alpha}-1}\frac{\pi}{6}\frac{h}{2}$.

When $\alpha > 1$, denote l_s as the arc length of $\Gamma_s \cap \{x_1 > 0\}$, by the above discussion, it is not hard to see that $l_s \approx C\overline{a}_h$. Then by a simple application of Jensen's inequality, we have

$$\begin{aligned} \frac{d}{ds}(|U_s|) &= \int_{\Gamma_s \cap \{x_1>0\}} \kappa^\alpha d\xi \\ &= l_s \int_{\Gamma_s \cap \{x_1>0\}} \kappa^\alpha \frac{1}{l_s} d\xi \\ &\geq l_s (\int_{\Gamma_s \cap \{x_1>0\}} \frac{\kappa}{l_s} d\xi)^\alpha \geq C l_s^{1-\alpha} \geq C\overline{a}_h^{1-\alpha}, \end{aligned}$$

then again by the simple fact that $\frac{3}{2} b_h \overline{a}_h \geq |U_h|$ we can finish the proof in the same way as the previous case.

From here until (2.51) we will prove the case $\frac{1}{2} < \alpha \leq 1$, and then we will give the detail for the case $\alpha > 1$.

Claim 2: Denote $h_k = 2^k$, $\overline{a}_k = \overline{a}_{h_k}$, $b_k = b_{h_k}$, $g_k = g_{h_k}$ and $D_k = D_{h_k}$. Then

$$g_k(0) \leq g_{k-1}(0) + C_0 2^{\frac{-k}{C}} \text{ for all } k \text{ large,} \tag{2.19}$$

where C_0 is a fixed constant, and C depends only on α.

Lemma 2 follows from Claim 1 and Claim 2 in the following way. Let the convex set P be the projection of the graph of g on the plane $\{x_3 = 0\}$, by Claim 2 and the fact that P contains x_1-axis (it follows from Claim 1), P must equal to $I \times \mathbb{R}$ for some interval $I \subset [0, \lim_{k\to\infty} g_k(0)]$. Then, by (2.15) $\mathcal{M}_u$ is also contained in a strip region as stated in Lemma 2.

To prove (2.19) , since g is positive and concave, $g_k(0) \leq h_k g_0(0) \leq 2^k(\delta + \delta^*)$. Hence, we can start from sufficiently large k_0, which satisfies $g_{k_0}(0) \leq 1$ and

$$g_{k_0} + C_0 \sum_{j=k_0}^{\infty} 2^{\frac{-j}{C}} \leq 2. \tag{2.20}$$

Suppose (2.19) holds up to k. Then by (2.20) we have $g_k(0) \leq 2$. By the concavity of g and $g \geq 0$, we have $g_{k+1}(0) \leq 2g_k(0) \leq 4$. By claim 1 we have $\overline{a}_{k+1} \geq \frac{1}{10000} h_k$. To prove (2.19) at $k+1$, we denote

$$L_k = \{x_1 \in \mathbb{R}^1 : -\frac{C_1}{4} h_k < x_1 < \frac{C_1}{4} h_k\}, C_1 = \frac{1}{10000}, \tag{2.21}$$

$$Q_k = L_k \times [h_k, h_{k+1}] \subset D. \tag{2.22}$$

Since $g > 0$, g is concave, we have the following estimates

$$g(x_1, h) \leq 8, \tag{2.23}$$

$$|\partial_h g(x_1, h)| \leq \frac{16}{h_k}, \tag{2.24}$$

$$|\partial_{x_1} g(x_1, h)| \leq \frac{16}{h_k}, \text{ for all } (x_1, h) \in Q_k \tag{2.25}$$

We denote $\mathcal{X}^{\pm} = \{(x_1, h) \in Q_k : |\partial_{x_1x_1} g^{\pm}(x_1, h)| \geq h_k^{-\beta}\}$, here β is chosen such that $\frac{1}{\alpha} < \beta < 2$. For any $h \in (h_k, h_{k+1})$, by (2.25) , we have

$$|\{x_1 \in L_k : (x_1, h) \in \mathcal{X}^+| h_k^{-\beta} \leq \int_{L_k} \partial_{x_1x_1} g^+ \tag{2.26}$$

$$\leq \int_{L_k} \partial_{x_1x_1} g \tag{2.27}$$

$$\leq 2 \sup_{L_k} |\partial_{x_1} g| \tag{2.28}$$

$$\leq \frac{C}{h_k}. \tag{2.29}$$

So $|\mathcal{X}^+| \leq C h_k^{\beta}$. Similarly we have$|\mathcal{X}^-| \leq C h_k^{\beta}$.

For any given $y_1 \in L_k$, denote $\mathcal{X}^{\pm}{}_{y_1} = \mathcal{X}^+ \cap \{x_1 = y_1\}$. Then by the above estimate there is a set $\widetilde{L}^{\pm} \subset L_k$ with measure $|\widetilde{L}^{\pm}| \leq C h_k^{\frac{\beta}{2}}$ such that for any $y_1 \in L_k - \widetilde{L}^{\pm}$, we have $|\mathcal{X}^{\pm}{}_{y_1}| \leq h_k^{\frac{\beta}{2}}$. When k is large, we can always find $y_1 = C h^{\frac{\beta}{2}} \in L_k - \widetilde{L}^{\pm}$, where the constant C is under control. For such y_1, we have

$$g(y_1, h_{k+1}) - g(y_1, h_k) = g^+(y_1, h_{k+1}) - g+(y_1, h_k) + |g^-(y_1, h_{k+1}) - g^-(y_1, h_k)|. \tag{2.30}$$

We will estimate $g^+(y_1, h_{k+1}) - g^+(y_1, h_k)$, another part can be estimated similarly.

Note that the distance from $y_1 = Ch^{\frac{\beta}{2}}$ to the origin is relatively very small comparing to the length of L_k when k is large, recall the fact that Ω_1 contains an ellipse E with center close to the origin and by taking δ, δ^* small enough we can make E as thin and as long as we need (note that later in the proof of case $\alpha > 1$, we can only find such ellipse with the size of short axis very small and with the size of the long axis bounded from below by some constant, but this is enough for the argument). By these facts, we can make the unit normal of Γ_s at the point $(y_1, g_s(y_1))$ very close to the x_2-axis direction, and in fact we can make them as close as we want by taking δ, δ^* enough small. By differentiating equation $u(x_1, g^+(x_1, h)) = h$ with respect to x_1 and h, and using the fact that $\partial_{x_1} g^+$ is small, we have

$$\begin{cases} (\partial_h g^+)^{-1} = (1+\varepsilon_1) u_\gamma \\ \partial_{x_1 x_1} g^+ = (1+\varepsilon_2)\kappa, \end{cases} \tag{2.31}$$

then by the equation $u_\gamma^{\frac{1}{\alpha}} \kappa = 1$ we have

$$\partial_h g_h^+(y_1, h) \le C(\partial_{x_1 x_1} g^+)^\alpha \le C h_k^{-\beta\alpha}. \tag{2.32}$$

Now

$$\begin{aligned} g^+(y_1, h_{k+1}) - g^+(y_1, h_k) &= \int_{h_k}^{h_{k+1}} \partial_h g^+(y_1, h) dh && (2.33) \\ &= \int_{\mathcal{X}_{y_1}^+} \partial_h g^+(y_1, h) dh + \int_{[h_k, h_{k+1}] - \mathcal{X}_{y_1}^+} dh && (2.34) \\ &\le C_1 h_k^{\frac{\beta}{2}} \frac{1}{h_k} + C_2 h_k^{-\beta\alpha} h_k. && (2.35) \end{aligned}$$

Recall that β satisfies $\frac{1}{\alpha} < \beta < 2$, we have $\eta := \min\{1 - \frac{\beta}{2}, \beta\alpha - 1\} > 0$. From (2.30) and

(2.35) , we have the estimate

$$g(y_1, h_{k+1}) - g(y_1, h_k) \le \frac{C}{h_k^{\eta}},$$

for some fixed constant C. Then, we will assume $\partial_{x_1} g(0, h_k) < 0$ (otherwise we can replace x_1 by $-x_1$), therefore by the above estimate we have

$$g(y_1, h_{k+1}) \le g(y_1, h_k) + \frac{C}{h_k^{\eta}} \le g(0, h_k) + \frac{C}{h_k^{\eta}}.$$

Since g is positive, concave and defined on the interval $[0, \bar{a}_{k+1}]$ with $\bar{a}_{k+1} \ge Ch_{k+1}$, we have

$$\frac{g_{k+1}(0)}{g_{k+1}(y_1)} \le \frac{\bar{a}_{k+1}}{\bar{a}_{k+1} - y_1} \le 1 + Ch_{k+1}^{\frac{\beta}{2}-1}.$$

Therefore, by the above two estimates we have

$$g_{k+1}(0) \le g_k(0) + Ch_k^{-\eta},$$

which implies (2.19) immediately.

For the proof of Lemma 2 when $\sigma \in (0, 1]$, we need to use (2.1) and (2.2). In fact, by (2.2) we see that Γ_h is moving at a velocity greater than or equal to its curvature to the power α. Hence, we still have the lower bound of $\frac{d}{ds}(|U_s|)$ as in the proof of Claim 1. Then we can follow the above proof for the case $\sigma = 0$ until (2.33) with the only change that replacing the equalities " $=$ " in (2.16) and (2.17) with inequalities " $\ge$ ". As in [69], when $\sigma = 0$, in order to control the second integral in (2.34) we used the equation $\kappa u_{\gamma}^{\frac{1}{\alpha}} = 1$. But when $\sigma \ne 0$, by (2.24) and (2.31) we have

$$u_{\gamma} \ge C(\partial_h g^+)^{-1} \ge Ch_k, \tag{2.36}$$

hence we can assume u_{γ} as large as we want, which means in the formula (2.1) the only important extra term is $(\sigma+u_\gamma^2)^{\frac{1}{2\alpha}-\frac{1}{2}}\frac{\sigma u_{\gamma\gamma}}{\sigma+u_\gamma^2}$. To handle this term we divide the integral (2.35) into three parts,

$$\begin{aligned} g^+(y_1,h_{k+1})-g+(y_1,h_k) &= \int_{h_k}^{h_{k+1}}\partial_h g^+(y_1,h)dh & (2.37)\\ &= (\int_{I_1}+\int_{I_2}+\int_{I_3})\partial_h g^+(y_1,h)dh, & (2.38)\end{aligned}$$

where

$$\begin{aligned} I_1 &= \mathcal{X}_{y_1}^+, & (2.39)\\ I_2 &= \{h\in[h_k,h_{k+1}]-I_1:(\sigma+u_\gamma^2)^{\frac{1}{2\alpha}-\frac{1}{2}}\frac{\sigma u_{\gamma\gamma}}{\sigma+u_\gamma^2}\le\frac{1}{2}\}, & (2.40)\\ I_3 &= [h_k,h_{k+1}]-I_1\cup I_2. & (2.41)\end{aligned}$$

For the first integral, we can do exactly the same thing as we have done from (2.34) to (2.35), namely $\int_{I_1}\partial_h g^+(y_1,h)dh\le\frac{C}{h_k}h_k^{\frac{\beta}{2}}=Ch_k^{\frac{\beta}{2}-1}$, note that the power $\frac{\beta}{2}-1$ is a negative number.

Then we estimate the second integral, note that when $(y_1,h)\in I_2$, $(\sigma+u_\gamma^2)^{\frac{1}{2\alpha}}\kappa u_\gamma\ge\frac{1}{2}$. By (2.36) u_γ is large, so we have $\kappa u_\gamma^{\frac{1}{\alpha}}\ge\frac{1}{4}$, then by (2.31) we have

$$\partial_h g^+\le C(\partial_{x_1x_1}g^+)^\alpha\le Ch_k^{-\alpha\beta}. \tag{2.42}$$

Hence $\int_{I_2}\partial_h g^+(y_1,h)dh\le Ch_k^{-\beta\alpha}h_k=Ch_k^{1-\beta\alpha}$, note that $1-\beta\alpha$ is a negative number. Observe that we can assume I_2 is on the right hand side of I_3, since by the concavity of g^+ we know that when $h\ge\inf I_2$, $\partial_h g^+(y_1,h)$ will satisfy the estimate (2.42).

For the third integral, as in [69] we need the following observation

$$\begin{cases} u_\gamma(y_1,h)=u_{x_2}(1+\varepsilon_1)\\ u_{\gamma\gamma}(y_1,h)=u_{x_2x_2}(1+\varepsilon_2)+\varepsilon_3u_{x_2}, \end{cases} \tag{2.43}$$

which can be proved by differentiating the equation $u(x_1, g^+(x_1, h)) = h$ twice with respect to x_1, and combining the facts discussed before (2.31). Note that by taking δ and δ^* sufficiently small, when k is large, we can make ε_i very small, for $i = 1, 2, 3$. Hence, by (2.43) we have

$$(\sigma + u_{x_2}^2)^{\frac{1}{2\alpha}-\frac{1}{2}} \frac{\sigma u_{x_2x_2}}{\sigma + u_\gamma^2} \geq \frac{1}{3}. \tag{2.44}$$

Since $\sigma \in [0, 1]$ and u_γ is large, we have

$$u'' = u_{x_2x_2} \geq \frac{1}{4}(u')^{3-\frac{1}{\alpha}}. \tag{2.45}$$

By differentiating the equation $u(x_1, g^+(x_1, h)) = h$ twice with respect to h, we have

$$(g^+)'' = -u''(g^+)'^3 \leq -\frac{1}{4}(g^+)'^{\frac{1}{\alpha}-3}(g^+)'^3 = -\frac{1}{4}(g^+)'^{\frac{1}{\alpha}}, \tag{2.46}$$

note (2.46) is for points with corresponding $h \in I_3$. By the discussion after (2.42) we need only to estimate $\int_{[h_k + h_k^{\frac{\beta+2}{4}}, \inf I_2]} (g^+)' dh$. Therefore by (2.46) and noticing that $(g^+)' \geq 0$ we have

$$\frac{\alpha}{\alpha - 1}(g^+)'^{\frac{\alpha-1}{\alpha}}(h) \leq \frac{\alpha}{\alpha - 1}(g^+)'^{\frac{\alpha-1}{\alpha}}(h_k) - \frac{1}{4}|I_3 \cap [h_k, h]|, \tag{2.47}$$

so when $h \in [h_k^{\frac{\beta+2}{4}}, \inf I_2]$, we have

$$(g^+)'(h) \leq ((g^+)'^{\frac{\alpha-1}{\alpha}}(h_k) + C(h - h_k))^{\frac{\alpha}{\alpha-1}}. \tag{2.48}$$

Finally we have

$$\int_{[h_k+h_k^{\frac{\beta+2}{4}},\inf I_2]}(g^+)'dh \leq \int_{h_k}^{h_{k+1}}((g^+)'^{\frac{\alpha-1}{\alpha}}(h_k)+C(h-h_k))^{\frac{\alpha}{\alpha-1}}dh \tag{2.49}$$

$$\leq \frac{\alpha-1}{2\alpha-1}((g^+)'^{\frac{\alpha-1}{\alpha}}(h_k)+C(h-h_k))^{\frac{\alpha}{\alpha-1}+1}|_{h_k}^{2h_k} \tag{2.50}$$

$$\leq C((g^+)'^{\frac{2\alpha-1}{\alpha}} \leq Ch_k^{\frac{1-2\alpha}{\alpha}}, \tag{2.51}$$

note that $\frac{1-2\alpha}{\alpha}<0$ when $\alpha>\frac{1}{2}$. Then we can complete our proof as the case $\sigma=0$.

When $\alpha>1$, we need to choose the constants and exponents more carefully. First of all, in view of the Lemma 2 for $\alpha>1$, in order to have properties (2.31) and (2.43) we need only to replace the number 2 in (2.20) with some number much smaller than the constant C in Lemma 2. The definition of L_k in (2.21) should be modified to

$$L_k=\{x_1\in\mathbb{R}^1:-\frac{C_1}{4}h_k^{\frac{1}{2\alpha-1}}<x_1<\frac{C_1}{4}h_k^{\frac{1}{2\alpha-1}}\}, C_1=\frac{1}{10000},$$

and the definition of Q_k in (2.22) remains the same. It is easy to see that we still have the estimates (2.23)-(2.24), and (2.25) becomes

$$|\partial_{x_1}g(x_1,h)|\leq\frac{16}{h_k^{\frac{1}{2\alpha-1}}}, \text{ for all } (x_1,h)\in Q_k.$$

Then for the definition of

$$\mathcal{X}^{\pm}=\{(x_1,h)\in Q_k:|\partial_{x_1x_1}g^{\pm}(x_1,h)|\geq h_k^{-\beta}\},$$

we need to choose the exponent β so that $\frac{1}{\alpha}<\beta<\frac{2}{2\alpha-1}$. By doing the same computation as (2.26)-(2.28) we have

$$|\{x_1\in L_k:(x_1,h)\in\mathcal{X}^+\}|h_k^{-\beta}\leq\int_{L_k}\partial_{x_1x_1}g^+\leq\frac{C}{h_k^{\frac{1}{2\alpha-1}}}.$$

So we have $|\mathcal{X}^+| \leq Ch_k^{1+\beta-\frac{1}{2\alpha-1}}$, and similarly we have $|\mathcal{X}^-| \leq Ch_k^{1+\beta-\frac{1}{2\alpha-1}}$. Then by the above estimate there is a set $\widetilde{L}^\pm \subset L_k$ with measure $|\widetilde{L}^\pm| \leq Ch_k^{\beta+\varepsilon-\frac{1}{2\alpha-1}}$ such that for any $y_1 \in L_k - \widetilde{L}^\pm$, we have $|\mathcal{X}^\pm{}_{y_1}| \leq h_k^{1-\varepsilon}$, where ε is chosen such that $\beta+\varepsilon < \frac{2}{2\alpha-1}$. Now, (2.31)-(2.34) remain the same, and (2.35) becomes

$$g^+(y_1, h_{k+1}) - g^+(y_1, h_k) \leq C_1 h_k^{1-\varepsilon}\frac{1}{h_k} + C_2 h_k^{-\beta\alpha} h_k.$$

By the choice of β, all the exponents on h_k is negative. We do not need to change anything from (2.36) to (2.45). Finally from (2.46) we need to replace the computation in the case $\alpha \leq 1$ with the following computation.

First we have $(g^+)'' \leq -\frac{1}{4}(g^+)'^{\frac{1}{\alpha}} \leq -\frac{1}{4}(g^+)'$, and we need only to bound $\int_{[h_k+h_k^{1-\frac{1}{2}\varepsilon}, \inf I_2]}(g^+)'dh$. Note that $(g^+)' \geq 0$, by integrating the above differential inequality we have $(g^+)'(h) \leq (g^+)'(h_k)e^{-\frac{1}{4}|I_3|} \leq (g^+)'(h_k)e^{\frac{1}{8}(h-h_k)}$, when $h \in [h_k + h_k^{1-\frac{1}{2}\varepsilon}, \inf I_2]$. Therefore, we have

$$\begin{aligned}\int_{[h_k+h_k^{1-\frac{1}{2}\varepsilon}, \inf I_2]}(g^+)'dh &\leq \int_{h_k}^{h_{k+1}}(g^+)'(h_k)e^{\frac{1}{8}(h-h_k)}dh \\ &\leq C(g^+)'(h_k) \leq \frac{C}{h_k}.\end{aligned}$$

2.2 Blow-down of an entire convex ancient solution converges to a power function

In this section we prove that the blow-down of an entire convex solution to (1.3) converges to a power function, and then by using this and a rescaling argument in next section we prove that, if a convex ancient solution to the generalized curve shortening flow sweeps the whole $\mathbb{R}^2$, it must be a shrinking circle.

Proof of Theorem 1 and the first part of Theorem 2. First we prove that there is a subsequence of u_h, where $u_h(x) = h^{-1}u(h^{\frac{1}{1+\alpha}}x)$, which converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$.

By adding a constant we may suppose $u(0) = 0$. Let $x_{n+1} = a \cdot x$ be the equation of the tangent plane of u at 0. By Corollary 3 and the convexity of u we have

$$a \cdot x \leq u(x) \leq C(1 + |x|^{1+\alpha}).$$

Hence,

$$h^{-\frac{\alpha}{1+\alpha}} a \cdot x \leq u_h(x) \leq C(\frac{1}{h} + |x|^{1+\alpha}).$$

By convexity we have that Du_h is locally uniformly bounded. Hence u_h sub-converges to a convex function u_0 which satisfies $u_0(0) = 0$, and

$$0 \leq u_0(x) \leq C|x|^{1+\alpha}.$$

It is easy to check that u_0 is an entire convex viscosity solution to equation (1.3) with $\sigma = 0$, and the comparison principle holds on any bounded domain.

Now we will prove $\{x|u_0(x) = 0\} = \{0\}$. In fact, if $\{x|u_0(x) = 0\}$ is a bounded set, then $\{x|u_0(x) = h\}$ is a closed, bounded convex curve which evolves under the generalized curve shortening flow, from [1] it follows that $\{x|u_0(x) = 0\} = \{0\}$. If $\{x|u_0(x) = 0\}$ contains a straight line, say the line $(t, 0), (t \in \mathbb{R})$, then by convexity, u is independent of x_1, which is impossible. So we need only to rule out the possibility that $\{x|u_0(x) = 0\}$ contains a ray but no straight lines. In this case, for fixed $h > 0$, we can find an ellipse E inside $\{x|u_0(x) < h\}$, with the short axis bounded from below by a constant depending only on h and with the long axis as long as we want (one needs only to look at the asymptotic cone of $\{x|u_0(x) = h\}$), but since $\{x|u_0(x) = h\}$ evolves under the generalized curve shortening flow and $E \subset \{x|u_0(x) \leq h\}$, which is impossible by comparison principle.

Then since $\{x|u_0(x) = 0\} = \{0\}$, $\Gamma_{1,u_0} = \{x|u_0(x) = 1\}$ is a bounded convex curve, and the level set $\{x|u_0(x) = -t\}$, with time $t \in (-\infty, 0)$, evolves under the generalized curve shortening flow, from [1], [2] we have the following asymptotic behavior of the convex solution u_0 of $L_0 u = 1$

$$u_0(x) = \frac{1}{1+\alpha}|x|^{1+\alpha} + \varphi(x), \text{ where } \varphi(x) = o(x^{1+\alpha}), \text{ for } x \neq 0 \text{ near the origin.} \quad (2.52)$$

In fact, if the initial level curve is in a sufficiently small neighborhood of circle, by Lemma 4 in the beginning of the fourth section, we have that $|\varphi(x)| \leq C|x|^{1+\alpha+\eta}$ for some small positive η, where C is a constant depending only on the initial closeness to the circle. Hence, given any $\epsilon > 0$, for sufficiently small $h' > 0$, we have

$$B_{(1-\epsilon)r}(0) \subset \Omega_{h',u_0} \subset B_{(1+\epsilon)r}(0),$$

where $r = ((1+\alpha)h')^{\frac{1}{1+\alpha}}$. Hence, there is a sequence $h_m \to \infty$ such that

$$B_{(1-\frac{1}{m})r_{m,i}}(0) \subset \Omega_{h_m,u} \subset B_{(1+\frac{1}{m})r_{m,i}}(0),$$

where $r_{m,i} = ((1+\alpha)ih_m)^{\frac{1}{1+\alpha}}$, $i = 1, \cdots, m$. Then u_{h_m} sub-converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$.

Since u_0 is an entire convex solution to $L_0 u = 1$, from the above argument, we can find a sequence h_m, such that $u_{0h_m}(x) = \frac{1}{h_m}u_0(h_m^{\frac{1}{1+\alpha}}x)$ locally uniformly converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$. Hence, the sublevel set $\Omega_{\frac{1}{1+\alpha},u_{0h_m}}$ satisfies

$$B_{1-\epsilon_m}(0) \subset \Omega_{\frac{1}{1+\alpha},u_{0h_m}} \subset B_{1+\epsilon_m}(0),$$

where $\epsilon_m \to 0$ as $m \to \infty$. By the discussion below (2.52), we have

$$u_{0h_m}(x) = \frac{1}{1+\alpha}|x|^{1+\alpha} + \varphi(x),$$

where $|\varphi(x)| \leq C|x|^{1+\alpha+\eta}$ for some fixed small positive η, and the constant C is independent of m. Replacing x by $h_m^{-\frac{1}{1+\alpha}}x$ in the above asymptotic formula, we have

$$u_0(x) = \frac{1}{1+\alpha}|x|^{1+\alpha} + h_m\varphi(h_m^{-\frac{1}{1+\alpha}}x),$$

where for any fixed x, $h_m\varphi(h_m^{-\frac{1}{1+\alpha}}x) \to 0$. Hence $u_0(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}$. So we have proved Theorem 1 and the first part of Theorem 2.

2.3 1-dimensional entire convex ancient solution must be a shrinking circle

We will follow the lines in the section 4 of [69]. It will be accomplished by the following lemma which is also true for higher dimensions, but we will only state it for $\mathbb{R}^2$.

Lemma 3. *Let Ω be a smooth, bounded, convex domain in $\mathbb{R}^2$. Let u be the solution of* (1.3) *with $\sigma = 0$, vanishing on $\partial\Omega$. Then for any constant h satisfying* $\inf_\Omega u < h < 0$, *the level set $\Gamma_{h,u} = \{u = h\}$ is convex. Moreover,* $\log(-u)$ *is a concave function.*

Proof. Observe $\varphi := -\log(-u)$ satisfies

$$|D\varphi|^{\frac{1}{\alpha}-1}\sum_{i,j=1}^{2}(\delta_{ij} - \frac{\varphi_i\varphi_j}{|D\varphi|^2})\varphi_{ij} = e^{\frac{1}{\alpha}\varphi}.$$

Since $\varphi(x) \to +\infty$ as $x \to \partial\Omega$, the result in [49](Theorem 3.13) implies φ is convex. One may notice that two of the conditions required in [49] are the strict convexity of domain and the C^2 smoothness of solution. The first one can be resolved by using strictly convex domains to approximate the convex domain. For the smoothness condition, one may worry about the minimum point where the gradient vanishes and the equation is singular. Moreover, in view of the solution $u = \frac{1}{1+\alpha}|x|^{1+\alpha}$, we see when $\alpha < 1$ it is not C^2 at the origin. However, by examining the proof in [49], one can see that the argument is made away from the minimum point, which means it can still be applied to our situation.

With the above lemma and the Lemma 4.4 in [69], we know that any convex compact ancient solution to the generalized curve shortening flow can be represented as a convex solution u to equation (1.3) with $\sigma = 0$, and if the solution to the flow sweeps the whole space, the corresponding u will be an entire solution. Thus Theorem 2 implies Corollary 1 immediately.

Remark 2.3.1. *We can also use the method in the section 4 of [69] to construct a non-rotationally symmetric convex compact ancient solution for generalized curve shortening flow with power $\alpha \in (\frac{1}{2}, 1)$, and in fact the solution will be defined in a strip region. All we need to do is replace Lemma 4.2, 4.3 and 4.4 in [69] for mean curvature flow by the corresponding lemmas for the generalized curve shortening flow.*

2.4 2-dimensional entire convex translating solution

In this section, by using the previous results and an delicate iteration argument we prove that under some extra condition on the asymptotic behavior of the solution at infinity the 2-d translating solution must be rotationally symmetric.

First of all, we would like to point out that instead of using Gage and Hamilton's exponential convergence of the curve shortening flow in [39] we need to use the corresponding exponential convergence for the generalized curve shortening flow and we will state it as a lemma which is corresponding to lemma 3.2 in [69].

Lemma 4. *Let $\{\ell_t\}$ be a convex solution to the generalized curve shortening flow with initial curve $\{\ell_0\}$ uniformly convex. Suppose $\{\ell_t\}$ is in the δ_0-neighborhood of a unit circle, $\{\ell_t\}$ shrinks to the origin at $t = \frac{1}{1+\alpha}$. Let $\widetilde{\ell}_t = (1-(1+\alpha)t)^{-\frac{1}{1+\alpha}}\ell_t$ be the normalization of ℓ_t. Then $\widetilde{\ell}_t$ is in the δ_t- neighborhood of the unit circle centered at the origin,*

$$\widetilde{\ell}_t \subset N_{\delta_t} S^1,$$

with

$$\delta_t \leq C\delta_0(\frac{1}{1+\alpha} - t)^\iota$$

for some small positive constant ι.

The proof of the above lemma is similar to the proof of lemma 3.2 in [69]. Using the condition that the initial curve is uniformly convex and the estimates in section II of [1], we can apply Schauder's estimates safely for $\alpha > \frac{1}{2}$ as in [69], which says that for $t \in (\frac{1}{4\alpha+4}, \frac{1}{2\alpha+2})$,

$$\|\widetilde{\ell}_t - S^1\|_{C^k} \leq C\delta_0.$$

Although the constant C will depend on the lower and upper bound of the curvature of the initial curve, it is not a problem for our purpose, since when we blow down the solution for $\sigma = 0$, the norm of the gradient Du_h on the curve $\{u_h(x) = 1\}$ approaches to 1. By the equation $\kappa u_\gamma^{\frac{1}{\alpha}} = 1$ we see that the curvature κ is also very close to 1 on that curve. However, the estimates in section II of [1] also shows that when $\alpha \leq 1$ the uniformly convex condition (though the convexity is still needed) is not needed, and the constant C in the above lemma is independent of the bound on the curvature of the initial curve. For the exponential decay rate of the derivatives of curvature, one can imitate the proof in Gage and Hamilton [39](5.7.10-5.7.15), and our corresponding estimate will be $|\kappa'(\tau)| \leq C\delta_0 e^{-\iota\tau}$ for some small positive number ι, where $\tau = -\frac{1}{1+\alpha}\log(\frac{1}{1+\alpha} - t)$. This estimate immediately implies our lemma.

An alternative way to see that is by writing down the normalized evolution equation for the generalized curve shortening flow by using support function $s(\theta, \tau)$ as following

$$s_\tau = -(s_{\theta\theta})^{-\alpha} + s,$$

here we still take the origin as the limiting point of the original generalized curve shortening flow. Then the linearized equation of the flow about the circle solution is

$$s_\tau = \alpha(s_{\theta\theta} + s) + s.$$

The rate of convergence is governed by the eigenvalues of the right hand side. The constant eigenfunction corresponds to scaling, which is factored out, while the $\sin\theta$ and $\cos\theta$ correspond to translations, which are also factored out. The next is $\cos(2\theta)$, which gives eigenvalue $1-3\alpha$. So when $\alpha > \frac{1}{3}$, we have exponential convergence of the normalized solution to the limiting circle with exponent $1-3\alpha$. The author learned this from professor Ben Andrews.

In the following we will consider the case when $\sigma = 1$ and $\alpha > 1$. By translating and adding some constant we can assume $u(0) = \inf u$. Let $u_h(x) = \frac{1}{h}u(h^{\frac{1}{1+\alpha}}x)$. Then u_h satisfies the equation $L_\sigma u_h = 1$ with $\sigma = h^{-\frac{2\alpha}{1+\alpha}}$. By Theorem 1, u_h converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$, and the level set $\Gamma_{\frac{1}{1+\alpha},u_h}$ converges to the unit circle as $h \to \infty$.

Lemma 5.

$$u(x) = \frac{1}{1+\alpha}|x|^{1+\alpha} + O(|x|^{1+\alpha-2\alpha\beta}) \tag{2.53}$$

where C is a fixed constant and the constant β is chosen such that $\frac{1}{2\alpha} < \beta < \min\{1, \frac{1+\alpha}{2\alpha}\}$.

For any given small $\delta_0 > 0$, taking h sufficiently large such that

$$\Gamma_{\frac{1}{1+\alpha},u_h} \subset N_{\delta_0}(S^1) \tag{2.54}$$

for unit circle S^1 with center p_0. Note that when h is large, δ_0 is very close to 0. Then we will prove the following claim,

Claim 3. For small fixed τ,

$$\Gamma_{\tau,u_h} \subset ((1+\alpha)\tau)^{\frac{1}{1+\alpha}} N_{\delta_\tau}((1+\frac{a_0}{\tau})^{\frac{1}{1+\alpha}} S^1) \tag{2.55}$$

with

$$\delta_\tau \le C_1(\tau)\sigma^\beta + C_2\delta_0\tau^\eta, \tag{2.56}$$

where the constants C_1 and C_2 are independent of δ_0 and h, and C_2 is also independent of τ, η is a small positive constant. u_0 is the solution of $L_0(u) = 1$ in $\Omega_{\frac{1}{1+\alpha},u_h}$ satisfying $u_0 = u_h = \frac{1}{1+\alpha}$ on $\partial\Omega_{\frac{1}{1+\alpha},u_h}$, $a_0 = |\inf u_0|$ and the center of $(1+\frac{a_0}{\tau})^{\frac{1}{1+\alpha}}S^1$ is the minimum point of u_0 times a factor $((1+\alpha)\tau)^{-\frac{1}{1+\alpha}}$.

Proof of Claim 3. We need only to prove

$$\operatorname{dist}\left((1+\alpha)^{\frac{1}{1+\alpha}}(\tau+a_0)^{\frac{1}{1+\alpha}}S^1, \Gamma_{\tau,u}\right) \le C_1(\tau)\sigma^\beta + C_2\delta_0\tau^{\frac{1}{1+\alpha}+\eta}, \tag{2.57}$$

where η is some small positive constant, C_2 is independent of τ. by Theorem 1 we know u_h converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$ uniformly on any compact subset of $\mathbb{R}^2$, then by the convexity of u_h, we have that when

$$x \in \{x \in \Omega_{\frac{1}{1+\alpha},u_h} : \tau_0 \le u_h < \frac{1}{1+\alpha}\},$$

$|Du_h|$ is bounded above and below by some constants depending on τ_0 for large h, by the growth condition for D^2u in Theorem 2 we have $\sigma(u_h)_{\gamma\gamma} \le C\sigma^\beta$, where C is a constant depending on τ_0. Therefore we have $\kappa(u_h)_\gamma^{\frac{1}{\alpha}} \approx 1 - C\sigma^\beta$ on $\{x \in \Omega_{\frac{1}{1+\alpha},u_h} : \tau \le u_h < \frac{1}{1+\alpha}\}$, where C depends on τ_0. Denote

$$\widetilde{u}_0 = (1 - C\sigma^\beta)^\alpha(u_0 - \frac{1}{1+\alpha}) + \frac{1}{1+\alpha},$$

then

$$L_0(\widetilde{u}_0) = 1 - C\sigma^\beta \text{ in } \Omega_{\frac{1}{1+\alpha},u_h}$$

with $\widetilde{u}_0 = u_h = \frac{1}{1+\alpha}$ on $\partial\Omega_{\frac{1}{1+\alpha},u_h}$. Now by comparison principle we have $\Omega_{\tau,u_0} \subset \Omega_{\tau,u_h} \subset$

$\Omega_{\tau,\widetilde{u}_0}$, and by the asymptotic behavior of u_0 we have

$$\Gamma_{\tau,u_0} \subset N_\zeta((\tau + a_0)^{\frac{1}{1+\alpha}} S^1) \text{ and } \Gamma_{\tau,\widetilde{u}_0} \subset N_\zeta((\tau + a_0 - C\sigma^\beta)^{\frac{1}{1+\alpha}} S^1),$$

where $\zeta = C\delta_0(\tau + a_0)^\eta$. Denote $\ell_1 = (\tau + a_0)^{\frac{1}{1+\alpha}} S^1$, $\ell_2 = (\tau + a_0 - C\sigma^\beta)^{\frac{1}{1+\alpha}} S^1$, both of them are centered at p_1, which is the minimum point of u_0. Hence $\mathrm{dist}((\tau + a_0)^{\frac{1}{1+\alpha}} S^1, \Gamma_{\tau,u_h}) \leq \mathrm{dist}(\ell_1, \ell_2) + C\delta_0(\tau + a_0)^{\frac{1}{1+\alpha}+\eta}$, where $\mathrm{dist}(\ell_1, \ell_2)$ can be bounded by $C_1(\tau)\sigma^\beta$, hence (2.56) follows from the above discussion. Now we will use an iteration argument to prove the following Claim 4, which will enable us to simplify (2.55) and (2.56).

Claim 4:

$$a_0 \leq \begin{cases} C\sigma|\log(\sigma)| & \text{if } \alpha \leq 1 \\ C\sigma^{\frac{1+\alpha}{2\alpha}} & \text{if } \alpha > 1 \end{cases} \tag{2.58}$$

Proof of Claim 4. We fix a large constant A such that $\{u_{\frac{A}{\tau}} = \frac{1}{1+\alpha}\}$ is very close to a unit circle. Let u_{0,τ^k} solve $L_0 u = 1$ with boundary condition $u = \tau^k$ on $\{u_h = \tau^k\}$. Denote $a_k = |\inf u_{0,\tau^k}|$. From the proof of Claim 3 we see that $\{u_0 < \tau\} \supset \{u_{0,\tau} < \tau\} \supset \{\widetilde{u}_0 < \tau\}$, by comparison principle, we have $\inf u_0 < \inf u_{0,\tau} < \inf \widetilde{u}_0$. So by the construction of $\widetilde{u}_0$ and a simple computation, we have $a_0 - a_1 \leq \inf \widetilde{u}_0 - \inf u_0 \leq C\sigma$. When $\tau^k \geq \frac{A}{h}$, we can iterate this argument for u_{0,τ^k} and $u_{0,\tau^{k+1}}$ by rescaling them to $\frac{1}{1+\alpha}\tau^{-k} u_{0,\tau^k}\left((1+\alpha)^{\frac{1}{1+\alpha}} \tau^{\frac{k}{1+\alpha}} x\right)$ and $\frac{1}{1+\alpha}\tau^{-k} u_{0,\tau^{k+1}}\left((1+\alpha)^{\frac{1}{1+\alpha}} \tau^{\frac{k}{1+\alpha}} x\right)$ respectively, after rescaling back, we have $a_k - a_{k+1} \leq C\sigma$. Note that the choice of A and the condition $\tau^k \geq \frac{A}{h}$ ensure the uniform gradient bound needed in the above argument. Let k_0 be an integer satisfying $\tau^{k_0} \geq \frac{A}{h} \geq \tau^{k_0+1}$, after k_0 steps we stop the iteration, and notice that $\{u_h = \frac{A}{h}\} = \frac{1}{h^{\frac{1}{1+\alpha}}}\{u = A\}$ is contained in a circle with radius $Ch^{-\frac{1}{1+\alpha}}$ for some constant C, so it takes at most time $Ch^{-1} = C\sigma^{\frac{1+\alpha}{2\alpha}}$ for $\{u_h = \frac{A}{h}\}$ shrink into a point. Claim 4 follows from the above discussion.

By omitting the lower order term we can rewrite (2.55) and (2.56) as

$$\Gamma_{\tau,u_h} \subset ((1+\alpha)\tau)^{\frac{1}{1+\alpha}} N_{\delta_\tau}(S^1)$$

with

$$\delta_\tau \leq C_1(\tau)\sigma^\beta + C_2\delta_0\tau^\eta. \tag{2.59}$$

If we take τ small such that $C_2\tau^\eta \leq \frac{1}{4}$, (2.59) becomes

$$\delta_\tau \leq C_1(\tau)\sigma^\beta + \frac{1}{4}\delta_0. \tag{2.60}$$

Now we can carry out an iteration argument similar as that in [69]. We start at the level $\frac{1}{1+\alpha}\tau^{-k_0}$ for some sufficient large k_0. Denote $\Omega_k = \tau^{\frac{k}{1+\alpha}}\Omega_{\frac{1}{1+\alpha}\tau^{-k},u}$ and $\Gamma_k = \partial\Omega_k$. Γ_k converges to unit circle as $k \to \infty$. Suppose Γ_k is in the δ_k neighborhood of S^1 centered at y_k, where $\delta_k \to 0$ as $k \to \infty$ and y_k is the minimum point of the solution of $L_0 u = 1$ in Ω_k with $u = \frac{1}{1+\alpha}$ on Γ_{k+1}. By (2.60) we have

$$\delta_{k-1} \leq C_1(\tau)\tau^{(k-1)\frac{2\alpha\beta}{1+\alpha}} + \frac{1}{4}\delta_k \tag{2.61}$$

for $k = k_0, k_0+1, \cdots$. Then we have

$$\Gamma_j \subset N_{\delta_j}(S^1) \tag{2.62}$$

with

$$\delta_j \leq C\tau^{j\frac{2\alpha\beta}{1+\alpha}} \tag{2.63}$$

It follows that

$$\Gamma_{\frac{1}{1+\alpha}\tau^{-j},u} \subset N_{\tilde{\delta}_j}(\tau^{\frac{-j}{1+\alpha}}S^1) \tag{2.64}$$

with

$$\widetilde{\delta}_j \leq C\tau^{\frac{2\alpha\beta-1}{1+\alpha}j} \tag{2.65}$$

where $\tau^{\frac{-j}{1+\alpha}}S^1$ is centered at $z_j = \tau^{\frac{-j}{1+\alpha}}y_j$. From Lemma 3 and (2.60) it is not hard to see that we have

$$|z_j - z_{j-1}| \leq C\tau^{\frac{2\alpha\beta-1}{1+\alpha}j} \tag{2.66}$$

Denote $z_0 = \lim_{j\to\infty} z_j$. Then

$$|z_j - z_0| \leq C\tau^{\frac{2\alpha\beta-1}{1+\alpha}j}, \tag{2.67}$$

which means in (2.64) we can assume the circle is centered at z_0 by changing the constant C a little bit. In fact when we choose different τ, the corresponding z_0 will not change, so we can assume $z_0 = 0$. Hence for $h = \frac{1}{1+\alpha}\tau^{-j}$,

$$\Gamma_{h,u} \subset N_\delta\left((1+\alpha)^{\frac{1}{1+\alpha}}h^{\frac{1}{1+\alpha}}S^1\right),$$

where

$$\delta \leq Ch^{\frac{1-2\alpha\beta}{1+\alpha}} \tag{2.68}$$

and S^1 is centered at the origin. By choosing different τ, we see that the estimate holds for all large h. Lemma 5 follows from the above estimates.

To finish the proof of Theorem 2 we need to use the following fundamental Liouville Theorem by Bernstein [63] (p.245).

Lemma 6. *Let u be an entire solution to the elliptic equation*

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = 0 \text{ in } \mathbb{R}^2.$$

If u satisfies the asymptotic estimate

$$|u(x)| = o(|x|), \text{ as } x \to \infty,$$

then u is a constant.

Proof of the second part of Theorem 2. Let u^* be the Legendre transform of u. Then u^* satisfies equation

$$G(x, D^2u^*) = \frac{\det D^2u^*}{(\delta_{ij} - \frac{x_ix_j}{1+|x|^2})F^{ij}(u^*)} = (1+|x|^2)^{\frac{1}{2\alpha}-\frac{1}{2}}, \tag{2.69}$$

where $F^{ij}(u^*) = \frac{\partial \det r}{\partial r_{ij}}$, at $r = D^2(u^*)$. We have

$$u^*(x) = C(\alpha)|x|^{1+\alpha} + O(|x|^{\frac{1+\alpha-2\alpha\beta}{\alpha}}), \tag{2.70}$$

where $C(\alpha)$ is a constant depending only on α. In fact, for big h, by Lemma 5 we have

$$u_h(x) = \frac{1}{1+\alpha}|x|^{1+\alpha} + O(|h|^{\frac{-2\alpha\beta}{1+\alpha}})$$

in $B_1(0)$. Denote u_h^* as the Legendre transforms of u_h. Then

$$u_h^*(x) = C(\alpha)|x|^{1+\frac{1}{\alpha}} + O(|h|^{\frac{-2\alpha\beta}{1+\alpha}}),$$

where $C(\alpha)$ is a constant depending only on α and in fact it is comes from the Legendre transform of the function $\frac{1}{1+\alpha}|x|^{1+\alpha}$. Note that $u_h^*(x) = h^{-1}u^*(h^{\frac{\alpha}{1+\alpha}}x)$, we obtain (2.70).

Let u_0 be the unique radial solution of (1.3) with $\sigma = 1$, and let u_0^* be the Legendre transform of u_0. Similar to (2.70) we have

$$u_0^*(x) = C(\alpha)|x|^{1+\alpha} + O(|x|^{\frac{1+\alpha-2\alpha\beta}{\alpha}}). \tag{2.71}$$

Since both u^* and u_0^* satisfy equation (2.69), $v = u^* - u_0^*$ satisfies the following elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x) v_{ij} = 0 \text{ in } \mathbb{R}^2,$$

where

$$a_{ij} = \int_0^1 G^{ij}(x, D^2 u_0^* + t(D^2 u^* - D^2 u_0^*))dt,$$

here $G^{ij} = \frac{\partial G(x,r)}{\partial r_{ij}}$ for any symmetric matrix r. Note that by the choice of β, $\frac{1+\alpha-2\alpha\beta}{\alpha} < 1$, so by (2.70) and (2.71) $v = O(|x|^{\frac{1+\alpha-2\alpha\beta}{\alpha}}) = o(|x|)$, as $|x| \to \infty$. By Lemma 6 we conclude that v is a constant.

Chapter 3

Conformally invariant integral inequalities and remainder terms in fractional sobolev inequality

In the first section of this chapter we prove some Carleman type sharp conformally invariant inequalities in unit ball. The inequalities hold for general dimensions, which extends the original Carleman's result in two dimension. The second section is devoted to some results about remainder terms in the fractional sobolev inequality; this section represents the joint work with Weth and Frank.

3.1 Carleman type conformally invariant integral inequalities

3.1.1 A family of conformally invariant integral inequalities

This section is devoted to the proof of Theorem 3. Since P_a enjoys very similar properties to the special case P_0 (classical harmonic extension), we are also able to use the method of symmetrization developed by Lieb [50] to prove the existence of maximizer as Hang, Wang and Yan did in [45]. The following Lemmas are parallel to those in [45], but notice that now we are dealing with poly-harmonic extension instead of harmonic extension.

Recall if Ω is a measurable set in $\mathbb{R}^n$, $p > 0$ and u is a measurable function on Ω, then

$$\|u\|_{L^p_w} = \sup_{t>0} t||u| > t|^{\frac{1}{p}}.$$

The weak-L^p space $L^p_w(\Omega)$ is defined as $\{u{:}u$ is measurable and $\|u\|_{L^p_w(\Omega)} < \infty\}$. More generally, for any $0 < p < \infty$ and $0 < q \leq \infty$, we have Lorentz norm $\|\cdot\|_{L^{p,q}}$ which is defined by $\|u\|_{L^{p,q}} = p^{\frac{1}{q}}(\int_0^\infty t^q||u| \geq t|^{\frac{q}{p}}\frac{dt}{t})^{\frac{1}{q}}$ and Lorentz space $L^{p,q}(\Omega)$. $L^p_w(\Omega) = L^{p,\infty}(\Omega)$ is a special case of such spaces.

Lemma 7. *For $a < 1$, defining P_a as in (1.9), there exist constants $c_{n,a}$ and $c_{n,a,p}$ such that*

$$\|P_a f\|_{L_w^{\frac{n}{n-1}}(\mathbb{R}^n_+)} \leq c_{n,a}\|f\|_{L^1(\mathbb{R}^{n-1})}$$

and

$$\|P_a f\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c_{n,a,p}\|f\|_{L^p(\mathbb{R}^{n-1})}$$

for all $1 < p \leq \infty$. Moreover for $1 < p < \infty$ we have

$$\|P_a f\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c_{n,a,p}\|f\|_{L^{p,\frac{np}{n-1}}(\mathbb{R}^{n-1})}.$$

Proof of Lemma 7. To prove the weak estimate, we may assume $f \geq 0$ and $\|f\|_{L^1(\mathbb{R}^{n-1})} = 1$. It is easy to see $(P_a f)(X, x_n) \leq \frac{d_{n,a}}{x_n^{n-1}}$ for $(X, x_n) \in \mathbb{R}^n_+$ and

$$\begin{aligned}
&\int_{(X,x_n)\in\mathbb{R}^n_+, 0<y<b} (P_a f)(X, x_n) d^{n-1}X dx_n \\
= \ & \int_{\mathbb{R}^{n-1}} d^{n-1}Y \left(f(Y) \int_0^b dx_n \int_{\mathbb{R}^{n-1}} d_{n,a} \frac{x_n^{1-a}}{((X-Y)^2 + x_n^2)^{\frac{n-a}{2}}} d^{n-1}X \right) \\
= \ & b
\end{aligned}$$

for $b > 0$. Hence for $t > 0$,

$$\begin{aligned} |P_a f > t| &= |\{(X, x_n) \in \mathbb{R}^n_+ : 0 < x_n < (d_{n,a}^{-1} t)^{-\frac{1}{n-1}}, (P_a f)(X, x_n) > t\}| \\ &\leq \frac{1}{t} \int_{X \in \mathbb{R}^{n-1}, 0 < x_n < (d_{n,a}^{-1} t)^{-\frac{1}{n-1}}} (P_a f)(X, x_n) d^{n-1} X dx_n \\ &= \frac{1}{t} (d_{n,a}^{-1} t)^{-\frac{1}{n-1}} \end{aligned}$$

The weak type inequality follows. The strong estimate follows from Marcinkiewicz interpolation theorem (see [64], p197) and the basic fact $\|P_a f\|_{L^\infty(\mathbb{R}^n_+)} \leq \|f\|_{L^\infty(\mathbb{R}^{n-1})}$. In fact, the Marcinkiewicz interpolation implies that if T is a linear bounded operator from $L^1(\mathbb{R}^{n-1})$ to $L^{p_0}_w(\mathbb{R}^n_+)$ and at the same time from $L^\infty(\mathbb{R}^{n-1})$ to $L^\infty(\mathbb{R}^n_+)$, we have that for each $p \in (1, \infty)$, T is a bounded operator from $L^{p,q}(\mathbb{R}^{n-1})$ to $L^{pp_0,q}(\mathbb{R}^n_+)$, where $1 < q < \infty$,. To complete the proof we need only to choose $p_0 = \frac{n}{n-1}$, $q = \frac{np}{n-1}$.

Remark 3.1.1. *In fact when $p = \frac{2(n-1)}{n-2}$ and $a = 0$, the second estimate was also proved by Brezis and Lieb [13] by using some elementary dual argument.*

Lemma 8. *If $n \geq 2$, $a < 1$ and $1 < p < \infty$, then the supremum*

$$c_{n,a,p}^{\frac{np}{n-1}} = \sup\{\|P_a f\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} : \|f\|_{L^p(\mathbb{R}^{n-1})} = 1\}, \tag{3.1}$$

is attained by some function. After multiplying by a nonzero constant, every maximizer f is nonnegative, radially symmetric with respect to some point, strictly decreasing in the radial direction and it satisfies the following Euler-Lagrange equation

$$f(Y)^{p-1} = \int_{\mathbb{R}^n_+} \frac{x_n^{1-a}}{((X-Y)^2 + x_n^2)^{\frac{n-a}{2}}} (P_a f)^{\frac{np}{n-1}-1}(X, x_n) d^{n-1} X dx_n.$$

In particular, if $n \geq 2$, $p = \frac{2(n-1)}{n-2+a}$ and $n - 2 + a > 0$, then every maximizer is of the form

$$f(Y) = \pm c(n, a) \left(\frac{\lambda}{\lambda^2 + |Y - Y_0|^2}\right)^{\frac{n-2+a}{2}} \tag{3.2}$$

for some $\lambda > 0$, $Y_0 \in \mathbb{R}^{n-1}$.

Proof of Lemma 8. First we recall the important Riesz rearrangement inequality. Let u be a measurable function on $\mathbb{R}^n$, the symmetric rearrangement of u is the nonnegative lower semi-continuous radial decreasing function u^* that has the same distribution as u. We have

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u(x)v(y-x)w(y)dy \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u^*(x)v^*(y-x)w^*(y)dy.$$

Using the fact $\|w\|_{L^p(\mathbb{R}^n)} = \|w^*\|_{L^p(\mathbb{R}^n)}$ for $p > 0$ and the standard duality argument, we see for $1 \leq p \leq \infty$,

$$\|u * v\|_{L^p(\mathbb{R}^n)} \leq \|u^* * v^*\|_{L^p(\mathbb{R}^n)}.$$

Moreover if u is nonnegative radially symmetric and strictly decreasing in the radial direction, v is nonnegative, $1 < p < \infty$ and

$$\|u * v\|_{L^p(\mathbb{R}^n)} = \|u^* * v^*\|_{L^p(\mathbb{R}^n)} < \infty,$$

then for some $x_0 \in \mathbb{R}^n$, we have $v(x) = v^*(x - x_0)$.

Now, assume f_i is a maximizing sequence in (3.1). Since $\|f_i^*\|_{L^p(\mathbb{R}^{n-1})} = \|f_i\|_{L^p(\mathbb{R}^{n-1})} = 1$ and

$$\begin{aligned}
\|P_a f_i\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} &= \int_0^\infty \|P_{a,x_n} * f_i\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})} dy \\
&\leq \int_0^\infty \|P_{a,x_n} * f_i^*\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})} dx_n \\
&= \|P_a f_i^*\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)},
\end{aligned}$$

where $P_{a,x_n} = d_{n,a} \frac{x_n^{1-a}}{(X^2+x_n^2)}$ and notice that it is symmetric and strictly decreasing in the radial direction of X variable for any fixed x_n. We see f_i is again a maximizing sequence. Hence we may assume f_i is a nonnegative radial decreasing function.

For any $f \in L^p(\mathbb{R}^{n-1})$ and any $\lambda > 0$, we let $f^\lambda(Y) = \lambda^{-\frac{n-1}{p}} f(\frac{Y}{\lambda})$, so that is clear that $(P_a f^\lambda)(X, x_n) = \lambda^{-\frac{n-1}{p}}(P_a f)(\frac{X}{\lambda}, \frac{x_n}{\lambda})$ and hence $\|f^\lambda\|_{L^p(\mathbb{R}^{n-1})} = \|f\|_{L^p(\mathbb{R}^{n-1})}$ and $\|P_a f^\lambda\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} = \|P_a f\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}$. For convenience, denote $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^{n-1}$ and

$$a_i = \sup\{f_i^\lambda(e_1) | \lambda > 0\} = \sup\{\lambda^{-\frac{n-1}{p}} f_i(\frac{e_1}{\lambda}) | \lambda > 0\}.$$

It follows that $0 \leq f_i(Y) \leq a_i |Y|^{-\frac{n-1}{p}}$, and hence $\|f_i\|_{L^{p,\infty}(\mathbb{R}^{n-1})} \leq |B_{n-1}|^{\frac{1}{p}} a_i$.

Now

$$\begin{aligned} \|P_a f_i\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} &\leq c(n,a,p)\|f_i\|_{L^{p,\frac{np}{n-1}}(\mathbb{R}^{n-1})} \\ &\leq c(n,a,p)\|f_i\|^{\frac{n-1}{n}}_{L^p(\mathbb{R}^{n-1})}\|f_i\|^{\frac{1}{n}}_{L^{p,\infty}(\mathbb{R}^{n-1})} \\ &\leq c(n,a,p) a_i^{\frac{1}{n}}, \end{aligned}$$

which implies $a_i \geq c(n,a,p) > 0$. We may choose $\lambda_i > 0$ such that $f_i^{\lambda_i}(e_1) \geq c(n,a,p) > 0$. Replacing f_i by $f_i^{\lambda_i}$ we may assume $f(e_1) \geq c(n,a,p) > 0$. On the other hand , since f_i is nonnegative radial decreasing and $\|f_i\|_{L^p(\mathbb{R}^{n-1})} = 1$, we see

$$|f_i(Y)| \leq |B_{n-1}|^{-\frac{1}{p}} |Y|^{-\frac{n-1}{p}}.$$

Hence after passing to a subsequence, we may find a nonnegative radial decreasing function f such that $f_i \to f$ a.e. It follows that $f(Y) \geq c(n,a,p) > 0$ for $|Y| \leq 1$, $f_i \rightharpoonup f$ in $L^p(\mathbb{R}^{n-1})$ and $\|f\|_{L^p(\mathbb{R}^{n-1})} \leq 1$. By Lieb [50](Lemma 2.6), we have

$$\int_{\mathbb{R}^{n-1}} ||f_i|^p - |f|^p - |f_i - f|^p| d^{n-1}Y \to 0.$$

It follows that

$$\begin{aligned}\|f_i - f\|^p_{L^p(\mathbb{R}^{n-1})} &= \|f_i\|^p_{L^p(\mathbb{R}^{n-1})} - \|f\|^p_{L^p(\mathbb{R}^{n-1})} + o(1)\\ &= 1 - \|f\|^p_{L^p(\mathbb{R}^{n-1})} + o(1).\end{aligned}$$

On the other hand, since $(P_a f_i)(X, x_n) \to (P_a f)(X, x_n))$ for $(X, x_n) \in \mathbb{R}^n_+$ and $\|P_a f_i\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c_{n,a,p}$, we see

$$\begin{aligned}\|P_a f_i\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} &= \|P_a f\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} + \|P_a f_i - P_a f\|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} + o(1)\\ &\leq c^{\frac{np}{n-1}}_{n,a,p}\|f\|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + c^{\frac{np}{n-1}}_{n,a,p}\|f_i - f\|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + o(1).\end{aligned}$$

Hence

$$1 \leq \|f\|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + \|f_i - f\|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + o(1).$$

Let $i \to \infty$, we see

$$1 \leq \|f\|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + (1 - \|f\|^p_{L^p(\mathbb{R}^{n-1})})^{\frac{n}{n-1}}.$$

Since $\frac{n}{n-1} > 1$ and $f \neq 0$, we see $\|f\|_{L^p(\mathbb{R}^{n-1})} = 1$. Hence $f_i \to f$ in $L^p(\mathbb{R}^{n-1})$ and f is a maximizer. This implies the existence of an extremal function.

Assume $f \in L^p(\mathbb{R}^{n-1})$ is a maximizer, then so is $|f|$. Hence $\|P_a f\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} = \|P_a|f|\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}$. On the other hand , since $|(P_a f)(X, x_n)| \leq (P_a|f|)(X, x_n)$ for $(X, x_n) \in \mathbb{R}^n_+$, we see $|P_a f| = P_a(|f|)$ and this implies either $f \geq 0$ or $f \leq 0$. Assume $f \geq 0$, then the Euler-Lagrange equation after scaling by a positive constant is given by

$$f(Y)^{p-1} = \int_{\mathbb{R}^n_+} \frac{y^{1-a}}{((X-Y)^2 + x_n^2)^{\frac{n-a}{2}}}(P_a f)^{\frac{np}{n-1}-1}(X, x_n)d^{n-1}X dx_n.$$

On the other hand, we know for $x_n > 0$,

$$\|P_{a,x_n} * f\|_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})} = \|P_{a,x_n} * f^*\|_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})}$$

which implies $f(Y) = f^*(Y - Y_0)$ for some Y_0. It follows from the above Euler-Lagrange equation and Lemma 2.2 of Lieb [50] that f must be strictly decreasing along the radial direction.

For the case when $p = \frac{2(n-1)}{n-2+a}$, we first observe that if $f \in L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})$, let $u = P_a f$, $\widetilde{f} = \frac{1}{|Y|^{n-2+a}} f(\frac{Y}{|Y|^2})$ and $\widetilde{u} = \frac{1}{|(X,x_n)|^{n-2+a}} f(\frac{(X,x_n)}{|(X,x_n)|^2})$, then we have $\widetilde{u} = P_a \widetilde{f}$, $\|\widetilde{f}\|_{L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})} = \|f\|_{L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})}$ and $\|\widetilde{u}\|_{L^{\frac{2n}{n-2+a}}(\mathbb{R}^n_+)} = \|u\|_{L^{\frac{2n}{n-2+a}}(\mathbb{R}^n_+)}$. This is the conformal invariance property for the particular power. As a consequence, if f is a maximizer which is nonnegative and radial, then $\frac{1}{|Y|^{n-2+a}} f(\frac{Y}{|Y|^2} - e_1)$ is also a maximizer. In particular, $\frac{1}{|Y|^{n-2+a}} f(\frac{Y}{|Y|^2} - e_1)$ is radial with respect to some points. To find such f, we need the following useful Proposition of Hang, Wang and Yan [45](Proposition 4.1).

Lemma 9. *Let $n \geq 2$, u be a function on $\mathbb{R}^n$ which is radial with respect to the origin, $0 < u(x) < \infty$ for $x \neq 0$, $e_1 = (1, 0, \cdots, 0)$, $\alpha \in R$, $\alpha \neq 0$. If $v(x) = |x|^\alpha u(\frac{x}{|x|^2} - e_1)$ is radial with respect to some point, then either $u(x) = (c_1|x|^2 + c_2)^{\frac{\alpha}{2}}$ for some $c_1 \geq 0, c_2 > 0$ or*

$$u(x) = \begin{cases} c_1|x|^\alpha & \text{if } x \neq 0 \\ c_2 & \text{if } x = 0. \end{cases} \tag{3.3}$$

Proof of Lemma 8 continued. Since $\|f\|_{L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})} = 1$ and it is strictly decreasing along the radial direction, we have $0 < f(Y) < \infty$ for all $Y \neq 0$. Note that since f satisfies the Euler-Lagrange equation, it is defined everywhere instead of almost everywhere. It follows from Lemma 3 that $f(Y) = (c_1|Y|^2 + c_2)^{-\frac{n-2+a}{2}}$ for some $c_1, c_2 > 0$ (since f can not be constant function and the scalar multiple of $|Y|^{2-n}$ is ruled out by the integrability). Using the condition $\|f\|_{L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})} = 1$, it is easy to see $c_1 c_2 = c_{n,a}$. Hence for some

$\lambda > 0$,

$$f(Y) = c(n,a)\left(\frac{\lambda}{\lambda^2 + |Y - Y_0|^2}\right)^{\frac{n-2+a}{2}}.$$

Proof of Theorem 3. For any $\widetilde{f} \in L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)$, let $\widetilde{u} = \widetilde{P}_a f$,

$$f = \frac{1}{|(Y,0) + (\mathbf{0}, \frac{1}{2})|^{n-2+a}} \widetilde{f} \circ \phi,$$

and

$$u = \frac{1}{|(X,x_n) + (\mathbf{0}, \frac{1}{2})|^{n-2+a}} \widetilde{u} \circ \phi.$$

By definition (1.12) we have $u = P_a f$ and by the discussion below (1.11) we have $\|\widetilde{f}\|_{L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)} = \|f\|_{L^{\frac{2(n-1)}{n-2+a}}(\mathbb{R}^{n-1})}$ and $\|\widetilde{u}\|_{L^{\frac{2n}{n-2+a}}(B_n)} = \|u\|_{L^{\frac{2n}{n-2+a}}(\mathbb{R}^n_+)}$. Then, Theorem 3 follows easily from the above facts and Lemma 8.

3.1.2 The limiting case

First we will discuss some conformal invariance properties of the operator $\widetilde{P}_a$. Let $\widetilde{\tau}$ be a conformal transform from B_n to itself, $\tau = \widetilde{\tau}|_{\partial B_n}$ is the induced conformal transform from ∂B_n to itself, $\widetilde{J}$ is the Jacobian of $\widetilde{\tau}$, J is the Jacobian of τ, $\varepsilon = n - 2 + a$. For $f \in L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)$, when $\varepsilon \neq 0$, we have

$$\widetilde{P}_a(J^{\frac{\varepsilon}{2(n-1)}} f \circ \tau) = \widetilde{J}^{\frac{\varepsilon}{2n}} (\widetilde{P}_a f) \circ \widetilde{\tau}. \tag{3.4}$$

It is straightforward to check this property by using the definition of $\widetilde{P}_a$ in (1.12).

Now, for smooth function f, when ε goes to 0 it is obvious that

$$\widetilde{P}_{2-n}(f \circ \tau) = (\widetilde{P}_{2-n} f) \circ \widetilde{\tau}. \tag{3.5}$$

By letting $f = 1$ and taking derivative with respect to ε at 0, we have

$$\frac{d(\widetilde{P}_a 1)}{d\varepsilon}|_{\varepsilon=0} + \widetilde{P}_{2-n}(\frac{1}{2(n-1)} \log J) = \frac{d(\widetilde{P}_a 1)}{d\varepsilon}|_{\varepsilon=0} \circ \widetilde{\tau} + \frac{1}{2n} \log \widetilde{J} \tag{3.6}$$

So the inequality in the Theorem 4 is invariant when F is replaced by $F \circ \tau + \frac{1}{n-1} \log J$.

Proof of Theorem 4. Recalling $\widetilde{P}_{2-n} 1 = 1$, let $f = 1 + \varepsilon F$, where F is some smooth function defined on ∂B_n. By Theorem 3, we have the inequality

$$\|\widetilde{P}_a(1+\varepsilon F)\|_{L^{\frac{2n}{n-2+a}}(B_n)} \leq S_{n,a} \|1+\varepsilon F\|_{L^{\frac{2(n-1)}{n-2+a}}(\partial B_n)},$$

which means

$$\left(\int_{B_n} (\widetilde{P}_a 1)^{\frac{2n}{\epsilon}} (1 + \frac{\varepsilon \widetilde{P}_a F}{\widetilde{P}_a 1})^{\frac{2n}{\varepsilon}} dx \right)^{\frac{1}{n}} \leq S_{n,a}^{\frac{2}{\epsilon}} \left(\int_{\partial B_n} (1+\varepsilon F)^{\frac{2(n-1)}{\varepsilon}} d\xi \right)^{\frac{1}{n-1}}.$$

Note that when $F = 0$ the above inequality becomes equality, then by the following estimates we will see in this case the integrals in both sides will converge to some finite numbers, which means the constant $S_{n,a}^{\frac{2}{\epsilon}}$ will also converge.

In order to take limit $\varepsilon \to 0$, we need to apply the Dominated Convergence Theorem. we will bound the term $\widetilde{P}_a 1$ from below by a constant A and bound $(\widetilde{P}_a 1)^{\frac{2n}{\varepsilon}}$ from above by a constant B, both A and B are independent of ε. Let us derive the lower bound of $\widetilde{P}_a 1$ first. From (1.9) and (1.12) we know

$$(\widetilde{P}_a 1) \circ \phi = (X^2 + (x_n + \frac{1}{2})^2)^{\frac{\varepsilon}{2}} d_{n,a} \int_{\mathbb{R}^{n-1}} \frac{x_n^{1-a}}{((X-Y)^2 + x_n^2)^{\frac{n-a}{2}}} \frac{1}{(Y^2 + \frac{1}{4})^{\frac{\varepsilon}{2}}} d^{n-1}Y,$$

by letting $U = \frac{Y-X}{x_n}$ in the integral, we have

$$\begin{aligned}
(\widetilde{P}_a 1)\circ\phi &= d_{n,a}\int_{\mathbb{R}^{n-1}}\frac{1}{(u^2+1)^{\frac{n-a}{2}}}\frac{(|X|^2+(x_n+\frac{1}{2})^2)^{\frac{\varepsilon}{2}}}{((Ux_n+X)^2+\frac{1}{4})^{\frac{\varepsilon}{2}}}d^{n-1}U\\
&\geq d_{n,0}\int_{|u|\leq 1}\frac{1}{(|U|^2+1)^{n-1}}\frac{(|X|^2+(x_n+\frac{1}{2})^2)^{\frac{\varepsilon}{2}}}{(2|X|^2+2x_n^2+\frac{1}{4})^{\frac{\varepsilon}{2}}}d^{n-1}U\\
&\geq d_{n,0}\int_{|U|\leq 1}\frac{1}{(|U|^2+1)^{n-1}}\frac{1}{2}d^{n-1}U\\
&= A.
\end{aligned}$$

In order to derive the upper bound of $(\widetilde{P}_a 1)^{\frac{2n}{\varepsilon}}$, it is enough to prove that $(\widetilde{P}_a 1)^{\frac{n-2}{\varepsilon}}$ is bounded from above by some constant B independent of ε. As in the proof of lower bound, after the same change of variable we have

$$\begin{aligned}
(\widetilde{P}_a 1)^{\frac{n-2}{\varepsilon}}\circ\phi &= g(X,x_n)\left(d_{n,a}\int_{\mathbb{R}^{n-1}}\frac{1}{(|U|^2+1)^{\frac{n-a}{2}}}\frac{1}{((Ux_n+X)^2+\frac{1}{4})^{\frac{\varepsilon}{2}}}d^{n-1}U\right)^{\frac{n-2}{\varepsilon}}\\
&\leq g(X,x_n)d_{n,a}\int_{\mathbb{R}^{n-1}}\frac{1}{(|U|^2+1)^{\frac{n-a}{2}}}\frac{1}{((Ux_n+X)^2+\frac{1}{4})^{\frac{n-2}{2}}}d^{n-1}U\\
&\leq g(X,x_n)\frac{d_{n,2-n}}{d_{n,0}}d_{n,0}\int_{\mathbb{R}^{n-1}}\frac{1}{(|U|^2+1)^{\frac{n}{2}}}\frac{1}{((Ux_n+X)^2+\frac{1}{4})^{\frac{n-2}{2}}}d^{n-1}U\\
&= g(X,x_n)\frac{d_{n,2-n}}{d_{n,0}}d_{n,0}\int_{\mathbb{R}^{n-1}}\frac{x_n}{((X-Y)^2+x_n^2)^{\frac{n}{2}}}\frac{1}{(|Y|^2+\frac{1}{4})^{\frac{n-2}{2}}}d^{n-1}Y\\
&= \frac{d_{n,2-n}}{d_{n,0}}\\
&= B,
\end{aligned}$$

where $g(X,x_n) = \left(|X|^2+(x_n+\frac{1}{2})^2\right)^{\frac{n-2}{2}}$. For the first inequality we applied Jensen's inequality, since $d_{n,a}\frac{1}{(u^2+1)^{\frac{n-a}{2}}}$ is a probability density in $\mathbb{R}^{n-1}$ and $g(t) = t^{\frac{n-2}{\varepsilon}}$ is convex when $t \geq 0$. The last identity holds because

$$d_{n,0}\int_{\mathbb{R}^{n-1}}\frac{x_n}{((X-Y)^2+x_n^2)^{\frac{n}{2}}}\frac{1}{(Y^2+\frac{1}{4})^{\frac{n-2}{2}}}d^{n-1}Y$$

is the harmonic extension of function $(Y^2+\frac{1}{4})^{-\frac{n-2}{2}}$ which is easy to verify that it is exactly $\frac{1}{(X^2+(x_n+\frac{1}{2})^2)^{\frac{n-2}{2}}}$.

Now we can take limit $\varepsilon \to 0$ safely. By denoting

$$\begin{aligned} I_n &= 2\frac{d(\widetilde{P}_a 1)}{d\varepsilon}|_{\varepsilon=0} \\ &= \left(\log(X^2+(x_n+\frac{1}{2})^2) - d_{n,2-n}\int_{\mathbb{R}^{n-1}} \frac{x_n^{n-1}}{((X-Y)^2+x_n^2)^{n-1}}\log(Y^2+\frac{1}{4})d^{n-1}Y\right)\circ\phi^{-1}, \end{aligned}$$

we get $\|e^{I_n+2\widetilde{P}_{2-n}F}\|_{L^n(B_n)} \leq S_n\|e^{2F}\|_{L^{n-1}(\partial B_n)}$. After replacing $2F$ with F, the inequality in Theorem 4 is proved. Since constant functions are optimizers for the above inequality, conformal invariance of the inequality tells us that the functions

$$F = C + \frac{1}{n-1}\log J$$

are also optimizers.

Remark 3.1.2. *The uniqueness is lost when taking limit in the proof of Theorem 4. It would be interesting to find a suitable method to prove that the optimizer is unique up to a conformal transform. In [9], Beckner proved the uniqueness of optimizers of the higher dimensional Beckner-Onofri's inequality (which is also a limiting inequality of a family of inequalities) by establishing a logarithm inequality which is dual to the original inequality, and for the new inequality one can use the symmetrization technique. The main difficulty for the uniqueness of optimizers of our inequality in Theorem 4 seems to us is that how to get a corresponding dual inequality which could enable us to use the symmetrization technique.*

3.1.3 Carleman type inequality for sub-bi-harmonic functions

Now we are in the situation where $n = 4$ and $a = -2$. By [41] we know that $(P_{-2}f)(X, x_n) = d_{4,-2}\int_{R^3}\frac{x_n^3}{((X-Y)^2+x_n^2)^{n-1}}f(Y)dY$ is the bi-harmonic extension of the function $f(Y)$ with boundary condition $\frac{\partial(P_{-2}f)(X,x_n)}{\partial x_n}|_{x_n=0} = 0$. It is straightforward to check that under the conformal map ϕ the bi-harmonic property and the Neumann boundary condition are preserved in dimension four, we have that $\widetilde{P}_{-2}g$ is a bi-harmonic extension of a function g defined on S^3 to a function on B_4 with boundary condition $\frac{\partial \widetilde{P}_{-2}g}{\partial \gamma}|_{y=0} = 0$. In view of Theorem 4, in order to prove Corollary 2 we only need to verify I_4 satisfies $\Delta^2 I_4 = 0$, $I_4 = 0|_{S^3}$ and $-\frac{\partial I_4}{\partial \gamma} = 1$.

From the formula for I_n, we have

$$I_4 = \left(\log(|X|^2 + (x_n + \frac{1}{2})^2) - d_{4,-2} \int_{R^3} \frac{x_n^3}{((X-Y)^2 + x_n^2)^3} \log(Y^2 + \frac{1}{4}) d^3Y \right) \circ \phi^{-1}.$$

By using the explicit formula of ϕ one can get

$$I_4 = 2 \log |\eta - S| - 2C \int_{S^3} \frac{(1-|\eta|^2)^3}{|\eta - \xi|^6} \log |\xi - S| d\xi,$$

where η is a point in B_4, ξ is a point on S^3, S is the south pole of S^3 and C is the normalizing constant such that $C \int_{S^3} \frac{(1-|\eta|^2)^3}{|\eta-\xi|^6} d\xi = 1$. Now from [41] we have the representation formula for bi-harmonic functions, in fact for a smooth function g on $\overline{B^4}$ we have

$$g(\eta) = \int_{B_4} G_{(\Delta^2, B_4)}(\eta, \varsigma) \Delta^2 g(\varsigma) d\varsigma + C \int_{S^3} \frac{(1-|\eta|^2)^3}{|\eta - \xi|^6} g(\xi) d\xi + D \int_{S^3} \frac{(1-|\eta|^2)^2}{|\eta - \xi|^4} (-\frac{\partial g}{\partial \gamma}(\xi)) d\xi \quad (3.7)$$

where $G_{(\Delta^2, B_4)}$ is the the Green's function with Dirichlet boundary condition and D is a known constant. Note that the positivity of $G_{(\Delta^2, B_4)}$ (see [11],[41]) enables us to use comparison principle to extend the inequality to sub-bi-harmonic case. Although the function $\log |\eta - S|$ is singular at the south pole S, If we apply the forthcoming approximation process we obtain

$$\begin{aligned} I_4 &= 2 \log |\eta - S| - 2C \int_{S^3} \frac{(1-|\eta|^2)^3}{|\eta - \xi|^6} \log |\xi - S| d\xi \\ &= -D \int_{S^3} \frac{(1-|\eta|^2)^2}{|\eta - \xi|^4} d\xi, \end{aligned}$$

since $-\frac{\partial \log|\xi - S|}{\partial\gamma}(\xi) = -1$. In the above equality $-D\int_{S^3}\frac{(1-|\eta|^2)^2}{|\eta-\xi|^4}d\xi$ is the bi-harmonic extension of constant function 0 with boundary condition $-\frac{\partial g}{\partial\gamma}(\xi) = 1$, so I_4 satisfies all three conditions mentioned above.

Since $\log|\eta - S|$ is singular, we use approximation to justify the previous formula for I_4. Take a sequence

$$S_t = (0,0,0,-t) \to S = (0,0,0,-1),$$

as $t \to 1+$. Then $\log|\eta - S_t|$ is a smooth bi-harmonic function on $\overline{B^4}$, so we have

$$\log(\eta - S_t) = C\int_{S^3}\frac{(1-|\eta|^2)^3}{|\eta-\xi|^6}\log|\xi - S_t|d\xi + D\int_{S^3}\frac{(1-|\eta|^2)^2}{|\eta-\xi|^4}\frac{1+ty}{1+t^2+2ty}d\xi,$$

here we use y to denote the last coordinate of ξ. For fixed $\eta \in B_4$, when t approximates 1 from the right, $|\log|\xi - S_t|| \le |\log|\xi - S||$ for ξ in a small neighborhood of S, since $|\log|\xi - S||$ is integrable on S^3, by the Dominated Convergence Theorem

$$C\int_{S^3}\frac{(1-|\eta|^2)^3}{|\eta-\xi|^6}\log|\xi - S_t|d\xi \to C\int_{S^3}\frac{(1-|\eta|^2)^3}{|\eta-\xi|^6}\log|\xi - S|d\xi,$$

as $t \to 1+$. Similarly, when t is close to 1 from right hand side, we have

$$\left|\frac{1+ty}{1+t^2+2ty}\right| \le \frac{1}{2} + \frac{10}{|\xi - S|^2}.$$

Since $\frac{1}{2} + \frac{10}{|\xi-S|^2}$ is integrable on S^3, by the Dominated Convergence Theorem again we have

$$D\int_{S^3}\frac{(1-|\eta|^2)^2}{|\eta-\xi|^4}\frac{1+ty}{1+t^2+2ty}d\xi \to \frac{1}{2}D\int_{S^3}\frac{(1-|\eta|^2)^2}{|\eta-\xi|^4}d\xi,$$

as $t \to 1+$. Now by taking limit $t \to 1+$, it is clear that we have the representation formula for $\log|\eta - S|$. Finally since the kernels in the representation formula (3.7) are positive, we conclude that the inequality in Corollary 2 is true for sub-biharmonic function u with boundary conditions $-\frac{\partial u}{\partial\gamma} = 1$ and $u = 0$ on ∂B_4. Note that the proof works due to the very specific fact that the Green's function of bi-laplacian with Dirichlet boundary condition on balls is positive, and its derivatives have the good sign which allows to apply a point-wise comparison principle. This is an old result due to Boggio [11] and has been extended to perturbations on ball in Grunau-Robert [44].

3.2 Remainder terms in fractional Sobolev inequality

This section is organized as follows. In Subection 3.2.1 we recall the conformal invariance of the problem, and we discuss the framework for an equivalent version of Theorem 5 on the sphere $\mathbb{S}^N \subset \mathbb{R}^{N+1}$, see Theorem 10. In Subsection 3.2.2 we prove this Theorem, thus completing the proof of Theorem 5. In Subsection 3.2.3 we give the proof of Theorem 6.

We conclude by pointing out the open problem to find an explicit constant $\alpha > 0$ in (1.22) via a constructive proof of Theorem 5. For a local version of Theorem 5 where the right hand side of (1.22) is replaced by $\alpha d^2(u, \mathcal{M}) + o(d^2(u, \mathcal{M}))$ and only $u \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ with $d(u, \mathcal{M}) < \|u\|_{s/2}$ is considered, the best constant is $\alpha = \frac{2s}{N+s+2}$. This follows from Proposition 2 below.

3.2.1 Prelinimaries

In the following, we will denote the scalar product in $\mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ by

$$\langle u, v\rangle_{s/2} = \int_{\mathbb{R}^N} |\xi|^s \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi,$$

so that $\|u\|^2_{s/2} = \langle u, u\rangle_{s/2}$ for $u \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$. In the remainder of this section, $0 < s < N$ is fixed and we abbreviate $q = 2n/(N-s)$. We recall that the group of conformal transformations on $\mathbb{R}^N$ is generated by translations, rotations, dilations and the inversion $x \mapsto \frac{x}{|x|^2}$. If h is one of these transformations and J_h is the modulus of its Jacobian determinant, then for any functions $u, v \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ we have $J_h^{\frac{1}{q}} u \circ h, J_h^{\frac{1}{q}} v \circ h \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ and

$$\langle J_h^{\frac{1}{q}} u \circ h, J_h^{\frac{1}{q}} v \circ h \rangle_{s/2} = \langle u, v \rangle_{s/2}. \tag{3.8}$$

This property is a consequence of the conformal covariance of the operator $(-\Delta)^{s/2}$, i.e., of the equality

$$(-\Delta)^{s/2}(J_h^{\frac{1}{q}} u \circ h) = J_h^{\frac{N+s}{2N}} [(-\Delta)^{s/2} u] \circ h \tag{3.9}$$

for all conformal transformations h on $\mathbb{R}^N$ and all Schwartz functions u. As stated in [59, Proposition 2.1], (3.9) is most easily derived by considering the inverse operator $(-\Delta)^{-s/2}$ given in (1.19). Indeed, the identity

$$(-\Delta)^{-s/2}(J_h^{\frac{N+s}{2N}} u \circ h) = J_h^{\frac{1}{q}} [(-\Delta)^{-s/2} u] \circ h \tag{3.10}$$

is equivalent to (3.9), and it can be verified case by case for dilations, rotations, translations and the inversion. In the latter form related to the Riesz potential, the conformal covariance had already been used by Lieb in [50].

Note that, if h is a conformal transformation on $\mathbb{R}^n$, it follows from (3.8) that the map $u \mapsto J_h^{\frac{1}{q}} u \circ h$ preserves distances with respect to the norm $\| \cdot \|_{s/2}$, i.e. we have

$$\| J_h^{\frac{1}{q}} u \circ h - J_h^{\frac{1}{q}} v \circ h \|_{s/2} = \| u - v \|_{s/2} \qquad \text{for all } u, v \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N). \tag{3.11}$$

Since the set $\mathcal{M}$ is also invariant under the transformations $u \mapsto J_h^{\frac{1}{q}} u \circ h$, we conclude that $d(J_h^{\frac{1}{q}} u \circ h, \mathcal{M}) = d(u, \mathcal{M})$ for all $u \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$. We also note that

$$|J_h^{\frac{1}{q}} u \circ h|_q = |u|_q \qquad \text{for any } u \in L^q(\mathbb{R}^N) \tag{3.12}$$

and any conformal transformation h on $\mathbb{R}^N$, which follows by an easy computation. In the following, we consider the inverse stereographic projection

$$\pi : \mathbb{R}^N \to \mathbb{S}^N \subset \mathbb{R}^{N+1}, \qquad \pi(x) = (\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}).$$

We recall that π is a conformal diffeomorphism. More precisely, if $g_{\mathbb{R}^N}$ denotes the flat euclidian metric on $\mathbb{R}^N$ and $g_{\mathbb{S}^N}$ denotes the metric induced by the embedding $\mathbb{S}^N \subset \mathbb{R}^{N+1}$, then the pullback of $g_{\mathbb{S}^N}$ to $\mathbb{R}^N$ satisfies

$$\pi^* g_{\mathbb{S}^N} = \frac{4}{(1+|\cdot|^2)^2} g_{\mathbb{R}^N}. \tag{3.13}$$

Moreover, the corresponding volume element is given by

$$J_\pi(x)dx = \Big(\frac{2}{1+|x|^2}\Big)^N dx, \tag{3.14}$$

For a function $v : \mathbb{S}^N \to \mathbb{R}$, we may now define

$$\mathcal{P}v : \mathbb{R}^N \to \mathbb{R}, \qquad [\mathcal{P}v](x) = J_\pi(x)^{\frac{1}{q}} v(\pi(x)) = \Big(\frac{2}{1+|x|^2}\Big)^{\frac{N-s}{2}} v(\pi(x)).$$

From (3.14), it is easy to see that $\mathcal{P}$ defines an isometric isomorphism between $L^q(\mathbb{S}^N)$ and $L^q(\mathbb{R}^N)$. We also note that

$$\mathcal{P}\,1 = 2^{(N-s)/2} U, \tag{3.15}$$

where 1 stands for unit function on $\mathbb{S}^N$ and U is defined in (1.20). Moreover, $H^{\frac{s}{2}}(\mathbb{S}^N)$ is the completion of the space of smooth functions on $\mathbb{S}^N$ under the norm $\|\cdot\|_*$ induced by scalar product

$$(u, v) \mapsto \langle u, v\rangle_* = \langle \mathcal{P}u, \mathcal{P}v\rangle_{s/2}.$$

We will always consider $H^{\frac{s}{2}}(\mathbb{S}^N)$ with the norm $\|\cdot\|_*$ induced by this scalar product (for matters of convenience, we suppress the dependence on s at this point). Hence, by construction,

$$\mathcal{P} \text{ is also an isometric isomorphism } (H^{\frac{s}{2}}(\mathbb{S}^N), \|\cdot\|_*) \to (\mathring{H}^{\frac{s}{2}}(\mathbb{R}^N), \|\cdot\|_{s/2}).$$

Next we note that $\langle \cdot, \cdot \rangle_*$ is the quadratic form of a unique positive self adjoint operator in $L^2(\mathbb{S}^N)$ which is commonly denoted by A_s in the literature. This operator is formally given by

$$[A_s w] \circ \pi = J_\pi^{-\frac{N+s}{2N}} (-\Delta)^{s/2}(\mathcal{P}w).$$

A key ingredient of the proof of Theorem 5 is the following representation of A_s as a function of the Laplace-Beltrami Operator $\Delta_{\mathbb{S}^N}$ on $\mathbb{S}^N$:

$$A_s = \frac{\Gamma(B + \frac{1+s}{2})}{\Gamma(B + \frac{1-s}{2})} \qquad \text{with } B = \sqrt{-\Delta_{\mathbb{S}^N} + (\frac{N-1}{2})^2}. \tag{3.16}$$

This formula is most easily derived by considering the inverse of A_s and using the Funk-Hecke formula, see [9] and also [59]. It also shows that the domain of A_s coincides with $H^s(\mathbb{S}^N)$. The following statement is a mere reformulation of (3.16).

Proposition 1. *The operator A_s is self adjoint and has compact resolvent. Its spectrum is given as the sequence of eigenvalues*

$$\lambda_k(s) = \frac{\Gamma(\frac{N+s}{2} + k)}{\Gamma(\frac{N-s}{2} + k)}, \qquad k \in \mathbb{N}_0,$$

and the eigenspace corresponding to the eigenvalue $\lambda_k(s)$ is spanned by the spherical harmonics $Y_{k,j}$, $j = 1, \ldots, \binom{k+N}{N} - \binom{k+N-2}{N}$, of degree k.

Next, we note that, via the isometric isomorphism $\mathcal{P}$, inequality (1.15) is equivalent to

$$\|u\|_*^2 \geq \mathcal{S}|u|_q^2 \qquad \text{for all } u \in H^{\frac{s}{2}}(\mathbb{S}^N), \tag{3.17}$$

with $q = \frac{2N}{N-s}$. Here, in accordance with the previous notation, we also write $|\cdot|_r$ for the L^r-norm of a function in $L^r(\mathbb{S}^N)$, $1 \le r \le \infty$. Equality is attained in (3.17) for nontrivial u if and only if $u \in \mathcal{M}_*$, where

$$\mathcal{M}_* := \mathcal{P}^{-1}(\mathcal{M}) = \{v \in H^{\frac{s}{2}}(\mathbb{S}^N) \ : \ \mathcal{P}v \in \mathcal{M}\}.$$

Moreover, the remainder term inequality (1.22) is equivalent to

$$d^2(u, \mathcal{M}_*) \ge \|u\|_*^2 - \mathcal{S}|u|_q^2 \ge \alpha\, d^2(u, \mathcal{M}_*) \quad \text{for } u \in H^{s/2}(\mathbb{S}^N), \tag{3.18}$$

where $d(u, \mathcal{M}_*) = \min\{\|u - \varphi\|_* \ : \ \varphi \in \mathcal{M}\}$. We may therefore reformulate Theorem 5 as follows.

Theorem 10. *There exists a positive constant α depending only on the dimension N and $s \in (0, N)$ such that (3.18) holds.*

We will prove Theorem 10 in Section 3.2.2 below, thus completing the proof of Theorem 5. We close this section with some comments on the conformal invariance of the reformulated problem and the geometry of $\mathcal{M}_*$. Via stereographic projection, the conformal transformations on $\mathbb{S}^N$ are in 1-1-correspondance with the conformal transformations on $\mathbb{R}^N$. So, if τ is an element of the conformal group of $\mathbb{S}^N$ and J_τ is the modulus of its Jacobian determinant, then (3.12) and (3.8) imply that

$$\langle J_\tau^{\frac{1}{q}} u \circ \tau, J_\tau^{\frac{1}{q}} v \circ \tau \rangle_{s/2} = \langle u, v \rangle_* \qquad \text{and} \qquad |J_h^{\frac{1}{q}} u \circ h|_q = |u|_q \tag{3.19}$$

for all $u, v \in H^{\frac{s}{2}}(\mathbb{S}^N)$. From (3.15), we deduce the representation

$$\mathcal{M}_* = \{cJ_\tau^{\frac{1}{q}} \,|\, \tau \text{ is an element of the conformal group of } \mathbb{S}^N, c \in \mathbb{R} \setminus \{0\}\}.$$

The modulus of the Jacobian determinant of a conformal transformation τ on $\mathbb{S}^N$ has the form $J_\tau(\xi) = \tilde{c}(1-\xi\cdot\theta)^{-N}$ for some $\theta \in B^{N+1} := \{x \in \mathbb{R}^{N+1} \ : \ |x| < 1\}$ and some $\tilde{c} > 0$ depending on $|\theta|$ (indeed, one can show that $\tilde{c} = (1-|\theta|^2)^{N/2}$, but we will not need this fact). Thus, $\mathcal{M}_*$ can be viewed as an $N+2$ dimensional smooth manifold embedded in $H^{\frac{s}{2}}(\mathbb{S}^N)$ via the mapping

$$\mathbb{R} \setminus \{0\} \times B^{N+1} \to H^{\frac{s}{2}}(\mathbb{S}^N), \qquad (c,\theta) \mapsto u_{c,\theta}, \tag{3.20}$$

where $u_{c,\theta}(\xi) = c(1-\xi\cdot\theta)^{-\frac{N-s}{2}}$ for $\xi \in \mathbb{S}^N$. This immediately implies that the tangent space $T_1\mathcal{M}_*$ at the function $1 = u_{1,0}$ is generated by the spherical harmonics $Y_0^0 = 1$ and Y_1^j, $j = 1, \ldots, N+1$, given by

$$Y_1^j(\xi) = \xi_j \qquad \text{for } \xi = (\xi_1, \ldots, \xi_{N+1}) \in \mathbb{S}^N \subset \mathbb{R}^{N+1}.$$

Hence $T_1\mathcal{M}_*$ coincides precisely with the generalized eigenspace of the operator A_s corresponding to the eigenvalues $\lambda_0(s)$ and $\lambda_1(s)$. Combining this fact with the minimax characterization of the eigenvalue $\lambda_2(s)$, we readily deduce that

$$\lambda_2(s) = \inf_{v \in T_1\mathcal{M}_*^\perp} \frac{\|v\|^2}{|v|_2^2} \tag{3.21}$$

with

$$T_1\mathcal{M}_*^\perp := \{v \in H^{\frac{s}{2}}(\mathbb{S}^N) \ : \ \langle v, w\rangle_* = 0 \text{ for all } w \in T_1\mathcal{M}_*\}. \tag{3.22}$$

The identity (3.21) will be of crucial importance for the local verification of (3.18) close to the manifold $\mathcal{M}_*$.

3.2.2 Proof of the remainder term inequality on the sphere

We briefly explain the strategy to prove this remainder term inequality which goes back to Bianchi and Egnell [10] in the case $s = 2$. First, the inequality is proved in a small neighborhood of the optimizer $U \in \mathcal{M}$ defined in (1.20). Considering a second order Taylor expansion of the difference functional

$$u \mapsto \Phi(u) := \|u\|_{s/2}^2 - \mathcal{S}\left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{2}{q}},$$

at U, it is not dificult to see that (1.22) holds in a neighborhood of U with some $\alpha > 0$ if and only if the second derivative $\Phi''(U)$ is positive definite on the $(N+2)-$codimensional normal space to the manifold $\mathcal{M}$ at U. This normal non-degeneracy property is the crucial step in the argument. Once inequality (1.22) is established in a neighborhood of U, it extends to a neighborhood of the whole manifold $\mathcal{M}$ as a consequence of the conformal invariance of all terms in (1.21). We will recall this conformal invariance in detail in Section 3.2.1 below. Finally, to obtain the global version of (1.22), a concentration compactness type argument is applied to show that sequences $(u_n)_n$ in $\mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ with $\Phi(u_n) \to 0$ as $n \to \infty$ satisfy $d(u_n, \mathcal{M}) \to 0$ as $n \to \infty$.

The general idea described here had already been used in [10, 56, 7], but the proofs of the normal non-degeneracy property in these papers strongly rely on the assumption that s is an even positve integer and therefore the eigenvalue problem for $\Phi''(U)$ can be written as a differential equation. In particular, ODE arguments are used to study the radial part of the corresponding eigenvalue problem. This method does not apply for general $s \in (0, N)$. On the other hand, one may observe that the eigenvalue problem has a much simpler form once inequality (1.22) is pulled back on the unit sphere $\mathbb{S}^N \subset \mathbb{R}^{N+1}$ via stereographic projection. The equivalent version of Theorem 5 on $\mathbb{S}^N$ is given in Theorem 10 below. The idea of studying (1.15) in its equivalent form on $\mathbb{S}^N$ also goes back to Lieb's paper [50] where the (equivalent) Hardy-Littlewood-Sobolev inequality was considered. Afterwards it has been applied in many related problems dealing with Sobolev type inequalities and corresponding Euler-Lagrange equations, see e.g. [6, 30, 59, 9] and the references therein. To our knowledge, its usefulness to identify remainder terms has not been noted so far.

About twenty years after the seminal work of Bianchi and Egnell [10], the topic of remainder terms in first order Sobolev inequalities (and isoperimetric inequalities) has again attracted a lot of attention in the last years. The recent works use techniques from symmetrization (see, e.g., [22, 38]), optimal transportation (see, e.g., [37]), and fast diffusion (see, e.g., [31, 32, 48]); see also [18] for a recent application of remainder terms. However, while these new methods lead to explicit constants and allow to treat non-Hilbertian Sobolev norms, the estimates for the remainder terms are typically weaker than in the result of Bianchi and Egnell. It is not clear to us whether the symmetrization and the optimal transportation approach can be extended to give remainder terms in the higher order case or in the case of arbitrary real powers of the Laplacian (see [48] for a fast diffusion approach in the fractional case). We therefore think it is remarkable that the original strategy of Bianchi–Egnell can be generalized to the full family of conformally invariant Hilbertian Sobolev inequalities.

We first prove a local variant of Theorem 10.

Proposition 2. *For all $u \in H^{\frac{s}{2}}(\mathbb{S}^N)$ with $d(u, \mathcal{M}_*) < \|u\|_*$, we have*

$$d^2(u, \mathcal{M}_*) \geq \|u\|_*^2 - \mathcal{S}|u|_q^2 \geq \frac{2s}{N+s+2} d^2(u, \mathcal{M}_*) + o(d^2(u, \mathcal{M}_*)). \tag{3.23}$$

Proof. We consider the functional

$$\Psi : H^{\frac{s}{2}}(\mathbb{S}^N) \to \mathbb{R}, \qquad \Psi(u) = \|u\|_*^2 - \mathcal{S}|u|_q^2. \tag{3.24}$$

It is easy to see that Ψ is of class $\mathcal{C}^2$ on $H^{\frac{s}{2}}(\mathbb{S}^N) \setminus \{0\}$. Moreover,

$$\Psi'(u)v = 2\langle u, v\rangle_* - 2\mathcal{S}|u|_q^{2-q} \int_{\mathbb{S}^N} |u|^{q-2} uv \, d\xi \tag{3.25}$$

and

$$\frac{1}{2}\Psi''(u)(v,w) = \langle v,w\rangle_* - \mathcal{S}(2-q)|u|_q^{2-2q}\int_{\mathbb{S}^N}|u|^{q-2}uv\,d\xi\int_{\mathbb{S}^N}|u|^{q-2}uw\,d\xi$$
$$-\mathcal{S}(q-1)|u|_q^{2-q}\int_{\mathbb{S}^N}|u|^{q-2}vw\,d\xi \qquad (3.26)$$

for $u \in H^{\frac{s}{2}}(\mathbb{S}^N)\setminus\{0\}$, $v,w \in H^{\frac{s}{2}}(\mathbb{S}^N)$.

Next, let $u \in H^{\frac{s}{2}}(\mathbb{S}^N)$ with $d(u,\mathcal{M}_*) < \|u\|_*$. It is easy to see that $d(u,\mathcal{M}_*)$ is achieved by some function $cJ_\tau^{\frac{1}{q}}$ in $\mathcal{M}_*$ with $c \in \mathbb{R}\setminus\{0\}$ and a conformal transformation τ on $\mathbb{S}^N$. Replacing u with $\frac{1}{c}J_{\tau^{-1}}^{\frac{1}{q}}u\circ\tau^{-1}$ and using (3.19), we may assume that $c=1$ and $\tau=\mathrm{id}$, hence we may write $u = 1+v$ with $v \in T_1\mathcal{M}_*^\perp$, the normal space of $\mathcal{M}_*$ at 1 defined in (3.22), and $d(u,\mathcal{M}_*) = \|v\|_*$. We note that $\Psi(1)=0$ and $\Psi'(1)=0$ (since the function 1 is a global minimizer of Ψ). Moreover, the condition $v \in T_1\mathcal{M}_*^\perp$ in particular implies – since $1 \in T_1\mathcal{M}_*$ – that

$$\langle 1,v\rangle_* = 0 \qquad \text{and} \qquad \int_{\mathbb{S}^N} v\,d\xi = 0. \qquad (3.27)$$

In particular, we find that

$$\begin{aligned}\Psi(u) &= \Psi(1+v) = \|1\|_*^2 + \|v\|_*^2 - \mathcal{S}|1+v|_q^2 \le \|1\|_*^2 + \|v\|_*^2 - \mathcal{S}|\mathbb{S}^N|^{\frac{2-q}{q}}|1+v|_2^2\\ &= \|1\|_*^2 + \|v\|_*^2 - \mathcal{S}|\mathbb{S}^N|^{\frac{2-q}{q}}(|\mathbb{S}^N| + |v|_2^2) = \Psi(1) + \|v\|_*^2 - \mathcal{S}|\mathbb{S}^N|^{\frac{2-q}{q}}|v|_2^2\\ &\le \|v\|_*^2 = d^2(u,\mathcal{M}_*),\end{aligned}$$

and this yields the first inequality in (3.23). Moreover, from (3.26) and (3.27) we infer that

$$\frac{1}{2}\Psi''(1)(v,v) = \|v\|_*^2 - (q-1)\mathcal{S}|\mathbb{S}^N|^{\frac{2-q}{q}}\int_{\mathbb{S}^N} v^2\,d\xi.$$

A second order Taylor expansion of Ψ at 1 thus yields

$$\begin{aligned}\Psi(u) = \Psi(1+v) &= \frac{1}{2}\Psi''(1)(v,v) + o(\|v\|_*^2)\\ &= \|v\|_*^2 - (q-1)\mathcal{S}|\mathbb{S}^N|^{\frac{2-q}{q}}|v|_2^2 + o(\|v\|_*^2).\end{aligned}$$

Using (1.16) and the identity $|\mathbb{S}^N| = 2\pi^{\frac{N+1}{2}}\Gamma(\frac{N+1}{2})^{-1}$, we find by a short computation (using the duplication formula for the Gamma function) that

$$(q-1)\mathcal{S}|\mathbb{S}^N|^{\frac{2-q}{q}} = \frac{N+s}{N-s}\,\mathcal{S}|\mathbb{S}^N|^{-\frac{s}{N}} = \frac{\Gamma(\frac{N+s}{2}+1)}{\Gamma(\frac{N-s}{2}+1)} = \lambda_1(s).$$

Noting moreover that $|v|_2^2 \le \frac{\|v\|_*^2}{\lambda_2(s)}$ as a consequence of (3.21), we conclude that

$$\Psi(u) \ge \|v\|_*^2\Big(1 - \frac{\lambda_1(s)}{\lambda_2(s)} + o(1)\Big) = d(u,\mathcal{M}_*)^2\Big(\frac{2s}{N+s+2} + o(1)\Big)$$

This shows the second inequality in (3.23). □

The next tool we need is the following property of optimizing sequences for (1.15).

Lemma 10. *Let* $(u_m)_m \subset \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)\setminus\{0\}$ *be a sequence with* $\lim_{m\to\infty}\frac{\|u_m\|_*^2}{|u_m|_q^2} = \mathcal{S}$. *Then* $\frac{d(u_m,\mathcal{M}_*)}{\|u_m\|_*} \to 0$ *as* $m\to\infty$.

Proof. By homogeneity, we may assume that $\|u_m\|_* = 1$ for all $m\in\mathbb{N}$, and we need to show that $d(u_m,\mathcal{M}_*)\to 0$ as $m\to\infty$. We let $v_m = \mathcal{P}u_m \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$ for $m\in\mathbb{N}$; then $\|v_m\|_{s/2} = 1$ for all m, and

$$\frac{1}{|v_m|_q^2} \to \mathcal{S} \qquad \text{as } m\to\infty. \tag{3.28}$$

By the profile decomposition theorem of Gérard (see [42, Théorème 1.1 and Remarque 1.2]) , there exists a subsequence – still denoted by $(v_m)_m$ – and

- a sequence $(\psi_j)_j$ of functions $\psi_j \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N)$,

- an increasing sequence of numbers $l_m \in \mathbb{N}$, $m \in \mathbb{N}$,
- a double sequence of values $h_m^j \in (0,\infty)$, $m, j \in \mathbb{N}$,
- a double sequence of points $x_m^j \in \mathbb{R}^N$, $m, j \in \mathbb{N}$

such that

$$\Big|v_m - \sum_{j=1}^{l_m} \big(h_m^j\big)^{-\frac{s}{2q}} \psi_j\big(\frac{\cdot - x_m^j}{h_m^j}\big)\Big|_q \to 0 \qquad \text{as } m \to \infty, \tag{3.29}$$

$$|v_m|_q^q \to \sum_{j=1}^{\infty} |\psi_j|_q^q \quad \text{as } m \to \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} \|\psi_j\|_{s/2}^2 \le 1. \tag{3.30}$$

Combining the Sobolev inequality (1.15) with (3.30) and using the concavity of the function $t \mapsto t^{2/q}$, we find that

$$1 \ge \mathcal{S} \sum_{j=1}^{\infty} |\psi_j|_q^2 \ge \mathcal{S}\Big(\sum_{j=1}^{\infty} |\psi_j|_q^q\Big)^{2/q} = \mathcal{S} \lim_{m\to\infty} |v_m|_q^2. \tag{3.31}$$

By (3.28), equality holds in all steps in (3.31). The strict concavity of the function $t \mapsto t^{2/q}$ then shows that $\psi_j \equiv 0$ for all but one $j \in \mathbb{N}$, say, $j = 1$, where $\mathcal{S}|\psi_1|_q^2 = 1$ and $\|\psi_1\|_{s/2} = 1$ as a consequence of (3.30), (3.31) and the Sobolev inequality (1.15). Hence $\Psi_1 \in \mathcal{M}$, and from (3.29) it now follows that

$$\Big|v_m - \big(h_m^1\big)^{-\frac{s}{2q}} \psi_1\big(\frac{\cdot - x_m^1}{h_m^1}\big)\Big|_q \to 0 \qquad \text{as } m \to \infty.$$

Therefore, defining

$$\tilde{v}_m \in \overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N), \quad \tilde{v}_m(x) = \big(h_m^1\big)^{\frac{s}{2q}} v_m(h_m^1 x + x_m^1) \qquad \text{for } m \in \mathbb{N},$$

we have $\tilde{v}_m \to \psi_1$ in $L^q(\mathbb{R}^N)$ for $m \to \infty$, but then also $\tilde{v}_m \to \psi_1$ in $\overset{\circ}{H}{}^{\frac{s}{2}}(\mathbb{R}^N)$ strongly since $\|\tilde{v}_m\|_{s/2} = \|v_m\|_{s/2} = 1 = \|\psi_1\|_{s/2}$ for all $m \in \mathbb{N}$. Consequently, $d(\tilde{v}_m, \mathcal{M}) \to 0$. By the invariance property (3.11), we then have $d(v_m, \mathcal{M}) \to 0$ and therefore also $d(u_m, \mathcal{M}_*) \to 0$ as $m \to \infty$, since $\mathcal{P}$ is an isometry. □

Remark 3.2.1. (i) In the proof given above, we do not need the full strength of Gérard's profile decomposition theorem. Inductively, Gérard writes v_m as an infinite sum of bubbles, see (3.29) and [42]. For our proof it is enough to stop this procedure after the very first step. As soon as one bubble is extracted, the strict concavity of the function $t \mapsto t^{2/q}$ implies the convergence.

(ii) In the case where $s \in (0, N)$ is an even integer, Lemma 10 follows directly from a classical concentration compactness result of Lions, see [52, Corollary 1]. For arbitrary $s \in (0, N)$, one could also use the duality between (1.15) and (1.18) and another concentration compactness result of Lions about optimizing sequences for (1.18), see [53, Theorem 2.1]. To us it seemed more natural to use a technique directly applicable to optimizing sequences for (1.15).

With the help of Proposition 2 and Lemma 10, we may now complete the

Proof of Theorem 10. Let $u \in H^{\frac{s}{2}}(\mathbb{S}^N)$. Since $0 \in \overline{\mathcal{M}_*}$, we have $d(u, \mathcal{M}_*) \leq \|u\|_*$. If $d(u, \mathcal{M}_*) < \|u\|_*$, then the first inequality in (3.18) follows from Proposition 2, and it is trivially satisfied if $d(u, \mathcal{M}_*) = \|u\|_*$. To prove the second inequality in (3.18) for some $\alpha > 0$, we argue by contradiction. For this we assume that there exists a sequence $(u_m)_m$ in $H^{\frac{s}{2}}(\mathbb{S}^N) \setminus \overline{\mathcal{M}_*}$ with

$$\frac{\|u_m\|_*^2 - \mathcal{S}|u_m|_q^2}{d^2(u_m, \mathcal{M}_*)} \to 0 \quad \text{as } m \to \infty. \tag{3.32}$$

By homogeneity we can assume that $\|u_m\|_* = 1$ for all $m \in \mathbb{N}$, then $d(u_m, \mathcal{M}_*) \leq 1$ for all $m \in \mathbb{N}$ and therefore (3.32) implies that $\lim_{m\to\infty} |u_m|_q^2 = \frac{1}{\mathcal{S}}$. Hence Lemma 10 gives $d(u_m, \mathcal{M}) \to 0$ as $m \to \infty$. But then Proposition 2 shows that (3.32) must be false. We conclude that there exists $\alpha > 0$ such that

$$\|u\|_*^2 - \mathcal{S}|u|_q^2 \geq \alpha\, d^2(u, \mathcal{M}_*) \qquad \text{for all } u \in H^{\frac{s}{2}}(\mathbb{S}^N),$$

as claimed. □

3.2.3 The weak $L^{q/2}$ remainder term inequality for domains of finite measure

In this section we give the proof of Theorem 6. For this we define

$$U_{\lambda,y} \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N), \qquad U_{\lambda,y}(x) := \lambda U(\lambda^{\frac{2}{N-s}}(x-y))$$

for $c \in \mathbb{R} \setminus \{0\}, \lambda > 0$ and $y \in \mathbb{R}^N$, so that

$$\mathcal{M} = \{cU_{\lambda,y} \,:\, c \in \mathbb{R} \setminus \{0\}, \lambda > 0, y \in \mathbb{R}^N\}.$$

It will be convenient to adjust the notation for the weak $L^{q/2}$-norm. We fix $q = \frac{2N}{N-s}$ from now on, and we write

$$|u|_{w,\Omega} = \sup_{\substack{A \subset \Omega \\ |A|>0}} |A|^{-\frac{s}{N}} \int_A |u|\, dx.$$

for the weak $L^{q/2}$-norm of a measurable function u defined on a measurable set $\Omega \subset \mathbb{R}^N$. We note the following scaling property, which follows by direct computation:

$$|U_{\lambda,y}|_{w,\mathbb{R}^N} = |U_{\lambda,0}|_{w,\mathbb{R}^N} = \frac{|U|_{w,\mathbb{R}^N}}{\lambda} \qquad \text{for } \lambda > 0,\ y \in \mathbb{R}^N. \tag{3.33}$$

Similarly, for a fixed domain $\Omega \subset \mathbb{R}^N$, $u \in \mathring{H}^{\frac{s}{2}}(\Omega)$ and $\lambda > 0$, define

$$\Omega_\lambda := \lambda^{-2/(N-s)}\Omega \subset \mathbb{R}^N \qquad \text{and} \qquad u_\lambda \in \mathring{H}^{\frac{s}{2}}(\Omega_\lambda), \quad u_\lambda(x) = \lambda u(\lambda^{\frac{2}{N-s}}x).$$

Then a direct computation shows

$$|\Omega_\lambda| = \lambda^{-q}|\Omega|, \quad |u_\lambda|_{w,\Omega_\lambda} = \frac{|u|_{w,\Omega}}{\lambda} \quad \text{and} \quad d(u_\lambda, \mathcal{M}) = d(u, \mathcal{M}). \tag{3.34}$$

Theorem 6 will follow immediately from the following Proposition.

Proposition 3. *There exists a constant C_0 depending only on N and $s \in (0, N)$ such that*

$$|u|_{w,\Omega} \le C_0 |\Omega|^{\frac{1}{q}}\, d(u, \mathcal{M}) \tag{3.35}$$

for all subdomains $\Omega \subset \mathbb{R}^N$ with $|\Omega| < \infty$ and all $u \in \mathring{H}^{\frac{s}{2}}(\Omega)$.

Proof. By the scaling properties noted in (3.34), it suffices to consider a subdomain $\Omega \subset \mathbb{R}^N$ with $|\Omega| = 1$ in the sequel. In this case we have, by Hölder's inequality and (1.15),

$$|u|_{w,\Omega} \le \|u\|_{L^q(\Omega)} \le \|u\|_{L^q(\mathbb{R}^N)} \le \frac{1}{\sqrt{\mathcal{S}}}\|u\|_{s/2} \quad \text{for every } u \in \mathring{H}^{\frac{s}{2}}(\mathbb{R}^N). \tag{3.36}$$

In the following, let $\rho \in (0, 1)$ be given by

$$\frac{\rho}{\sqrt{\mathcal{S}}(1-\rho)} = \Big(|\mathbb{S}^{N-1}| \int_1^\infty \frac{r^{N-1}}{(1+r^2)^N}\, dr\Big)^{\frac{1}{q}} \tag{3.37}$$

Let $u \in \mathring{H}^{\frac{s}{2}}(\Omega)$. If $\rho\|u\|_{s/2} \le d(u, \mathcal{M})$, then

$$|u|_{w,\Omega} \le \frac{1}{\rho\sqrt{\mathcal{S}}} d(u, \mathcal{M}) \tag{3.38}$$

as a consequence of (3.36). So in the remainder of this proof we assume that

$$\rho\|u\|_{s/2} > d(u, \mathcal{M})\,. \tag{3.39}$$

By homogeneity we may assume that $\|u\|_{s/2} = 1$. Since $\rho < 1$, the infimum in the definition of $d(u, \mathcal{M})$ is attained as a consequence of (3.39), and we have $d(u, \mathcal{M}) = \|u - cU_{\lambda,y}\|_{s/2}$ for some $c \in \mathbb{R}$, $\lambda > 0$ and $y \in \mathbb{R}^n$. Moreover, (3.39) implies that

$$|1 - c| = \big|\|u\|_{s/2} - \|cU_{\lambda,y}\|_{s/2}\big| \le d(u, \mathcal{M}) \le \rho\,,$$

that is, $1-\rho \leq c \leq 1+\rho$. We note that

$$\begin{aligned} d(u,\mathcal{M})^2 &= \|u - cU_{\lambda,y}\|_{s/2}^2 \geq \mathcal{S}\|u - cU_{\lambda,y}\|_{L^q(\mathbb{R}^N)}^2 \\ &\geq \mathcal{S}|c|^2 \|U_{\lambda,y}\|_{L^q(\mathbb{R}^N\setminus\Omega)}^2 \geq \mathcal{S}(1-\rho)^2\|U_{\lambda,y}\|_{L^q(\mathbb{R}^N\setminus\Omega)}^2 . \end{aligned}$$

Now let $B \subset \mathbb{R}^N$ denote the open ball centered at zero with $|B| = 1$, and let $r_0 > 0$ denote the radius of B. Since the function U in (1.20) is radial and strictly decreasing in the radial variable, the bathtub principle [51, Theorem 1.14] implies that

$$\|U_{\lambda,y}\|_{L^q(\mathbb{R}^N\setminus\Omega)}^2 \geq \|U_{\lambda,y}\|_{L^q(\mathbb{R}^N\setminus(B+y))}^2 = \|U_{\lambda,0}\|_{L^q(\mathbb{R}^N\setminus B)}^2 ,$$

and hence

$$\|U_{\lambda,0}\|_{L^q(\mathbb{R}^N\setminus B)}^q \leq \Big(\frac{d(u,\mathcal{M})}{\sqrt{\mathcal{S}}(1-\rho)}\Big)^q \leq \Big(\frac{\rho}{\sqrt{\mathcal{S}}(1-\rho)}\Big)^q = |\mathbb{S}^{N-1}| \int_1^\infty \frac{r^{N-1}}{(1+r^2)^N}\, dr \tag{3.40}$$

by our choice of ρ in (3.37). On the other hand, we compute

$$\|U_{\lambda,0}\|_{L^q(\mathbb{R}^N\setminus B)}^q = |\mathbb{S}^{N-1}| \int_{r_0}^\infty \frac{r^{N-1}\lambda^q}{\big[1+(\lambda^{\frac{2}{N-s}}r)^2\big]^N}\, dr = |\mathbb{S}^{N-1}| \int_{\lambda^{\frac{2}{N-s}}r_0}^\infty \frac{r^{N-1}}{(1+r^2)^N}\, dr$$

This implies that $\lambda^{\frac{2}{N-s}}r_0 \geq 1$ and therefore

$$\begin{aligned} \|U_{\lambda,0}\|_{L^q(\mathbb{R}^N\setminus B)}^q &= |\mathbb{S}^{N-1}| \int_{\lambda^{\frac{2}{N-s}}r_0}^\infty \frac{r^{N-1}}{(1+r^2)^N}\, dr \\ &\geq 2^{-N}|\mathbb{S}^{N-1}| \int_{\lambda^{\frac{2}{N-s}}r_0}^\infty \frac{dr}{r^{N+1}} = \frac{|\mathbb{S}^{N-1}|}{N\,(2r_0)^N}\,\lambda^{-q}. \end{aligned} \tag{3.41}$$

Combining (3.40) and (3.41), we conclude that

$$d(u,\mathcal{M}) \geq \frac{C_1}{\lambda} \qquad \text{with } C_1 := \sqrt{\mathcal{S}}(1-\rho)\Big(\frac{|\mathbb{S}^{N-1}|}{N\,(2r_0)^N}\Big)^{\frac{1}{q}}. \tag{3.42}$$

Using (3.33), (3.36) and (3.42), we find that

$$\begin{aligned}|u|_{w,\Omega} &\leq |cU_{\lambda,y}|_{w,\Omega} + |u - cU_{\lambda,y}|_{w,\Omega} \leq (1+\rho)|U_{\lambda,y}|_{w,\mathbb{R}^N} + \frac{1}{\sqrt{\mathcal{S}}}\|u - cU_{\lambda,y}\|_{s/2} \\ &= \frac{1+\rho}{\lambda}|U|_{w,\mathbb{R}^N} + \frac{1}{\sqrt{\mathcal{S}}}d(u,\mathcal{M}) \leq C_2 d(u,\mathcal{M})\end{aligned}$$

with $C_2 := \frac{(1+\rho)}{C_1}|U|_{w,\mathbb{R}^N} + \frac{1}{\sqrt{\mathcal{S}}}$. Combining this with (3.38), we thus obtain the claim with $C_0 := \max\{C_2, \frac{1}{\rho\sqrt{\mathcal{S}}}\}$. □

Finally, Theorem 6 now simply follows by combining Theorem 5 and Proposition 3 and setting $C := \alpha C_0^{-2}$.

Chapter 4

Regularity results in some applications of optimal transportation

In the first section of this chapter we give the proof for the C^1 regularity of the solution to the principal-agent problem. The proof is based on a perturbation argument. The second section represents joint work with Indrei; we obtain some regularity results of the free boundary in optimal partial transport with general cost.

4.1 Regularity of the solution to the principal-agent problem

Before embarking on the proof, we will give an interesting lemma that will be used later.

Lemma 11. [*21*] *Suppose $u(p)$ is a convex function defined on a bounded convex domain $\Omega \subset \mathbb{R}^n$ which contains the origin 0. If u is singular at 0 (namely $\partial u(0)$ contains more than one point) and $0 \in ri\partial u(0)$, where $ri\partial u(0)$ denotes the relative interior of the set*

$\partial u(0)$ (a set consists of all sub-gradients of u at 0). Then we have the estimate

$$\int_{S_\epsilon} |Du(p)|^2 dp \geq C|S_\epsilon|, \tag{4.1}$$

where $S_\epsilon := \{p \in \Omega | u(p) \leq \epsilon\}$, $|S_\epsilon|$ is its volume, and C is a constant independent of ϵ.

This lemma was first proved by Carlier and Lachand-Robert in [21] by using a blow-up analysis and a compactness argument, because of the independent interest of this lemma, we will provide a more direct constructive proof of it, and in particular, the constant C can be explicitly estimated. The proof is very much inspired by [14].

Proof of Lemma 11. Since $0 \in \mathrm{ri}\partial u(0)$, by rotating the coordinate system we can assume $u(p) \geq k|p_1|$ for some positive k, where $p = (p_1, p')$, and $p' = (p_2, p_3, \cdots, p_n)$. In the following, $\mathrm{Proj}(S_\epsilon)$ denotes the orthogonal projection of S_ϵ on the hyperplane $\{p | p_1 = 0\}$, and $\frac{1}{2}S_\epsilon$ denotes a $\frac{1}{2}$-dilation of S_ϵ with respect to the center 0. Now, we estimate the integration of $|Du|^2$ along the segment $I_{p'} := \{p | \mathrm{Proj}(p) = p', p \in S_\epsilon\}$, where $p' \in \mathrm{Proj}(\frac{1}{2}S_\epsilon)$. Let $\tilde{I}_{p'} := [a, a+d] \times \{p'\}$ be one of of the two components of $I_{p'} - \frac{1}{2}S_\epsilon$, without loss of generality we take the upper one, so by convexity we have $u(a, p') \leq \frac{1}{2}\epsilon$, and $u(a+d, p') = \epsilon$. Then we have

$$\begin{aligned} \int_{I_{p'}} |Du|^2 dp_1 &\geq \int_{\tilde{I}_{p'}} |\frac{\partial u}{\partial p_1}|^2 dp_1 \\ &= \int_a^{a+d} |\frac{\partial u}{\partial p_1}|^2 dp_1 \\ &\geq (\frac{\frac{\epsilon}{2}}{d})^2 d = \frac{\epsilon^2}{4d}, \end{aligned}$$

the last " $\geq$ " follows from the convexity of u. Since $u(p) \geq k|p_1|$, we have $S_\epsilon \subset$

$[-\frac{\epsilon}{k}, \frac{\epsilon}{k}] \times \mathrm{Proj}(S_\epsilon)$, and $d \leq \frac{2\epsilon}{k}$. Therefore

$$\begin{aligned}
\int_{S_\epsilon} |Du|^2 dp &\geq \int_{\mathrm{Proj}(\frac{1}{2}S_\epsilon)} \frac{\epsilon^2}{4\frac{2\epsilon}{k}} dp' \\
&\geq \frac{k}{8}\epsilon |\mathrm{Proj}(\frac{1}{2}S_\epsilon)| \\
&= \frac{k}{2^{n+2}}\epsilon |\mathrm{Proj}(S_\epsilon)| \\
&\geq \frac{k}{2^{n+2}}\epsilon \frac{|S_\epsilon|}{\frac{2\epsilon}{k}} = \frac{k^2}{2^{n+3}}|S_\epsilon|.
\end{aligned}$$

So the Lemma 1 is proved with the constant $\frac{k^2}{2^{n+3}}$.

Now we start the proof of Theorem 8, the main ingredient in the argument is that we do the change of variables as Figalli, Kim and McCann did in [36]. In the new variables, a b-convex function u in the original variables becomes a $\tilde{b}$-convex function with an extra property that it is also convex in the usual sense. Another advantage is that constant functions are $\tilde{b}$-convex, which allows us to perturb the minimizer by using constant functions safely.

Firstly, we do a change of variable exactly same as that in [36]. By **(B0)**-**(B1)**, the map $x \in \overline{\mathbf{X}} \mapsto p = D_y b(x, y_0) \in \overline{\mathbf{X}}_{y_0}$ and its inverse $p \in \overline{\mathbf{X}}_{y_0} \mapsto x = x_b(y_0, p) \in \overline{\mathbf{X}}$ are C^3 diffeomorphisms, where y_0 is the null product as in the minimization problem. Then $\tilde{u}(p) := u(x_b(y_0, p)) - b(x_b(y_0, p), y_0) + c(y_0)$ is a non-negative $\tilde{b}$-convex function, where $\tilde{b}(p, y) := b(x_b(y_0, p), y) - b(x_b(y_0, p), y_0) + c(y_0)$. In fact, the above correspondence between u and $\tilde{u}$ defines a $1-1$ map between the space $\mathcal{U}_0$ and the space $\widetilde{\mathcal{U}}_0$ of non-negative $\tilde{b}$-convex function on $\overline{\mathbf{X}}_{y_0}$. By switching x and y in Remark 1.3.1, we see that $\tilde{u}(p)$ is a convex function. One should also notice that $\tilde{b}$ satisfies the same condition **(B0)** $-$ **(B3)** as b does, except that $\tilde{b}$ might be only C^3 on the first variable.

Now, suppose $\tilde{u}$ is minimizer and it is not C^1 at a point $p_0 \in \mathbf{X}_{y_0}$, namely $\partial\tilde{u}(p_0)$ contains more than one point. We claim that *one can find a point* $y_1 \in \overline{\boldsymbol{Y}}$, *such that* $D_p\tilde{b}(p_0, y_1) \in ri\partial\tilde{u}(p_0)$.

Proof of the claim. Note that $\partial\tilde{u}(p_0)$ is a bounded convex set, so it is the convex hull of its extreme points. Since $\tilde{u}$ is a convex function, for any extreme point ω of $\partial\tilde{u}(p_0)$ we can find a sequence of points p_i at which u is differentiable, so that $p_i \to p_0$ and $D\tilde{u}(p_i) \to \omega$, as $i \to \infty$. By **(B1)**, we have $D\tilde{u}(p_i) = D_p\tilde{b}(p_i, z_i)$ for a unique $z_i \in \mathbf{Y}$. Therefore, by passing to a subsequence we can assume $z_i \to y_1$ for some $y_1 \in \overline{\mathbf{Y}}$, then by taking limit we have $\omega = D_p(p_0, y_1)$. From the above discussion we see that all extreme points of $\partial\tilde{u}(p_0)$ are contained in $D_p\tilde{b}(p_0, \overline{\mathbf{Y}})$, which is a convex set by **(B2)**, so $\partial\tilde{u}(p_0) \subset D_p\tilde{b}(p_0, \overline{\mathbf{Y}})$.

Then we perform the second change of variables. By **(B1)** we have that the C^2 diffeomorphism $p \in \overline{\mathbf{X}}_{y_0} \mapsto \bar{p} := D_y\tilde{b}(p_0, y_1) \in \overline{\mathbf{X}}_{y_1}$ and its inverse $\bar{p} \mapsto p_{\tilde{b}}(y_1, \bar{p})$ are well defined. Then $\bar{u}(\bar{p}) := \tilde{u}(p_{\tilde{b}}(y_1, \bar{p})) - \tilde{b}(p_{\tilde{b}}(y_1, \bar{p}), y_1) - \lambda_1$ is a $\bar{b}$-convex function, where $\lambda_1 = \tilde{u}(p_0) - \tilde{b}(p_0, y_1)$, and $\bar{b} := \tilde{b}(p_{\tilde{b}}(y_1, \bar{p}), y) - \tilde{b}(p_{\tilde{b}}(u_1, \bar{p}), y_1) - \lambda_1$ satisfies **(B0)-(B3)**, except that it might be only C^2 on its first variable. By the same reason as in the first change of variables, $\bar{u}$ is not only $\bar{b}$-convex but also convex in the usual sense. Note that by the choice of y_1 in the above claim, $0 \in \mathrm{ri}\partial\bar{u}(\bar{p}_0)$, where $\bar{p}_0 = D_y\tilde{b}(p_0, y_1)$. In these new variables, the principal's net losses are given by

$$\widetilde{L}(\bar{u}) = \int_{\mathbf{X}_{y_1}} [c(y_{\bar{b}}(\bar{p}, D\bar{u})) - \bar{b}(\bar{p}, y_{\bar{b}}(\bar{p}, D\bar{u})) + \bar{(}u)]f(x(\bar{p}))\det(\frac{\partial x^i}{\partial \bar{p}_j})d\bar{p},$$

and the space of admissible functions becomes

$$\overline{\mathcal{U}}_0 = \{\bar{u} | \bar{u} \text{ is } \bar{b} - \text{convex and } \bar{u} \geq \tilde{b}(p_{\tilde{b}}(y_1, \bar{p})), y_1 + \lambda_1\}.$$

Note that in the new variables, $c(y)$ still satisfies Condition 1, namely there exists some positive δ so that $c(y_{\bar{b}}(\bar{p}, q)) - \bar{b}(\bar{p}, y_{\bar{b}}(\bar{p}, q)) - \delta|q|^2$ is convex in the q variable for all $\bar{p} \in \mathbf{X}_{y_1}$. It is also easy to see that after the above change of variables, constant functions are $\bar{b}$-convex.

In the following, for simplicity of notations, we will use $b(p, y)$ instead of $\bar{b}(\bar{p}, y)$, p instead of $\bar{p}$, u instead of $\bar{u}$, and $f(p)dp$ instead of $f(x(\bar{p}))\det(\frac{\partial x^i}{\partial \bar{p}_j})d\bar{p}$. We will also omit the subscript $\bar{b}$. So the functional in the minimization problem looks like

$$\widetilde{L}(u) = \int_{\mathbf{X}_{y_1}} [c(y(p, Du)) - b(p, y(p, Du)) + u] f(p) dp.$$

Now suppose u is a minimizer, by the second change of variables, we see that $u \geq 0$. Let $u_\epsilon := \max\{u, \epsilon\}$ and $S_\epsilon := \{u \leq \epsilon\}$. Since constant function ϵ is b-convex, u_ϵ is still in the space of admissible functions. Denote $G(p, q) = c(y(p, q)) - b(p, y(p, q)) - \delta|q|^2$. We have

$$\widetilde{L}(u_\epsilon) - \widetilde{L}(u) = I_1 + I_2 + I_3,$$

where

$$I_1 = -\delta \int_{S_\epsilon} |Du|^2 f(p) dp, \quad I_2 = \int_{S_\epsilon} (G(p, 0) - G(p, Du)) f(p) dp,$$

and

$$I_3 = \int_{S_\epsilon} (\epsilon - u) f(p) dp.$$

Since $0 \leq u \leq \epsilon$ in S_ϵ, we have $I_3 \leq C_1 \epsilon |S_\epsilon|$, for some constant C_1. For I_2, we have

$$\begin{aligned} I_2 &\leq \int_{S_\epsilon} D_q G(p, 0) \cdot D(\epsilon - u) f(p) dp \\ &= \int_{\partial S_\epsilon \cap \partial \mathbf{X}_{y_1}} (\epsilon - u) f(p) D_q G(p, 0) \cdot \overrightarrow{n} d\xi \\ &- \int_{S_\epsilon} (\epsilon - u) f(p) \mathrm{div}_p (D_q G(p, 0)) dp - \int_{S_\epsilon} (\epsilon - u) Df(p) \cdot D_q G(p, 0) dp, \end{aligned}$$

where the inequality follows from the convexity of G in q variable, and the equality follows from the divergence theorem. Since b is at least C^2 and f is $W^{1,\infty}$, all the integrand in the above three integrals are bounded by $C\epsilon$, for some constant C. For the area of $\partial S_\epsilon \cap \partial \mathbf{X}_{y_1}$, we need to use a simple estimate

$$|\partial S_\epsilon \cap \partial \mathbf{X}_{y_1}| \leq C |S_\epsilon|, \tag{4.2}$$

where C is a constant. This estimate was proved by Carlier and Lachand-Robert [21] and for reader's convenience we include their proof here.

Proof of the estimate 4.2 [21].

$$|S_\epsilon| = \frac{1}{n}\int_{S_\epsilon} \operatorname{div}(p-p_0)dp \geq \frac{1}{n}\int_{\partial S_\epsilon \cap \partial \mathbf{X}_{y_1}} (p-p_0)\cdot \overrightarrow{n}\, d\xi \geq \frac{1}{n} C|\partial S_\epsilon \cap \partial \mathbf{X}_{y_1}|,$$

where p_0 is the assumed singular point of u, and the last inequality follows from the convexity of X_{y_1} (by the convexity, $(p-p_0)\cdot\overrightarrow{n}$ is bounded from below by some positive constant).

By this estimate (4.2), we have $I_2 \leq C_2\epsilon|S_\epsilon|$, for some constant C_2. Then by Lemma 11, we have

$$\widetilde{L}(u_\epsilon) - \widetilde{L}(u) \leq -C_1\delta|S_\epsilon| + C_2\epsilon|S_\epsilon| + C_3\epsilon|S_\epsilon|,$$

note that the constants C_1, C_2, C_3 are all independent of ϵ. Recall that δ is a fixed positive constant in Condition 1, so when ϵ is sufficiently small, we see that $\widetilde{L}(u_\epsilon) < \widetilde{L}(u)$, which contradicts the fact that u is a minimizer. So u must be C^1.

4.2 Regularity of the free boundary in the optimal partial transport problem for general cost functions

4.2.1 Preliminaries

Definition 4.2.1. *Given an $(m-1)$-plane π in $\mathbb{R}^m$, we denote a general cone with respect to π by*

$$C_\alpha(\pi) := \{z \in \mathbb{R}^m : \alpha|P_\pi(z)| < P_{\pi^\perp}(z)\},$$

where $\pi \oplus \pi^{\perp} = \mathbb{R}^m$, $\alpha > 0$, *and* $P_{\pi}(z)$ & $P_{\pi^{\perp}}(z)$ *are the orthogonal projections of* $z \in \mathbb{R}^m$ *onto* π *and* $\pi^{\perp}$, *respectively.*

Definition 4.2.2. *A domain* D *is said to satisfy the uniform interior cone condition if there exists* $\alpha > 0$ *and* $\delta > 0$ *such that for all* $x \in \partial D$ *there exists* $\nu_x \in \mathbb{S}^{n-1}$ *so that*

$$(y + C_{\alpha}(\nu_x^{\perp})) \cap B_{\delta}(x) \subset D \cap B_{\delta}(x),$$

for all $y \in D \cap B_{\delta}(x)$. *We define the profile of such domains to be the ordered pair* (δ, α).

Definition 4.2.3. *A domain* $D \subset \mathbb{R}^n$ *is said to satisfy a uniform interior ball condition if there exists* $r > 0$ *such that for all* $x \in \partial D$, *there exists* $\nu_x \in \mathbb{S}^{n-1}$ *for which* $B_r(x + r\nu_x) \subset D$.

Definition 4.2.4. *We denote by* $\mathcal{F}$, *the collection of cost functions* $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *that satisfy the following three conditions:*

1. $c \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$;

2. $c(x, y) \geq 0$ *and* $c(x, y) = 0$ *only for* $x = y$;

3. *For* $x, p \in \mathbb{R}^n$, *there exists a unique* $y = y(x, p) \in \mathbb{R}^n$ *such that* $\nabla_x c(x, y) = p$; *similarly, for any* $y, q \in \mathbb{R}^n$, *there exists a unique* $x = x(y, q) \in \mathbb{R}^n$ *such that* $\nabla_y c(x, y) = q$.

We denote by $\mathcal{F}_0$, *the set of* $C^4(\mathbb{R}^n \times \mathbb{R}^n)$ *cost functions in* $\mathcal{F}$ *that satisfy:*

4. $\det(\nabla_{(x,y)} c) \neq 0$ *for all* $x, y \in \mathbb{R}^n$;

5.(A3S) For $x, p \in \mathbb{R}^n$,

$$A_{ij,kl}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \ \ \forall \ \ \xi, \eta \in \mathbb{R}^n, \langle \xi, \eta \rangle = 0, c_0 > 0.$$

Definition 4.2.5. *A set $U \subset \mathbb{R}^n$ is called c-convex with respect to another set $V \subset \mathbb{R}^n$ if the image $c_y(U, y)$ is convex for each $y \in V$.*

Lemma 4.2.6. *Let $c \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$, and consider two domains $\Omega \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^n$ with disjoint closures, and set*

$$b_0 = \inf_{x \in \overline{\Omega}, y \in \overline{\Lambda}} c(x, y) > 0. \tag{4.3}$$

Then for any $b \geq b_0$ and $y \in \mathbb{R}^n$, the domain $E_y^b := \{x : c(x, y) < b\}$ satisfies a uniform interior cone condition with radius $r = r(b_0, ||c||_{C^2}) > 0$.

Proof. Let

$$c_1 := \inf_{x \in \overline{\Omega}, y \in \overline{\Lambda}} |\nabla_x c(x, y)|, \tag{4.4}$$

and note that since $c \in \mathcal{F}$ we have $c_1 > 0$. Indeed, suppose on the contrary that there exists $(\bar{x}, \bar{y}) \in \overline{\Omega} \times \overline{\Lambda}$ for which $\nabla_x c(\bar{x}, \bar{y}) = 0$. Let $\phi(x) := c(x, \bar{x})$; using condition 2 in the definition of $\mathcal{F}$, $\phi(x) \geq 0$ and $\phi(x) = 0$ only for $x = \bar{x}$. Therefore, $\nabla_x c(\bar{x}, \bar{x}) = 0$, but by uniqueness, we must have $\bar{x} = \bar{y}$ (using condition 3), contradicting the disjointness assumption (hence, $c_1 > 0$ depends on $b_0 > 0$). Now for a fixed $y_0 \in \Lambda$, denote $\phi(x) := c(x, y_0)$. Then for a fixed point $x_0 \in \{x : \phi(x) = b\}$, we choose a coordinate system such that x_n is the direction of the normal to the level set pointing into the sublevel set $\{x : \phi(x) \leq b\}$ and x_0 is the origin. Now let $r := \frac{c_1}{c_2}$, where $c_2 = ||c||_{C^2}$, and consider the ball B_r centered at $(0, \ldots, r)$ with radius r. In particular ∂B_r touches the origin. Now we will show that $B_r \subset \{x : \phi(x) < b\}$: indeed, it is simple to see (by forming similar triangles) that $\cos(\theta) > \frac{|x|}{2r} = \frac{|x|c_2}{2c_1}$, where θ is the angle between x and e_n. Therefore,

$$\begin{aligned}\phi(x) &\leq \phi(0) + \nabla\phi(0)\cdot x + \frac{c_2}{2}|x|^2\\ &= b - |\nabla\phi(0)|e_n \cdot x + \frac{c_2}{2}|x|^2\\ &< b - (c_1|x|)\left(\frac{|x|c_2}{2c_1}\right) + \frac{c_2}{2}|x|^2 = b.\end{aligned}$$

Remark 4.2.7. *By interchanging the roles of x and y in Lemma 4.2.6, a similar statement holds for $E_x^b := \{y : c(x,y) < b\}$.*

Remark 4.2.8. *By the positivity of c_1 in (4.4), it follows that we may take $\nu_x := \frac{c_x(x,y)}{|c_x(x,y)|}$ as the direction of the ball at each point $x \in \partial E_y^b \cap \overline{\Omega}$ and $y \in \overline{\Lambda}$. Thus, for $c \in \mathcal{F}$, all sublevel sets with height $b \geq b_0$ as in Lemma 4.2.6 satisfy a uniform interior cone condition with profile depending only on the positive separation of $\overline{\Omega}$ and the norm of c, and we may take ν_x as the direction of the cone.*

□

Lemma 4.2.9. *Let $c \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$, and suppose it satisfies Conditions 2 and 3 in Definition 4.2.4. Let Ω and Λ be be two domains with disjoint closures, and set*

$$b_0 = \inf_{x\in\overline{\Omega}, y\in\overline{\Lambda}} c(x,y) > 0.$$

Then for any $b > b_0$ and $y \in \mathbb{R}^n$, the domain $E_b^y := \{x : c(x,y) < b\}$ satisfies a uniform interior cone condition with the openning of the cone as close to π as we want by taking the height of the cone sufficiently small.

Proof. First, note that since c satisfies Conditions 2 and 3,

$$c_1 := \inf_{x\in\overline{\Omega}, y\in\overline{\Lambda}} |\nabla_x c(x,y)| > 0,$$

(as in the proof Lemma 4.2.6). Now fix $y \in \overline{\Lambda}$, and consider $\phi(x) := c_x(x, y)$. Then Then for a fixed point $x_0 \in \{x : \phi(x) = b\}$ we choose a coordinate system such that x_n is the direction of the normal to the level set pointing into the sublevel set $\{x : \phi(x) \leq b\}$ and x_0 is the origin. Let $0 < \theta < \frac{\pi}{2}$ and note that if x has angle θ with the x_n direction, then

$$\phi(x) = \phi(0) + \nabla\phi(0) \cdot x + o(|x|) \leq \phi(0) - c_1|x| \cos\theta + o(|x|).$$

Now since $c \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$, by the uniform continuity of c_x we have $o(x) \leq \frac{1}{2}c_1|x| \cos\theta$, for $x \in B_\delta(0)$ and $\delta > 0$ (depending only on θ and the modulus of continuity of c_x). Thus, $\phi(x) < b$ when x has angle at most θ from x_n direction and is in the δ-ball around the origin.

4.2.2 Regularity theory

In this section, we will prove various regularity results on the free boundary under minimal assumptions on the cost function.

Theorem 4.2.10. *(Rectifiability) Let $f = f\chi_\Omega$ and $g = g\chi_\Lambda$ be two nonnegative integrable functions. Assume that Ω and Λ are bounded. If $c : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathcal{R}^+$ is semiconvex, satisfies Condition 2 in Definition 4.2.4, and*

$$\inf_{x\in\overline{\Omega}, y\in\overline{\Lambda}} |\nabla_x^- c(x, y)| > 0,$$

where $\nabla_x^- c(x, y)$ is the subdifferential of c in the variable x, then $\partial U_m \cap \Omega$ is $(n-1)$-rectifiable.

Proof. First, note that our assumptions imply Conditions 1-2” in [[33], Remark 2.11]. In particular, 2” is implied by our semiconvexity assumption, see e.g. [[35], Proposition 2.3] (here the author proves that the optimal map is approximatively differentiable a.e. but since our domains are bounded, the map is truly differentiable a.e.). Thus, utilizing

[[33], Remark 3.2] we obtain

$$U_m \cap \Omega := \bigcup_{(\bar{x},\bar{y})\in\gamma_m} \{x \in \Omega : c(x,\bar{y}) < c(\bar{x},\bar{y})\}.$$

Next, thanks to the semiconvexity of c combined with (3.1), we may apply the nonsmooth implicit function theorem [[67], Theorem 10.50] to conclude that the level set

$$E_a := \{x \in \mathbb{R}^n : c(x,\bar{y}) = a\}$$

is locally an $(n-1)$-dimensional Lipschitz graph. Thus, for $x \in \partial U_m \cap \Omega$, it follows that

$$x \in \partial\{x \in \Omega : c(x,\bar{y}) < c(\bar{x},\bar{y})\},$$

for some $(\bar{x},\bar{y}) \in \gamma_m$. Hence, there exists a profile (δ_x, α_x) such that $(x + C_{\alpha_x}(\nu_x^{\perp})) \cap B_{\delta_x}(x) \subset (U_m \cap \Omega) \cap B_{\delta_x}(x)$, for some $\nu_x^{\perp} \in \mathcal{S}^{n-1}$. Now consider the sets

$$A_j^x := \{z \in (\partial U_m \cap \Omega) \cap B_{\delta_x}(x) : \delta_z \geq \frac{1}{j}, \alpha_z \leq j\},$$

(recall that each $z \in \partial U_m \cap \Omega$ has a profile(δ_z, α_z)). For each $j \in \mathcal{N}$,we may select $\epsilon_j > 0$ so that $P := \{\nu_i\}_{i=1}^{m_{\epsilon_j}}$ is a sufficiently fine partition of $\mathcal{S}^{n-1}$ (i.e. for each $\nu \in \mathcal{S}^{n-1}$, there exists $\nu_i \in P$ so that $|\nu - \nu_i| < \epsilon_j$), and for all

$$\omega \in A_{ij}^x := \{z \in (\partial U_m \cap \Omega) \cap B_{\delta_x}(x) : |\nu_z - \nu_i| < \epsilon_j, \delta_z \geq \frac{1}{j}, \alpha_z \leq j\},$$

we have

$$(\omega + C_{\alpha_j}(\nu_i^{\perp})) \cap B_{\delta_j}(z) \subset (U_m \cap \Omega) \cap B_{\delta_j}(z)$$

for some $\alpha_j > 0$ and $\delta_j > 0$. Thanks to this cone property, it is not difficult to show that for each $i, j \in \mathcal{N}$, A_{ij}^x is contained on the graph of a Lipschitz function (generated by superema of the cones with fixed opening given by α_j). Note,

$$\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{m_{\epsilon_j}} A_{ij}^x = (\partial U_m \cap \Omega) \cap B_{\delta_x}(x),$$

(without loss of generality, we may assume $\epsilon \searrow 0$ as $j \to \infty$). This shows that A_{ij}^x is $(n-1)$-rectifiable. Next, let $(\partial U_m \cap \Omega)_s := \{x \in \partial U_m \cap \Omega : \mathrm{dist}(x, \partial\Omega) > s\}$. Now by compactness, there exists $\{x_k\}_{k=1}^{n(x)} \subset (\partial U_m \cap \Omega)_s \subset \partial U_m \cap \Omega$ so that

$$(\partial U_m \cap \Omega)_s = \bigcup_{k=1}^{n(s)} (\partial U_m \cap \Omega)_s \cap B_{\delta_{x_k}}(x_k).$$

From what we proved, it follows that

$$(\partial U_m \cap \Omega)_s = \bigcup_{k=1}^{n(s)}\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{m_{\epsilon_j}} A_{ij}^{x_k},$$

where each $A_{ij}^{x_k}$ is $(n-1)$-rectifiable. Thus, by taking $s \to 0$, we obtain the result.

Theorem 4.2.11. *(Lipschitz regularity) Let $f = f\chi_\Omega$ and $g = g\chi_\Lambda$ be a nonnegative integrable functions. Assume that Ω and Λ are bounded with $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$ and Λ is c-convex with respect to Ω. If $c \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ is semiconvex and satisfies Conditions 1, 2, and 3 in Definition 2.4, then $\partial U_m \cap U$ is locally Lipschitz.*

Proof. By our assumptions, we have

$$U_m \cap \Omega := \bigcap_{(\bar{x},\bar{y}) \in \gamma_m} \{x \in \Omega : c(x, \bar{y}) < c(\bar{x}, \bar{y})\}.$$

Next, let $x \in \partial U_m \cap \Omega$ and note that since

$$x \in \partial\{x \in \Omega : c(x, \bar{y}) < c(\bar{x}, \bar{y})\},$$

By Lemma 4.2.9 there exists a profile (δ, α) so that

$$(x + C_\alpha(\nu_x^\perp)) \cap B_\delta(x) \subset (U_m \cap \Omega) \cap B_\delta(x),$$

where $\nu_x := -\frac{c_x(x,T_m(x))}{|c_x(x,T_m(x))|}$ and T_m is the optimal partial transport with mass m. Note that as in Lemma 4.2.9 we can take α as close to 0 as we want by taking δ suitably small. For $x \in \partial U_m \cap \Omega \cap B_\delta(x)$, consider the convex set $E_z = c_x(z, \Lambda)$ (note that convexity follows by the c-convexity assumption of Λ). By the positive separation we see that the origin is not in E_z, from this and the convexity of E_z we can easily find a cone $C_{\tilde\alpha}(\nu^\perp)$ which contains E_z, where $\tilde\alpha$ depends only on the positive separate and the C^1 norm of c. As mentioned above, we can assume $\alpha < \tilde\alpha$ by taking δ small. Next, note that by C^1 regularity of c, $c_x(z, \Lambda) \to c_x(x, \Lambda)$ as $z \to x$; hence,we may select $0 < \delta_x \leq \delta$ and $\alpha_x \geq \tilde\alpha$ so that $B_{\delta_x}(x) \subset B_\delta(z)$ and $C_{\alpha_x}(\nu^\perp) \subset C_{\tilde\alpha}(\nu_z^\perp)$; thus for all $z \in \partial U_m \cap B_{\delta_x}(x)$,

$$(z + C_{\alpha_x}(\nu^\perp)) \cap B_{\delta_x}(x) \subset (U_m \cap \Omega) \cap B_{\delta_x}(x).$$

Therefore, as in the proof of Theorem 4.2.14 (see the argument below (4.9)), we obtain that the free boundary locally coincides with the graph of a Lipschitz function (generated by superema of the cones as above).

Remark 4.2.12. *By localizing the problem as in Corollary ?, one may prove an analogous results of Theorem 4.2.10 and 4.2.11 for non-disjoint domains.*

Corollary 4.2.13. *(Semiconvexity) Let $f = f\chi_\Omega$ and $g = g\chi_\Lambda$ be a nonnegative integrable functions. Assume that Ω and Λ are bounded with $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$ and Λ isc-convex with respect to Ω. If $c \in \mathcal{F}$, then $\partial U_m \cap \Omega$ is locally semiconvex.*

Proof. By Theorem 4.2.11, it follows that $\partial U_m \cap \Omega$ is locally a Lipschitz graph, and Lemma 4.2.6 implies that each point on the graph has a ball touching it from below. Thus, locally, the free boundary may be represented as a suprema of spherical caps, and this readily implies semiconvexity.

Theorem 4.2.14. *Let* $f = f\chi_\Omega \in L^p(\mathbb{R}^n)$ *be a non-negative function with* $p \in (\frac{n+1}{2}, \infty]$*, and* $g = g\chi_\Lambda \in L^1(\mathbb{R}^n)$ *a positive function bounded away from zero. Moreover, assume that* Λ *is relatively c-convex with respect to a neighborhood of* $\Omega \cup \Lambda$*, and separated from* Ω *by a hyperplane. Let* $c \in \mathcal{F}_0$ *and* $m \in (0, \min ||f||_{L^1}, ||g||_{L^1}]$*. Then for there exists an explicit* α *for which* $\partial U_m \cap \Omega$ *is locally a* $C^{1,\alpha}$ *graph, where* $U_m \cap \Omega$ *is the free boundary arising from the partial optimal transport problem.*

Proof. First, note that by [33, Remark 2.11], there exists a unique solution to the optimal partial transport problem. Moreover, this solution has the form

$$\gamma_m := (Id \times T_m)_\# f_m = (T_m^{-1} \times Id)_\# g_m.$$

By [33, Proposition 2.4] and [33, Remark 2.5],

$$\gamma = \gamma_m + (Id \times Id)_\#((f - f_m) + (g - g_m)), \tag{4.5}$$

where γ solves the classical optimal transport problem between the densities $f + (g - g_m)$, $g + (f - f_m)$, and with the given cost function $c(x, y) \in \mathcal{F}_0$. From the classical theory, we know that γ is supported on the graph of a function T, and there exists a potential function Ψ which satisfies:

$$\nabla_x c(x, T(x, \nabla\Psi(x))) = \nabla\Psi(x). \tag{4.6}$$

Now by [33, Theorem 2.6 and Remark 2.11], it follows that $T_\#(f_m + (g - g_m)) = g$ (i.e. T will not move the points in the inactive region); hence, it coincides with the partial transport T_m in the active region $U_m \cap \Omega$. Now let $f' := f_m + (g - g_m)$ and note that by our assumptions,

$$|det(D^2_{xy}c)| \frac{f'}{g(T_m)} \in L^p(U_m \cap \Omega),$$

(indeed, $g = g_m$ on $U_m \cap \Omega$). Thus, we may apply [54, Theorem 1] to obtain $\Psi \in C^{1,\alpha}(\overline{U_m \cap \Omega})$. Next, we utilize [33, Remarks 3.2 and 3.3] to note that the active region of Ω may be identified as

$$U_m \cap \Omega := \bigcup_{(\bar{x},\bar{y}) \in \gamma_m} \{x \in \Omega : c(x,\bar{y}) < c(\bar{x},\bar{y})\}. \tag{4.7}$$

Let $\bar{x} \in \partial U_m \cap \Omega$. By (4.7) it follows that $\bar{x} \in \partial E_{\bar{y}}^{b_{\bar{y}}}$, where $(\bar{x},\bar{y}) \in \gamma_m$ and $b_{\bar{y}} := c(\bar{x},\bar{y}) > 0$. Note that since $d(\Omega,\Lambda) > 0$, all level sets of c inside Ω satisfy the uniform interior cone condition with a uniform profile depending only on $d(\Omega,\Lambda) > 0$ by Lemma 4.2.6. Thus, there exists $\delta = \delta(dist(\Omega,\Lambda))$, $\tilde{\alpha} = \tilde{\alpha}(dist(\Omega,\Lambda))$ such that

$$(x + C_{\tilde{\alpha}}(\nu_x^{\perp})) \cap B_\delta(x) \subset U_m \cap \Omega,$$

where $\nu_x = \frac{\nabla\Psi(x)}{|\nabla\Psi(x)|}$. Moreover, choose $R_x > 0$ such that

$$B_{R_x}(x) \cap \partial\Omega = \emptyset,$$

and set $r_x := \min\{\delta, R_x\}$. Since $\nabla\Psi \in C^{0,\alpha}(\overline{U_m \cap \Omega})$, for any given $\epsilon > 0$ there exists $\eta > 0$ such that $|\nu_x - \nu_z| < \epsilon$ for any $z \in \partial U_m \cap B_\eta(x)$, where $\nu_z = \frac{\nabla\Psi(z)}{|\nabla\Psi(z)|}$ is the direction of the cone at z (see Remark 4.2.8 and (4.6)). Thus, we may take the same profile $(\delta,\tilde{\alpha})$ for all cones touching $\partial U_m \cap B_{r_x}(x)$ by picking $\eta_x > 0$ small enough (depending on $dist(\Omega,\Lambda)$ and r_x), and there exists $0 < \alpha_x \leq \tilde{\alpha}$ so that for all $z \in \partial U_m \cap B_{\eta_x}(x)$,

$$(z + C_{\alpha_x}(\nu_x^{\perp})) \cap B_{\eta_x}(x) \subset U_m \cap \Omega. \tag{4.8}$$

In fact, by possibly taking α_x and η_x smaller, this statement holds for all $z \in \overline{U}_m \cap B_{\epsilon_x}(x)$ (this is due to the fact that by (4.7) all interior points of the active region lie on a level set of the cost function and the normal to this level set is close to ν_x for interior points close to x). Indeed, let $z \in B_{\eta_x}(x) \cap U_m$. Then $(z, T_m(z)) \in \gamma_m$ and by (4.7), $z \in \partial E^b_{T_m(z)}$ with $b = c(z, T_m(z)) > 0$ (by the positive separation assumption). Now by Remark 4.2.8, there exists $\nu_z \in \mathbb{S}^{n-1}$ so that

$$(y + C_\alpha(\nu_z^\perp)) \cap B_\delta(z) \subset E^b_{T_m(z)} \cap \Omega \subset U_m \cap \Omega,$$

for all $y \in \overline{E^b_{T_m(z)}} \cap B_\delta(x)$. In particular,

$$(z + C_\alpha(\nu_z^\perp)) \cap B_\delta(z) \subset U_m \cap \Omega.$$

Thus, by possibly choosing η_x smaller, if necessary, we may assume $\overline{B}_{\eta_x}(x) \subset B_\delta(z)$ hence, by the continuity of the gradient of the potential, $|\nu_x - \nu_z| < \eta_x$ (as above), and so we may choose α_x small so that

$$C_{\alpha_x}(\nu_x^\perp) \subset C_\alpha(\nu_z^\perp).$$

Hence,

$$(z + C_{\alpha_x}(\nu_x^\perp)) \cap B_{\eta_x}(x) \subset (z + C_\alpha(\nu_z^\perp)) \cap B_\delta(z) \subset U_m \cap \Omega.$$

Therefore, we proved that there exists $\eta_x > 0$ and $0 < \alpha_x \leq \tilde{\alpha}$ so that for all $z \in \overline{B}_{\eta_x}(x) \cap \overline{U}_m$,

$$(z + C_{\alpha_x}(\nu_x^\perp)) \cap B_{\eta_x}(x) \subset U_m \cap \Omega. \tag{4.9}$$

Now by rotating the coordinate system, we may assume that $x = 0$, $\nu_x = -e_n$, $\pi := \nu_x^\perp = \mathbb{R}^{n-1}$, and that the cone $C_{\alpha_0}(\pi)$ is symmetric with respect to the e_n axis. Define $\phi : \tilde{B}_{\eta_0}(0) \subset \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\phi(z') := \sup_{y:=(y', y_n) \in \partial U_m \cap \overline{B_{\eta_0}(0)}} K_y(z'),$$

where K_y is the cone function at the point y on the free boundary. Note that ϕ is Lipschitz since it is the supremum of Lipschitz functions with bounded Lipschitz constant (depending on the opening of the cones). Moreover, by construction we have

$$\partial U_m \cap B_{\eta_0}(0) \subset \operatorname{graph} \phi|_{\tilde{B}_{\eta_0}(0)}. \tag{4.10}$$

Now we claim that there exist constants $d, \tilde{d} \in (0,1)$ with $\tilde{d}$ depending on d and d depending on the profile of the level sets of the cost function, so that

$$\operatorname{graph} \phi|_{\tilde{B}_{d\eta_0}(0)} \subset \partial U_m \cap \overline{B_{\tilde{d}\eta_0}(0)}. \tag{4.11}$$

Indeed, pick any $\tilde{d} \in (0,1)$; we may select a constant $d = d(\tilde{d}, \alpha_x) > 0$ small enough, so that the graph of $\phi(Proj_{\pi}(B_{d\eta_0}(0)))$ is contained in $B_{\tilde{d}\eta_0}(0)$ (this is possible, since ϕ has a uniform Lipschitz constant in $B_{\eta_0}(0)$ which depends only on the profile of the level sets). Let $y \in \operatorname{graph} \phi|_{\tilde{B}_{d\eta_0}(0)} \subset B_{\tilde{d}\eta_0}(0)$. If $y \notin \partial U_m \cap \overline{B_{\tilde{d}\eta_0}(0)}$, then since y is on an open cone with opening inward to $U_m \cap \Omega$, it follows that $y \in U_m \cap \Omega$. Since $\partial U_m \cap \overline{B}_{\tilde{d}\eta_0}$ is compact, for $\theta > 0$ small, it follows that $Q_\theta(y) \cap \partial U_m \cap \overline{B}_{\tilde{d}\eta_0} = \emptyset$, where Q_θ is a small cylinder whose interior is centered at y and whose base diameter and height is equal to θ; in particular, $Q_\theta \cap \operatorname{graph} \phi|_{\tilde{B}_{d\eta_0}(0)}$ does not contain any free boundary points. Next we consider a general fact: let $w \in \operatorname{graph} \phi|_{\tilde{B}_{\eta_0}(0)} \setminus \partial U_m$, $L_t(w) := w + te_n$, and

$$s(w) := \sup_{\{t \geq 0 : L_t(w) \in U_m \cap \Omega\}} t;$$

note that since $w \in \operatorname{graph} \phi|_{\tilde{B}_{\eta_0}(0)}$,

$$s(w) \geq \tilde{s}(w) := \sup_{\{t \geq 0 : L_t(w) \in B_{\eta_0}(0)\}} t, \tag{4.12}$$

(otherwise it would contradict the definition of ϕ as a suprema of cones in $B_{\eta_0}(0)$ and

w as a point on the graph). Next, keeping the base fixed, we enlarge the height of the cylinder along the $\{y + te_n : t \in \mathbb{R}\}$ axis in a symmetric way (with respect to the plane $y_n + \pi = \mathbb{R}^{n-1}$) so that it surpasses $4\eta_0$; we denote the resulting cylinder by $\tilde{Q}_\theta$. By (4.12) we have $\tilde{Q}_\theta \cap B_{\eta_0} \subset U_m \cap \Omega$. Then we increase its base diameter, θ, until the first time when $\tilde{Q}_\theta$ hits the free boundary $\partial U_m \cap \Omega$ inside $B_{\eta_0}(0)$, and denote the time of first contact by θ and a resulting point of contact by y_θ (note that since $0 \in \partial U_m \cap B_{\tilde{\eta}_0}(0)$, this quantity is well defined). Since ϕ is a continuous graph in $B_{\eta_0}(0)$, and both y and y_θ are on the graph, we may select a sequence of points $y_k \in \operatorname{graph} \phi|_{\tilde{B}_{\eta_0}(0)} \cap \tilde{Q}_\theta$ such that $y_k \to y_\theta$ (by connectedness of $\phi|_{\tilde{B}_{\eta_0}(0)} \cap \tilde{Q}_\theta$). Since $y_\theta \in B_{\tilde{\eta}_0}(0)$ is an interior point, for k sufficiently large we will have $y_k \in B_{\eta_0}(0) \cap \tilde{Q}_\theta$. Thus, by definition of θ, we will have that the y_k are not free boundary points but on the graph of ϕ; thus, by (4.12), $s(y_k) \geq \tilde{s}(y_k)$, and this implies $\tilde{y}_k := y_k + \tilde{s}(y_k)e_n \in \partial B_{\eta_0}(0) \cap \overline{U}_m$. By (4.9) we have

$$(\tilde{y}_k + C_{\alpha_0}(\pi)) \cap B_{\eta_0}(0) \subset U_m \cap \Omega.$$

However, for large k, $y_\theta \in (\tilde{y}_k + C_{\alpha_0}(\pi))$ and this contradicts that y_θ is a free boundary point, thereby establishing (4.11). Thus, combining (4.10) and (4.11) we obtain that in a neighborhood around the around the origin, the free boundary is the graph of the Lipschitz function ϕ. Hence, the normal of its graph exists $\mathcal{H}^{n-1}$ for a.e. $z' \in \tilde{B}_{\eta_0}(0)$ and coincides with the Hölder function $\frac{\nabla\Psi}{|\nabla\Psi|}$ outside of the set of measure zero. As the normal can be represented by $\frac{(D\phi(z'),-1)}{\sqrt{1+|D\phi(z')|^2}}$ at a point $(z', \phi(z'))$ on the graph, it is easy to see that it is in fact Hölder, and this concludes the proof. □

Remark 4.2.15. *By reverse symmetry, we may interchange the roles of f and g in Theorem 4.2.14 (and adjusting the assumptions accordingly) in order to obtain $C^{1,\alpha}_{loc}$ regularity of $\partial V_m \cap \Lambda$.*

Remark 4.2.16. *(Geometry of c-convex domains) For a geometric description of c-convex domains, see [66]. For example, based on a calculation therein, one can prove the* *following: Suppose* $D \subset \mathbb{R}^n$ *is a bounded domain and* K *a convex subset with smooth boundary. Let*

$$c_1 := \inf_{x\in\Omega, y\in K} \det c_{x,y}(x,y),$$

and $c_2 := \|c(\cdot,\cdot)\|_{C^3}$*. Then for a fixed* y*, consider* $c_y(x,y) : K \to c_y(K,y)$*. If the principle curvatures of* ∂K *are greater than* $\frac{c_2^n}{c_1}$*, then* $c_y(K,y)$ *is convex.*

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