Failure of local equi-Lipschitzness for families of Lorentz distances to Cauchy surface foliations

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Abstract

The families of Lorentzian distance functions to and from points along a complete timelike line in a spacetime are known to be locally equi-Lipschitz continuous in a neighborhood of the line. This is an essential component in the proof of the classical Lorentzian splitting theorems. We show that this property fails in general for families of Lorentzian distances to and from the level sets of a given Cauchy temporal function. Moreover, we formulate conjectures based on the existence of Cauchy temporal functions in cosmological and timelike geodesically complete spacetimes such that the Lorentz distances to its level sets are equi-Lipschitz, which are equivalent to Bartnik's splitting conjecture.

Keywords: Lorentz distance, Cauchy temporal function, equi-Lipschitz, Bartnik's cosmological splitting conjecture

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Contents

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1 Introduction

The Lorentzian splitting theorems [8, 9, 13] are by now a classical topic in Lorentzian geometry and mathematical general relativity. They assert that a timelike geodesically complete or globally hyperbolic spacetime (M, g) which contains a complete timelike line (i.e., a proper time-maximizing geodesic $\gamma : \mathbb{R} \to M$) and satisfies the strong energy condition (i.e., $\mathrm{Ric}(v, v) \geq 0$ for all timelike $v \in TM$) splits isometrically as a product spacetime $(\mathbb{R} \times S, dt^2 - \tilde{g})$, where $(S, \tilde{g})$ is a complete Riemannian manifold of nonnegative Ricci curvature. Recently, new methods were developed in [4] based on the ellipticity of the p-d'Alembertian studied in [2], which lead to a vastly improved Lorentz–Finsler splitting theorem [6] as well as a splitting for spacetimes whose metric and volume weight can be of C^1 regularity [5].

A central ingredient in the proof is the following: Given a complete timelike line $\gamma : \mathbb{R} \to M$ in a timelike geodesically complete or globally hyperbolic spacetime (M, g), there exists a neighborhood U of $\gamma(\mathbb{R})$ such that the functions

$$\{\ell(\cdot, \gamma(t))\}_{t>0}, \quad \{\ell(\gamma(-t), \cdot)\}_{t>0}, \tag{1.1}$$

are locally equi-Lipschitz continuous on U.

While this result already appears in Eschenburg [8], it is most elegantly understood in terms of the *(generalized) timelike co-ray condition* introduced in Galloway and Horta [10], which, if it holds at a point q, states that curves $\alpha : [0, \infty)$ obtained as subsequential limits of (almost) maximizers connecting q to $\gamma(t_n)$ (or $\gamma(-t_n)$ to q), are timelike. This is an open condition; on γ itself all such limits only produce parts of the timelike line γ, thus this condition holds in a neighborhood of γ.

In situations where one would like to prove a splitting of spacetime, but there is no complete timelike line present a priori, such as in the setting of Bartnik's [1] cosmological splitting conjecture, one might be tempted to consider families of Lorentz distances $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ as well as $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$, where $\Sigma_s := \{\tau = s\}$ are the level sets of a given Cauchy temporal function $\tau \in C^\infty(M)$ and are thus smooth spacelike Cauchy hypersurfaces. In this note, we show that there is in general an issue with such an approach, even for rather nice (i.e., steep with respect to a background complete Riemann metric) Cauchy temporal functions, by way of two counterexamples.[1] These examples also contradict Lemma 3.2 in [14] and subsequently the results therein which assert the existence of global viscosity solutions to the Lorentzian eikonal equation $g(\nabla u, \nabla u) = 1$.

It is worth noting that the existence of a Cauchy temporal function for which the aforementioned families are locally equi-Lipschitz is equivalent to Bartnik's splitting conjecture.

1.1 Notation and conventions

Lorentzian metrics have signature $(+, -, \ldots, -)$, so $g(v, v) > 0$ means v is timelike, $g(v, v) = 0$ means v is null, and $g(v, v) < 0$ means v is spacelike. Spacetimes are connected, time oriented Lorentzian manifolds with smooth metric tensors $g_{ij} \in C^\infty$ and have dimension $\dim M =: n \geq 2$. We fix a background complete Riemannian metric h throughout. If X is the time orientation vector field, then $g(v, X) \geq 0$ means v is future, $g(v, X) \leq 0$ means v is past. If v is causal, we write $|v|_g := \sqrt{g(v, v)}$ for its Lorentzian norm. A spacelike

[1] Some partial efforts to deal with these issues for a certain class of Cauchy surfaces have been made in [11, 12], whereby certain splitting results are obtained, subject to certain additional conditions.

hypersurface is a hypersurface such that the restriction of g on it is negative definite. The function $\ell : M^2 \to \{-\infty\} \cup [0,\infty)$ denotes the (extended) time separation function, by convention $\ell(x,y) = -\infty$ if $x \not\leq y$. Moreover, we write $\ell(A,B) := \sup_{x\in A, y\in B} \ell(x,y)$. A function $\tau \in C^\infty(M)$ is called *temporal* if $\nabla\tau$ is future directed timelike. A temporal function τ is *Cauchy* if each of its level sets $\{\tau = s\}$, $s \in \mathbb{R}$, is a Cauchy hypersurface.

2 Busemann functions associated to timelike lines

Let (M,g) be globally hyperbolic or timelike geodesically complete, and let $\gamma : \mathbb{R} \to M$ be a complete timelike line. Let us briefly describe how to obtain the local Lipschitz regularity of the family $\{\ell(\cdot,\gamma(t))\}_{t>0}$ in a neighborhood of $\gamma(\mathbb{R})$, following Galloway–Horta [10].

Given a point $q \in I(\gamma) := I^+(\gamma) \cap I^-(\gamma)$, a *(generalized) co-ray* is an inextendible causal curve α starting at q which is obtained as a subsequential limit of (almost) maximizing timelike curves α_n which connect q to $\gamma(t_n)$, where $t_n \to +\infty$. Such a (generalized) co-ray is always causal and maximizing. We say the *(generalized) timelike co-ray condition* holds at q if every (generalized) co-ray starting at q is timelike. The essential result now is the following

Proposition 2.1 ([10]).

(i) The (generalized) timelike co-ray condition is an open condition.

(ii) Every (generalized) co-ray starting at $\gamma(t)$, for any t, corresponds to $\gamma|_{[t,\infty)}$. In particular, the (generalized) timelike co-ray condition holds at every point on γ.

The above result implies that the (generalized) timelike co-ray condition holds in a neighborhood U of $\gamma(\mathbb{R})$. In particular, it is now easily seen that $\{\ell(\cdot,\gamma(t)\}_{t>0}$ is equi-Lipschitz on this neighborhood. If there existed $q \in U$ such that this family was not equi-Lipschitz in any neighborhood of q, one could find (almost) maximizing timelike geodesics in g-arclength parametrization from q_n to $\gamma(t_n)$, where $q_n \to q$ and $t_n \to \infty$, such that $|\gamma_n'(0)|_h \to \infty$. In this case, it is easily seen that the γ_n must converge subsequentially to a null ray at q, thus contradicting the (generalized) timelike co-ray condition.

In fact, the family $\{\ell(\cdot,\gamma(t)\}_{t>0}$ is not only locally equi-Lipschitz but also locally equi-semiconvex on U [4, Prop. 5]. These ingredients are essential in the use of p-d'Alembertian techniques, where we recall that $\Box_p f := -\mathrm{div}(|\nabla f|^{p-2}\nabla f)$, $0 \neq p < 1$. In [4], we give a streamlined argument (the result was also shown in [2]) for the fact that the standard smooth d'Alembertian comparison for a Lorentz distance $f := \ell(\cdot,y)$ on $I^-(y) \setminus \mathrm{Cut}^-(y)$ extends to a weak comparison inequality across the cut locus, in the sense that

$$\int_M \left(\frac{(n-1)\varphi}{f} + g(\nabla\varphi, |\nabla f|^{p-2}\nabla f) \right) \geq 0 \tag{2.1}$$

for all $0 \leq \varphi \in C_c^1(I^-(y))$. Applying this result now to the sequence of approximate Busemann functions

$$b_t^+(x) := \ell(\gamma(0),\gamma(t)) - \ell(x,\gamma(t)) \tag{2.2}$$

and keeping in mind the local equi-Lipschitzness and local equi-semiconvexity of the family $\{\ell(\cdot,\gamma(t))\}_{t>0}$ in the neighborhood of the line where the (generalized) timelike co-ray condition holds, we may pass to the limit to obtain

$$\int_M g(\nabla\varphi, |\nabla b^+|_g^{p-2}\nabla b^+)\, dvol_g \leq 0, \tag{2.3}$$

where $b^+ := \lim_{t\to\infty} b_t^+$. This can be understood as the weak form of the inequality $\Box_p b^+ \le 0$. Similarly, $\Box_p b^- \ge 0$, where $b^- := \lim_{t\to\infty} b_t^-$ and

$$b_t^-(x) := \ell(\gamma(-t), x) - \ell(\gamma(-t), \gamma(0)). \tag{2.4}$$

Since $b^+ \ge b^-$ and $b^+(\gamma(t)) = b^-(\gamma(t))$ for every t due to the fact that γ is a maximizing timelike line, one can obtain $b^+ = b^-$ on a neighborhood of $\gamma(\mathbb{R})$ by the maximum principle for quasilinear elliptic operators, cf. [4, Prop. 9]. An application of a nonlinear Bochner-type identity then yields $\mathrm{Hess}_g(b^+) = 0$ and in particular that b^+ is smooth near γ, see [4, Cor. 14]. From here, the remaining arguments to obtain the Lorentzian splitting theorem follow the classical sources [8, 9, 13, 10].

One may be tempted to follow this line of thought using Busemann-type functions associated to Cauchy surface foliations, but, as the next section shows, the equi-Lipschitz property of Lorentz distances fails in this case in general, thus the analogous arguments do not go through due to lack of regularity.

3 Busemann type functions associated to a Cauchy surface foliation

Let (M, g) be a globally hyperbolic spacetime. Let $\tau \in C^\infty(M)$ be a given h-steep temporal function, i.e.,

$$d\tau(V) \ge \max(|V|_h, |V|_g). \tag{3.1}$$

Bernard–Suhr [3] show that any globally hyperbolic spacetime possesses such an h-steep Cauchy temporal function. Now, [14, Lemma 3.2] (and subsequent results) assert the local equi-Lipschitzness of the families of Lorentz distances $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ as well as $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$, where $\Sigma_s := \{\tau = s\}$. The aforementioned reference uses this in an essential way in their construction of global viscosity solutions to the Lorentzian eikonal equation $g(\nabla u, \nabla u) = 1$. However, we show that it does not hold in that generality without further assumptions. In fact, one can even find a temporal function which is steep with respect to the background Euclidean metric on 2-dimensional Minkowski spacetime which contradicts the claim. Before coming to it, we note the following elementary observation:

Lemma 3.1. *The family $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ is locally equi-Lipschitz on M if and only if M is covered by open sets U such that for each $q \in U$, the h-norms of the initial tangents of g-unit speed maximizing timelike geodesics $\gamma_{(t)}$ from q to Σ_t satisfy $|\gamma'_{(t)}(0)|_h \le C$ for some C which is independent of t, where t is large enough (depending on U).*

Example 3.2. We give a counterexample to [14, Lemma 3.2]. Consider Minkowski space $\mathbb{R}^{1,1}$ with metric $dt^2 - dx^2$. If we transform to null coordinates (u, v) which are defined via

$$u = \frac{t-x}{\sqrt{2}}, \quad v = \frac{t+x}{\sqrt{2}}, \tag{3.2}$$

the Minkowski metric is $g = 2dudv$ in these coordinates. A vector $X = A\partial_u + B\partial_v$ is future causal if and only if $A, B \ge 0$, since $g(X, X) = 2AB$. We write $h = du^2 + dv^2$ for the standard Euclidean metric in these coordinates. Define now

$$\tau(u, v) := F(u) + v, \tag{3.3}$$

where F is smooth function satisfying $F'(u) \geq 1$ for all F (we will specify a choice of F further below). We claim that τ is an h-steep temporal function: Indeed, $d\tau = F'(u)du + dv$, thus $\nabla\tau = \partial_u + F'(u)\partial_v$, so that $g(\nabla\tau, \nabla\tau) = 2F'(u) \geq 2$, so $\nabla\tau$ is temporal. To see that it is h-steep, take any future causal X as above, then

$$d\tau(X) = F'(u)du(X) + dv(X) = F'(u)A + B \geq A + B \geq \sqrt{A^2 + B^2} = |X|_h. \tag{3.4}$$

Consider now the level set $\Sigma_s = \{(u, v) : F(u) + v = s\}$. Let us calculate the intersection of $J^+(0)$ with Σ_s. Take any straight line $\gamma(t) = (tu, tv)$ with $(u, v) \in \Sigma_s$, for γ to be future causal we must have $u, v \geq 0$, but $v = s - F(u)$, so that $F(u) \leq s$, as well as $u \geq 0$. By monotonicity, there is a unique u_{max} such that $F(u_{max}) = s$, and $F(u) \leq s$ for all $u \in [0, u_{max}]$. All in all,

$$J^+(0) \cap \Sigma_s = \{(u, v) : u \in [0, u_{max}], 0 \leq v = s - F(u)\}, \tag{3.5}$$

which is compact. Take now a maximizing causal line from $\gamma(t) = (tu, tv) = (tu, t(s - F(u)))$ connecting the origin to some point on Σ_s, we want to determine at which point on Σ_s the time separation to Σ_s from the origin is maximized. To this end, note that

$$\frac{1}{2}\ell((0,0),(u,v))^2 = uv = u(s - F(u)) =: f(u). \tag{3.6}$$

Let us now make a specific choice $F(u) = u + u^3/3$, which indeed satisfies $F'(u) = 1 + u^2 \geq 1$ as required. Moreover, $F''(u) = 2u$. We claim that with this choice of F, $f(u)$ is strictly concave: indeed

$$f'(u) = s - F(u) - uF'(u), \tag{3.7}$$

$$f''(u) = -2F'(u) - uF''(u) = -(2 + 2u^2 + 2u^2) < 0. \tag{3.8}$$

Let s be large enough so that $s > F(0)$. Also, note that $f'(u_{max}) = s - F(u_{max}) - u_{max}F'(u_{max}) = -u_{max}F'(u_{max}) < 0$. Thus, $f'(u) = 0$ uniquely at some $u_s \in (0, u_{max})$, where u_s is implicitly given by

$$s = F(u_s) + u_sF'(u_s) \tag{3.9}$$

and the pair $(u_s, s - F(u_s) = v_s)$ is the point on Σ_s at which a maximizer from the origin to Σ_s lands. The maximizing segment is precisely $t \in [0, 1] \mapsto \gamma(t) = (tu_s, tv_s)$. Note that, using the explicit form of F,

$$v_s = s - F(u_s) = u_sF'(u_s) = u_s(1 + u_s^2). \tag{3.10}$$

Thus,

$$\gamma'(0) = u_s\partial_u + u_s(1 + u_s^2)\partial_v. \tag{3.11}$$

Hence

$$g(\gamma'(0), \gamma'(0)) = 2u_s^2(1 + u_s^2). \tag{3.12}$$

If we write $W_s := \gamma'(0)/|\gamma'(0)|_g$, then

$$W_s = \frac{1}{\sqrt{2(1 + u_s^2)}}\partial_u + \sqrt{\frac{1 + u_s^2}{2}}\partial_v. \tag{3.13}$$

Then

$$h(W_s, W_s) = \frac{1}{2(1 + u_s^2)} + \frac{1 + u_s^2}{2}. \tag{3.14}$$

Clearly, the above goes to $+\infty$ as $u_s \to +\infty$, or equivalently, as $s \to +\infty$ since

$$s = F(u_s) + u_sF'(u_s) = u_s + u_s^3/3 + u_s(1 + u_s^2) = 2u_s + 4u_s^3/3. \tag{3.15}$$

The result continues to be false even if one assumes the Cauchy surfaces to be compact. Before coming to this, we observe that the equi-Lipschitzness already forces the existence of timelike lines in this setting.

Lemma 3.3. *Let (M, g) be a globally hyperbolic spacetime with compact Cauchy surfaces. Let $\tau \in C^\infty(M)$ be a Cauchy temporal function with level sets $\Sigma_s := \{\tau = s\}$. If either of the families $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ or $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$ is locally equi-Lipschitz, then there exists a (not necessarily complete) timelike line in M.*

Proof. Let us assume $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ is locally equi-Lipschitz, the other case is analogous.

Compactness yields a g-unit speed maximizing timelike geodesic γ_n connecting Σ_{-n} to Σ_n. Moreover, γ_n intersects Σ_0 uniquely in a point z_n (normalized so that $z_n = \gamma_n(0)$). By compactness, upon passing to a subsequence, we may assume that $z_n \to z \in \Sigma_0$. By assumption, since γ_n also maximizes from z_n to Σ_n, $\{\gamma_n'(0)\}_{n\in\mathbb{N}}$ is contained in a compact subset of TM, thus, after passing to a further subsequence, $\gamma_n'(0) \to v \in T_zM$. By standard ODE theory, $\gamma_n \to \gamma$ in C^∞_{loc}, where γ is the maximally extended geodesic with initial data $\gamma(0) = z$ and $\gamma'(0) = v$. As all of the γ_n are maximizing, γ is thus a timelike line. □

Example 3.4. In [7], Ehrlich–Galloway construct a timelike (and null) geodesically complete, globally hyperbolic spacetime which has no timelike lines. Thus, by the previous result, there is no Cauchy temporal function τ on this spacetime (no matter if h-steep or not) for which either of the families $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ or $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$ is locally equi-Lipschitz.

Let us note the following connection of the equi-Lipschitz property of the families of Lorentz distances with Bartnik's cosmological splitting conjecture:

Corollary 3.5. *Let (M, g) be a globally hyperbolic, timelike geodesically complete spacetime with compact Cauchy surfaces satisfying the strong energy condition. Then (M, g) splits isometrically as a Lorentzian product if and only if there exists a Cauchy temporal function $\tau \in C^\infty(M)$ such that either of the families $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ or $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$ is locally equi-Lipschitz.*

Proof. The 'if' statement follows from Lemma 3.3. The 'only if' statement is elementary to verify since in this case the geometrical splitting is assumed; (for a stronger conclusion under a stronger hypothesis, see Proposition 4.3 below). □

4 Outlook

In light of the relationship between equi-Lipschitzness of Lorentz distances to Cauchy surfaces and Bartnik's splitting conjecture, let us formulate the following conjecture which is equivalent to Bartnik's:

Conjecture 4.1. Let (M, g) be a globally hyperbolic, timelike geodesically complete spacetime with compact Cauchy surfaces which satisfies the strong energy condition. Then there exists a Cauchy temporal function $\tau \in C^\infty(M)$ such that one of the families $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ or $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$ is locally equi-Lipschitz, where $\Sigma_t := \{\tau = t\}$.

The counterexamples we gave to the equi-Lipschitzness of these families were (1) in a setting where the strong energy condition holds but the Cauchy surfaces are not compact

(Minkowski), and (2) in a setting where Cauchy surfaces are compact but the strong energy condition does not hold (the example by Ehrlich–Galloway). In light of this, we formulate the following conjecture, which would imply our previous one:

Conjecture 4.2. Let (M, g) be a globally hyperbolic, timelike geodesically complete spacetime with compact Cauchy surfaces which satisfies the strong energy condition. Then for every Cauchy temporal function $\tau \in C^\infty(M)$ both of the families $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ or $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$ are locally equi-Lipschitz.

To see that this is not unreasonable, let us prove the following:

Proposition 4.3. *Let $(M, g) = (\mathbb{R} \times S, dt^2 - \tilde{g})$ be a product spacetime, where $(S, \tilde{g})$ is a compact Riemannian manifold. Then for every Cauchy temporal function $\tau \in C^\infty(\mathbb{R} \times S)$, the families $\{\ell(\cdot, \Sigma_t)\}_{t>0}$ and $\{\ell(\Sigma_{-t}, \cdot)\}_{t>0}$ are locally equi-Lipschitz.*

Proof. We show this for $\{\ell(\cdot, \Sigma_t)\}_{t>0}$, the case of the other family is analogous.

Given any $s \in \mathbb{R}$, since Σ_s is a smooth spacelike acausal Cauchy hypersurface it is easily seen that it must be a global graph over S, i.e.,

$$\Sigma_s = \{(f_s(x), x) \in \mathbb{R} \times S : x \in S\} \tag{4.1}$$

for some smooth function $f_s \in C^\infty(S)$. The fact that Σ_s is spacelike translates to the condition $|df_s|_{\tilde{g}} < 1$, i.e., f_s is 1-Lipschitz on S. Let $D := \mathrm{diam}(S) < +\infty$ (by compactness), then

$$\max_S f_s - \min_S f_s \leq D < +\infty \tag{4.2}$$

for every $s \in \mathbb{R}$. We write $M_s := \max_S f_s$, so that $f_s \geq M_s - D$. Fix a compact set $K \subseteq \mathbb{R} \times S$ and take any $p := (t_0, x_0) \in K$. For s large enough, $K \subseteq I^-(\Sigma_s)$. Connect p to Σ_s by the vertical segment $\alpha : t \mapsto (t + t_0, x_0)$. This segment uniquely meets Σ_s at $t = f_s(x_0)$, so that its Lorentzian arclength $L_g(\alpha)$ is

$$\ell(p, \Sigma_s) \geq L_g(\alpha) = f_s(x_0) - t_0 \geq M_s - D - t_0. \tag{4.3}$$

Let $\gamma_s(t) = (t, c_s(t))$ be any timelike geodesic realizing $\ell(p, \Sigma_s)$, so that $v_s := |c_s'|_{\tilde{g}} < 1$. This geodesic intersects Σ_s at some $(f_s(y_s), y_s)$, so that

$$\ell(p, \Sigma_s) = L_g(\gamma_s) = (f_s(y_s) - t_0)\sqrt{1 - v_s^2} \leq (M_s - t_0)\sqrt{1 - v_s^2}. \tag{4.4}$$

In total,

$$M_s - D - t_0 \leq (M_s - t_0)\sqrt{1 - v_s^2}. \tag{4.5}$$

Note that $M_s \to +\infty$ as $s \to +\infty$, and moreover t_0 remains uniformly bounded for $p = (t_0, x_0) \in K$ since the latter is compact. Thus, dividing by M_s and sending $s \to \infty$, we get $1 \leq \liminf_{s\to\infty} \sqrt{1 - v_s^2}$, so that $\lim_{s\to\infty} v_s^2 \to 0$. This shows that $|\gamma_s'(0)|_h^2 = 1 + v_s^2$ remains uniformly bounded in s, where $h := dt^2 + \tilde{g}$. □

The conjectures formulated above provide a different perspective on Bartnik's conjecture, which could be helpful in its study. In light of the previous result, they are equivalent to each other and to Bartnik's conjecture itself.

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