

ON THE CARDINALITY OF SETS IN \mathbf{R}^d OBEYING A SLIGHTLY OBTUSE ANGLE BOUND*

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Abstract. In this paper, we explicitly estimate the number of points in a subset $A \subseteq \mathbf{R}^d$ as a function of the maximum angle $\angle A$ that any three of these points form, provided $\angle A < \theta_d := \arccos(-\frac{1}{d}) \in (\pi/2, \pi)$. We also show $\angle A < \theta_d$ ensures that A coincides with the vertex set of a convex polytope. This study is motivated by a question of Paul Erdős and indirectly by a conjecture of László Fejes Tóth.

Key words. combinatorial geometry, effective bounds, obtuse angle bounds, acute sets, cardinality, criterion for convex position, Erdős, Füredi, Danzer, Grünbaum

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1. Introduction. Let us begin with a simple definition.

DEFINITION 1.1 (angle bound). *For $A \subseteq \mathbf{R}^d$, let $\angle A$ denote the smallest value of $\theta \in [0, \pi]$, such that no triple of points $x, y, z \in A$ determine an angle $\angle xyz$ greater than θ , i.e., $\frac{x-y}{|x-y|} \cdot \frac{z-y}{|z-y|} \geq \cos \theta$ for all $x, y, z \in A$.*

For a set A , we denote by $|A| \in \{0, 1, \dots, \infty\}$ its cardinality. In the 1950s, Paul Erdős raised the following conjecture [6].

Conjecture: If $A \subseteq \mathbf{R}^d$ satisfies $\angle A \leq \pi/2$, then $|A| \leq 2^d$.

The conjectured bound 2^d is obviously sharp, being achieved by the vertices of the hypercube in \mathbf{R}^d . For $d = 3$, he had advertised the problem a decade earlier [5], which he claimed [6] led to an unpublished solution by Kuiper and Boerdijk. For $d \geq 3$, the conjecture was resolved affirmatively by Danzer and Grünbaum [3], who established it through a chain of remarkable inequalities reproduced in [1]. They also asked the following natural question: if the angle bound θ is acute, i.e., $\theta \in [0, \frac{\pi}{2}]$, what would be the optimal upper bound for $|A|$ subject to the strict inequality $\angle A < \theta$? Danzer and Grünbaum had raised a conjecture on this question which was disproved two decades later by Erdős and Füredi [7]. Since then, the question has remained both interesting and challenging, as, e.g., Gerencsér and Harangi [10], [11] and Aigner and Ziegler [1] discuss; a significant stream of research has focused on how large a set A can be while satisfying the strictly acute bound $\angle A < \frac{\pi}{2}$.

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Now what if the angle bound θ is obtuse, i.e., $\theta \in (\pi/2, \pi)$? In this case, much less appears to be known. It is known, however, that the set $A \subseteq \mathbf{R}^d$ subject to $\angle A \leq \theta$ must be finite; see, e.g., Otsetarova [16]. In the plane, the number of obtuse angles determined by a set is known to grow as a cubic function of its cardinality [2], [8]. Here we address the following natural question.

QUESTION 1.2. *Given $\theta \in (\pi/2, \pi)$, estimate the smallest $N = N(\theta, d) \in \mathbf{N}$ (if any exists) such that for all $A \subseteq \mathbf{R}^d$ satisfying $\angle A \leq \theta$, we have $|A| \leq N$.*

Since $N(\pi, d) = +\infty$, it is not obvious that $\theta < \pi$ makes $N(\theta, d)$ finite; we describe in Appendix A how this fact can be inferred by refining the conclusion of Erdős and Füredi [7, Theorem 4.3], who used sphere packing and covering asymptotics to derive two-sided bounds on $N(\theta, d)$ for θ near π ; their statement is predicated on the unstated requirement that our N (their n) be sufficiently large; we clarify how large in Theorem A.1. After the current manuscript was first posted to the arXiv, we also learned that Kupavskii and Zakharov [13] had used a method inspired by Erdős and Füredi's to show for each $\frac{\pi}{2} < \theta < \pi$ the growth of $N(\theta, d)$ to be *doubly* exponential in d , but with θ dependent rates which are not estimated explicitly. In this paper, we use an entirely different approach to give the first explicit upper bound for $N(\theta, d)$ in the region $\theta < \theta_d$, when

$$(1.1) \quad \theta_d := \arccos\left(-\frac{1}{d}\right) \in \left(\frac{\pi}{2}, \pi\right)$$

shrinks to $\pi/2$ as $d \rightarrow \infty$. Instead of relying on the sphere packing and covering arguments (see Appendix A for more details), we observe that $\theta := \angle A < \theta_d$ implies the points of A form the vertices of a convex polytope. Our bound for the number of vertices of this polytope in terms of θ and d then follows from the Gauss–Bonnet theorem. This argument is a higher dimensional version of the planar statement that if all angles of a convex polygon are at most $2\pi/3$, then the polygon has at most six vertices.

In the first circulated draft of our companion work [15], we exploited the present result to attack a conjecture of Fejes Tóth, concerning the placement of a large number of lines through the origin so as to maximize the expected acute angle between them. There instead we were interested in maximizing a power $\alpha > 1$ of the (renormalized) angle between each pair of lines. Having already established the analogous conjecture in the limiting case $\alpha = \infty$ [14], we used the present results to extend this conclusion to large finite values of α . In subsequent drafts of [15], however, this argument has been replaced by a different approach (based in part on an appendix authored by Bilyk, Glazyrin, Matzke, Park, and Vlasiuk) which allows us to extend our conclusion to the larger range $\alpha > \alpha_{\Delta^d}$, where $\alpha_{\Delta^d} < 2$ in certain cases.

Changing the ambient space from \mathbf{R}^d to \mathbf{R}^{d+1} hereafter, let \mathcal{H}^d be the area measure on $\mathbf{S}^d = \{x \in \mathbf{R}^{d+1} \mid |x| = 1\}$, and let $\omega_d = \mathcal{H}^d(\mathbf{S}^d)$ denote the total area of \mathbf{S}^d , e.g., $\omega_1 = 2\pi$, $\omega_2 = 4\pi$. For $\eta \in (0, \frac{\pi}{2})$, let $f_d(\eta)$ denote the fraction of \mathbf{S}^d covered by generalized normals to the cone

$$(1.2) \quad C_{u,\eta} := \{x \in \mathbf{R}^{d+1} \mid u \cdot x \geq |u||x| \cos \eta\}$$

of half-angle η around $0 \neq u \in \mathbf{R}^{d+1}$, so that

$$\begin{aligned}
 f_d(\eta) &:= \frac{1}{\omega_d} \mathcal{H}^d(\{x \in \mathbf{S}^d \mid x \cdot z \leq 0 \text{ for every } z \in C_{u,\eta}\}) \\
 (1.3) \quad &= \frac{\omega_{d-1}}{\omega_d} \int_0^{\frac{\pi}{2}-\eta} \sin^{d-1}(t) dt \\
 &= \left(\int_0^\pi \sin^{d-1}(t) dt \right)^{-1} \int_0^{\frac{\pi}{2}-\eta} \sin^{d-1}(t) dt.
 \end{aligned}$$

Equivalently, $f_d(\eta)$ is the zeroth curvature measure that the cone $C_{u,\eta}$ assigns to its vertex, in the terminology of Federer (see [9], [17]).

Our result is the following.

THEOREM 1.3 (cardinality under possibly obtuse angle bound). *Fix $d \geq 1$ and $0 < \theta < \theta_{d+1}$ and $\eta_d(\theta) := \arcsin\left(\frac{\sin(\theta/2)}{\sin(\theta_d/2)}\right) \in (0, \frac{\pi}{2})$. If $A \subseteq \mathbf{R}^{d+1}$ satisfies $\angle A \leq \theta$, then (1.1) and (1.3) yield*

$$(1.4) \quad |A| \leq \frac{1}{f_d(\eta_d(\theta))}.$$

Remark 1.4 (dimensional monotonicity). For fixed θ , our cardinality bound $1/f_d(\eta_d(\theta))$ increases with dimension since $f_d(\eta)$ is monotone decreasing with respect to both η and d : indeed, the difference of averages

$$\begin{aligned}
 \frac{\partial \log f_d(\eta)}{\partial d} &= \left(\int_0^{\frac{\pi}{2}-\eta} \sin^{d-1}(t) dt \right)^{-1} \int_0^{\frac{\pi}{2}-\eta} (\log \sin t) \sin^{d-1}(t) dt \\
 &\quad - \left(\int_0^\pi \sin^{d-1}(t) dt \right)^{-1} \int_0^\pi (\log \sin t) \sin^{d-1}(t) dt
 \end{aligned}$$

is negative since $t \in [0, \pi] \mapsto \log \sin t$ is symmetric about $\pi/2$ and increasing on $[0, \pi/2]$.

Example 1.5 (explicit bounds in low dimensions). If $d = 1$, then $\eta_1(\theta) = \frac{\theta}{2}$ and $f_1(\eta) = \frac{1}{2} - \frac{\eta}{\pi}$, so $f_1(\eta_1(\theta))^{-1} = \frac{2\pi}{\pi-\theta}$ yields the known sharp values $\{2, 3, 4, \infty\}$ corresponding to $\theta \in \{0, \frac{\pi}{3}, \frac{\pi}{2}, \pi\}$. However, $f_2(\eta) = \frac{1}{2}(1 - \cos(\frac{\pi}{2} - \eta))$ yields a bound $f_2(\eta_2(\frac{\pi}{2}))^{-1} \approx 10.9$ worse than the sharp value 8 attained by the vertices of the cube in \mathbf{R}^3 .

For nonobtuse sets, the bound (1.4) becomes less and less accurate with increasing dimension, growing slightly faster than exponentially.

CLAIM 1.6 (asymptotic bounds as dimension increases).

$$f_d\left(\eta_d\left(\frac{\pi}{2}\right)\right)^{-1} < 2\left(\frac{\pi}{2}\right)^{2d-1} e^{(d \log d)/2} [1 + O(d^{-1})].$$

This growth, while faster than the known sharp value of 2^{d+1} , is slow compared to the doubly exponential growth of $N(\theta, d)$ observed for each fixed $\theta \in (\frac{\pi}{2}, \pi)$ in [13], suggesting a rich range of intermediate asymptotic behavior in the narrowing region $\frac{\pi}{2} < \theta < \theta_d$ of the (θ, d) plane. It is also smaller than the doubly exponential upper bound obtained in Appendix A using Erdős–Füredi sphere-covering arguments (with C_d independent of d), which confirms that our bounds represent a significant improvement on the state of the art, at least in their limited range of validity.

Proof of Claim 1.6. From the identity

$$\sin(\theta_d/2) := \sqrt{\frac{d+1}{2d}}$$

which follows from (1.1) (and which is reasserted below in Theorem 2.3) we derive

$$\begin{aligned} \eta_d\left(\frac{\pi}{2}\right) &= \arcsin\left(\sqrt{1 - \frac{1}{d+1}}\right) \\ &< \frac{\pi}{2} - \sqrt{2(1 - \sqrt{1 - (d+1)^{-1}})} \\ &= \frac{\pi}{2} - d^{-1/2}(1 + O(d^{-1})), \end{aligned}$$

where $\arcsin(t) < \frac{\pi}{2} - \sqrt{2 - 2t}$ has been used. Estimating (1.3) using $\frac{2}{\pi} < t^{-1} \sin(t) < 1$ on $t \in (0, \frac{\pi}{2})$ yields

$$f_d(\eta) > \left(2 \int_0^{\pi/2} t^{d-1} dt\right)^{-1} \int_0^{\frac{\pi}{2} - \eta} \left(\frac{2}{\pi} t\right)^{d-1} dt,$$

whence

$$f_d\left(\eta_d\left(\frac{\pi}{2}\right)\right)^{-1} < 2\left(\frac{\pi}{2}\right)^{2d-1} d^{d/2}[1 + O(d^{-1})]$$

as desired. □

In the next section, we will prove Theorem 1.3 using two propositions which may have independent interest. Proposition 2.1 shows that the strict inequality $\angle A < \theta_d$ cannot hold unless no point in $A \subseteq \mathbf{R}^d$ is a convex combination of $d+1$ others. On the other hand, if no point in A is a convex combination of other points, then A consists precisely of the vertices of a convex polytope. Proposition 2.4 combines a spherical diameter-to-radius inequality [4] with the generalized Gauss–Bonnet theorem to estimate $|A|$ in terms of the angle bound $\angle A \leq \theta$ in this case. In Appendix A, we review the sphere-packing and covering arguments of Erdős and Füredi for comparison and show how they can be extended to sets which need not be too large.

2. Proofs. Let $\text{conv}(A)$ denote the convex hull and $\text{int}(A)$ the interior of any subset $A \subseteq \mathbf{R}^d$.

PROPOSITION 2.1 (angle estimates from an interior point of a simplex). *Let $d \geq 2$, and let $\{v_0, \dots, v_d\} \subseteq \mathbf{R}^d \setminus \{0\}$ be vertices of a d -dimensional simplex containing the origin. Let $w_i = \frac{v_i}{|v_i|}$. Then*

$$(2.1) \quad \min_{0 \leq i < j \leq d} w_i \cdot w_j \leq -\frac{1}{d}$$

and equality holds if and only if $\text{conv}\{w_0, \dots, w_d\}$ is a regular simplex.

Proof. The proposition clearly holds for $d = 2$ since at least one of the three angles at the origin must exceed $2\pi/3$ unless $\{w_0, w_1, w_2\}$ is equilateral. We will proceed by an induction on dimension d . Let $V = \{v_0, \dots, v_d\}$, so that $\text{conv}(V)$ is the d -dimensional simplex. If the origin lies on the boundary of $\text{conv}(V)$, then the induction hypothesis yields $\min_{i \neq j} w_i \cdot w_j \leq -\frac{1}{d-1}$. So let us assume $0 \in \text{int}(\text{conv}(V))$. Let $w_i = \frac{v_i}{|v_i|}$. We

claim $W = \{w_0, \dots, w_d\}$ also forms vertices of a d -dimensional simplex containing the origin in its interior. To see this, observe

$$\begin{aligned} 0 \in \text{int}(\text{conv}(V)) & \\ \iff \text{for every } i, v_i = - \sum_{j \neq i} \lambda_j v_j \text{ for some } \lambda_j > 0 & \\ \iff \text{for every } i, w_i = - \sum_{j \neq i} \lambda_j w_j \text{ for some } \lambda_j > 0 & \\ \iff 0 \in \text{int}(\text{conv}(W)), & \end{aligned}$$

which proves the claim and gives $a_i > 0$ such that $\sum_{i=0}^d a_i w_i = 0$. Without loss of generality, assume $a_0 = \min_i \{a_i\}$, $a_1 = \max_i \{a_i\}$. Set $b_i = \frac{a_i}{a_0}$, so that $w_0 = -\sum_{i=1}^d b_i w_i$. Now we claim

$$\min_{1 \leq i < j \leq d} w_i \cdot w_j \geq -\frac{1}{d} \text{ implies } w_0 \cdot w_1 \leq -\frac{1}{d}.$$

To see this, observe

$$\begin{aligned} w_0 \cdot w_1 &= - \sum_{i=1}^d b_i w_1 \cdot w_i \\ &\leq -b_1 + \frac{1}{d} \sum_{i=2}^d b_i \\ &\leq -b_1 + \frac{1}{d} \sum_{i=2}^d b_1 = -\frac{b_1}{d} \\ &\leq -\frac{1}{d} \end{aligned}$$

since $b_1 \geq 1$. We have shown (2.1), and in view of the above inequalities, we see that equality holds in (2.1) if and only if $w_i \cdot w_j = -\frac{1}{d}$ for all $1 \leq i < j \leq d$ and $b_1 = 1$, that is, $\sum_{i=0}^d w_i = 0$. By taking dot product with w_j , $j \neq 0$, on the last identity, we conclude that equality holds in (2.1) if and only if $w_i \cdot w_j = -\frac{1}{d}$ for all $0 \leq i < j \leq d$, that is, $\text{conv}\{w_0, \dots, w_d\}$ forms a regular d -dimensional simplex. \square

COROLLARY 2.2 (deciding when points lie in convex position). *Let $d \geq 2$ and $A \subseteq \mathbf{R}^d$. If $\angle A < \theta_d := \arccos(-\frac{1}{d})$, then A consists of the vertices of a convex polytope (not necessarily d -dimensional).*

Proof. If not, there exist $v \in A$ and $\{v_0, \dots, v_k\} \subseteq A \setminus \{v\}$, which forms vertices of a k -dimensional simplex such that $v \in \text{conv}\{v_0, \dots, v_k\}$ by Carathéodory's theorem. Proposition 2.1 then yields i, j such that $\angle(v_i, v, v_j) \geq \theta_k \geq \theta_d$, a contradiction. \square

To derive the desired cardinality bound for convex polytopes, we use the following spherical version of Jung's theorem [12] relating diameter to radius bounds in flat space, established on the unit sphere \mathbf{S}^d equipped with its standard round metric by Dekster [4, Theorem 2].

THEOREM 2.3 (Dekster, 1995). *Let $R \in (0, \frac{\pi}{2})$. If $H \subseteq \mathbf{S}^d$ satisfies $\text{diam } H \leq 2 \arcsin(\sin(\theta_d/2) \sin R) \in (0, \pi)$, then H can be contained in a closed ball of radius R in \mathbf{S}^d . Here $\sin(\theta_d/2) = \sqrt{\frac{d+1}{2d}}$ from (1.1).*

PROPOSITION 2.4 (cardinality bound for convex polytopes). *Let $d \geq 1$ and $0 < \theta < \theta_d$. If $V = \{v_1, \dots, v_n\}$ consists of the vertices of a convex polytope $\text{conv}(V) \subseteq \mathbf{R}^{d+1}$ with nonempty interior and satisfies $\angle V \leq \theta$, then $n \leq \frac{1}{f_d(\eta)}$ with $\eta = \eta_d(\theta) = \arcsin\left(\frac{\sin(\theta/2)}{\sin(\theta_d/2)}\right) \in (0, \frac{\pi}{2})$.*

Proof. For $\kappa \in (0, \frac{\pi}{2})$ and $u, v \in \mathbf{R}^{d+1}$, use (1.2) to define a family of cones $v + C_{u,\kappa} \subseteq \mathbf{R}^{d+1}$ with vertex v , direction $u \neq 0$, and half-angle κ . Now consider any of the given vertices v_i ; by translating the polytope we may assume $v_i = 0$. By dilating λV with sufficiently large $\lambda > 0$ we also assume that only those edges emanating from $v_i = 0$ meet \mathbf{S}^d , and we set $H \subseteq \mathbf{S}^d$ to be the intersection points of those edges with \mathbf{S}^d . Notice $\angle V \leq \theta$ implies $\text{diam } H \leq \theta$, and hence Dekster’s theorem implies, with η in place of R , that H (and hence V) is contained in the cone $v_i + C_{u_i,\eta}$ for some unit vector u_i . Since the conclusion is invariant under translations and dilations, we deduce there exist unit vectors u_1, \dots, u_n such that

$$(2.2) \quad \text{conv } V \subseteq \bigcap_{i=1}^n v_i + C_{u_i,\eta}.$$

Let f_i denote the fraction of \mathbf{S}^d occupied by the generalized normals to $\text{conv}(V)$ at v_i :

$$f_i := \frac{1}{\omega_d} \mathcal{H}^d(\{x \in \mathbf{S}^d \mid (x - v_i) \cdot (z - v_i) \leq 0 \text{ for every } z \in \text{conv } V\}).$$

Then

$$\sum f_i = 1,$$

an intuitive fact which can also be seen as a consequence of, e.g., Federer’s generalization of the Gauss–Bonnet formula, which asserts that for any convex body the zeroth curvature measure of the entire convex body is unity, coinciding with its Euler–Poincaré characteristic [9]; in the case of a convex polytope $\text{conv}(V)$, the zeroth curvature measure vanishes, except at the vertices of the body, and assigns mass f_i to v_i .

Now by the covering property (2.2), we have

$$(2.3) \quad 1 = \sum_{i=1}^n f_i \geq \sum_{i=1}^n f_d(\eta) = n f_d(\eta),$$

whence $n \leq 1/f_d(\eta)$. □

Proof of Theorem 1.3. Let $A \subseteq \mathbf{R}^{d+1}$ satisfy $\angle A \leq \theta < \theta_{d+1}$. We may assume that A is not contained in any d -dimensional hyperplane since otherwise we may apply an induction on dimension using the monotonicity of $d \mapsto f_d(\eta_d(\theta))$ established in Remark 1.4. Then by Corollary 2.2, A is the set of vertices of a $(d + 1)$ -dimensional convex polytope, and the theorem follows from Proposition 2.4 and $\theta_{d+1} < \theta_d$. □

Appendix A. Finite-size Erdős and Füredi bounds. Erdős and Füredi relate the quantity we have estimated to sphere-packing and sphere-covering bounds on the unit sphere \mathbf{S}^{d-1} in [7, Theorem 4.3]. Since their published statement contains at least one misprint (and some tacit hypotheses, as observed in [13]), let us recount their argument and refine their conclusions, so that they apply to all sets, and not only to large ones. Using our angle bound

$$\angle P := \sup_{x,y,z \in P} \angle xyz,$$

let

$$\alpha_d(n) := \inf_{\substack{P \subseteq \mathbf{R}^d \\ |P|=n}} \angle P$$

denote the minimal maximal angle made by n points in \mathbf{R}^d , e.g., $\alpha_d(3) = \pi/3 = 60^\circ$, $\alpha_3(4) \approx 109.5^\circ$, etc. Note $P \subseteq P'$ implies $\angle P \leq \angle P'$, yielding monotonicity of this minimax $\alpha_d(n) \leq \alpha_d(n+1)$ with respect to the size of the sets being considered. This means

$$N_d(\theta) := \sup_{\alpha_d(n) \leq \theta} n$$

is a nondecreasing inverse to $\alpha_d(n)$ (and in fact agrees with $N(\theta, d)$ from our Question 1.2). Any upper (respectively, lower) bound for $\alpha_d(n)$ can therefore be translated into a lower (respectively, upper) bound for $N_d(\theta)$, and vice versa. Theorem 4.3 of [7] asserts

$$(A.1) \quad \frac{\pi}{(\log_2 n)^{1/(d-1)}} < \pi - \alpha_d(n) < \frac{4\pi}{(\log_2 n)^{1/(d-1)}}$$

under the tacit assumption that n needs to be sufficiently large. Inspection of the proof suggests the intended statement should perhaps have been that

$$(A.2) \quad \frac{c_d}{(1 + \log_2 n)^{1/(d-1)}} < \pi - \alpha_d(n) < \frac{C_d}{(-1 + \log_2 |n - 1|)^{1/(d-1)}},$$

where c_d and C_d are certain constants independent of d , to be estimated presently, and the lower bound is only asserted for large n . Note (A.2) is equivalent to the bound

$$(A.3) \quad 2^{(c_d/(\pi-\theta))^{d-1}-1} < N_d(\theta) < 1 + 2^{(C_d/(\pi-\theta))^{d-1}+1}$$

while the requirement that $N_d(\theta)$ be large translates to θ being close to π . For $\theta > \pi - c_d$, the bounds (A.3) grow doubly exponentially with dimension d . Kupavskii and Zakharov established doubly exponential bounds for $N_d(\theta)$ in the full range $\frac{\pi}{2} < \theta < \pi$, but their rate constants are not explicit and must evidently depend on θ near $\frac{\pi}{2}$ [13].

Let us now recall the arguments which establish (A.2). They begin with two crucial statements concerning the packing and covering of the unit sphere in \mathbf{R}^d with balls of radius $\rho/2$, namely that there exist positive functions $c_d(\rho)$ and $C_d(\rho)$ such that we have the following.

(Packing): For all $0 < \rho < \bar{\rho} < \frac{\pi}{2}$, there exist more than $(c_d(\bar{\rho})/\rho)^{d-1}$ (undirected) lines through the origin in \mathbf{R}^d such that any two of them determine an angle greater than ρ .

(Covering): For all $0 < \rho < \bar{\rho} < \pi$, there exist fewer than $(C_d(\bar{\rho})/\rho)^{d-1}$ (undirected) lines through the origin in \mathbf{R}^d such that any line determines an angle less than $\rho/2$ with at least one of them.

Clearly it costs no generality to suppose $C_d(\rho)$ and $-c_d(\rho)$ to be nondecreasing and to choose them to have finite nonzero limits $C_d(0^+)$ and $-c_d(0^+)$. We may take them to be independent of dimension, or we may take them to be minimal, i.e., to be the actual dimension- and radius-dependent packing and covering profiles of \mathbf{S}^{d-1} . Erdős and Füredi assert more, namely that we can take $c_d(\rho) = 1$ and $C_d(\rho) = 4$. However, since they give neither reference nor proof, we have not attempted to confirm these precise values, but we note that they can only hold under a smallness hypothesis on $\bar{\rho}$ which they presumably had in mind since **(Covering)** contradicts $C_d(\rho) = 4$ when $\rho \lesssim \frac{\pi}{2}$ if $d = 2$. In any case, it makes no difference to the remainder of their

argument. Indeed, if we accept these two facts, the proof of the lower bound is a simple iterative construction while the upper bound relies on an elementary lemma from graph theory. We now recall this to refine the conclusions of [7]. See [7] or [13] for a fuller development of other parts of the proof.

THEOREM A.1 (Finite-size Erdős–Füredi-type bounds). *Fix $d \geq 2$, $n \geq 3$, and $0 < \bar{\rho} < \pi$. If $C_d : [0, \infty) \rightarrow (0, \infty)$ satisfies **(Covering)**, then the second inequality in (A.2) holds with $C_d = C_d(\bar{\rho})$. If $\bar{\rho} < \frac{\pi}{2}$ and $c_d : [0, \infty) \rightarrow (0, \infty)$ satisfies **(Packing)**, then the first inequality in (A.2) holds with $c_d = c_d(\bar{\rho})$ provided $n > (d-1) \log_2 \frac{c_d(\bar{\rho})}{\bar{\rho}}$.*

Proof.

Packing bound. Given $0 < \rho < \bar{\rho} < \frac{\pi}{2}$, fix $m > (c_d(\bar{\rho})/\rho)^{d-1}$ and lines L_1, \dots, L_m such that any two of them make an angle larger than ρ . If we regard them instead as directed lines, the angle between any two of them is less than $\pi - \rho$. Take two points A, B parallel to L_1 , and then translate them parallel to L_2 far enough that any vector from the original to the translated pair almost parallels L_2 . The largest angle between these four points is at worst the directed angle between L_1 and L_2 and hence less than $\pi - \rho$. Now iterate the construction to obtain 2^m points having the largest angle less than $\pi - \rho$. This shows

$$\alpha_d(2^m) \leq \pi - \rho < \pi - \frac{c_d(\bar{\rho})}{m^{1/(d-1)}},$$

and the same bound extends to $\alpha_d(n)$ for all $n \leq 2^m$. Thus we have shown the first inequality in (A.2) holds with $c_d = c_d(\bar{\rho})$ as soon as $n > (d-1) \log_2 \frac{c_d(\bar{\rho})}{\bar{\rho}}$.

Covering bound. Given $0 < \rho < \bar{\rho} < \pi$, fix $m < (C_d(\bar{\rho})/\rho)^{d-1}$ and lines L_1, \dots, L_m such that any other line makes an angle less than $\rho/2$ with one of them. For any set P with $2^m + 1$ points, color the complete graph on P by assigning to each edge xy a color i such that xy is within angle $\rho/2$ of L_i . Then a lemma of Szekeres [18] asserts the existence of an odd cycle in some color. (Alternatively, the last paragraph of the proof of Kupavskii and Zakharov’s Proposition 1 [13] gives a self-contained proof of an even simpler statement which also suffices for the present purpose.) This cycle contains points x, y, z such that all edges of the triangle Δxyz are within angle $\rho/2$ of the same L_i . The largest angle in this triangle is at least $\pi - \rho$. Thus $n \geq 2^m + 1$ implies

$$\alpha_d(n) \geq \alpha_d(2^m + 1) \geq \pi - \rho > \pi - \frac{C_d(\bar{\rho})}{m^{1/(d-1)}}.$$

Now letting $\rho \rightarrow 0$ yields that the second inequality in (A.2) holds with $C_d = C_d(\bar{\rho})$ for all m and hence all n . \square

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