TOWARDS THE SMOOTHNESS OF OPTIMAL MAPS ON RIEMANNIAN SUBMERSIONS AND RIEMANNIAN PRODUCTS (OF ROUND SPHERES IN PARTICULAR)

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Abstract. The variant **A3w** of Ma, Trudinger and Wang's condition for regularity of optimal transportation maps is implied by the nonnegativity of a pseudo-Riemannian curvature — which we call *cross-curvature* — induced by the transportation cost. For the Riemannian distance squared cost, it is shown that (1) cross-curvature nonnegativity is preserved for products of two manifolds; (2) both **A3w** and cross-curvature nonnegativity are inherited by Riemannian submersions, as is domain convexity for the exponential maps; and (3) the n-dimensional round sphere satisfies cross-curvature nonnegativity. From these results, a large new class of Riemannian manifolds satisfying cross-curvature nonnegativity (thus **A3w**) is obtained, including many whose sectional curvature is far from constant. All known obstructions to the regularity of optimal maps are absent from these manifolds, making them a class for which it is natural to conjecture that regularity holds. This conjecture is confirmed for certain Riemannian submersions of the sphere such as the complex projective spaces $\mathbf{CP}^n$.

1. Introduction

This paper addresses questions in optimal transportation theory and Riemannian geometry. For a general introduction to these subjects we refer to the books by Villani [V1] [V2] for optimal transport theory and the book by Cheeger and Ebin [CE] for Riemannian geometry.

1.1. Background: optimal transport and pseudo-Riemannian geometry.

In optimal transportation theory one is interested in phenomena which occur when moving mass distributions so as to minimize the transportation cost. Mathematically, there are source and target domains, M, $\bar{M}$, two differential manifolds equipped with a lower semi-continuous cost function $c : M \times \bar{M} \to \mathbf{R} \cup \{\infty\}$. Given two positive Borel probability measures ρ, $\bar{\rho}$ on M, $\bar{M}$, respectively, one wants to understand the optimal map $F : M \to \bar{M}$, which minimizes the average

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cost

$$\int_M c(x, F(x)) d\rho(x) \tag{1.1}$$

among all such maps pushing-forward ρ to $\bar{\rho}$, denoted $F_{\#}\rho = \bar{\rho}$ and meaning that

$$\rho(F^{-1}(E)) = \bar{\rho}(E) \qquad \forall \text{ Borel } E \subset \bar{M}. \tag{1.2}$$

Particular attention has been devoted to the case $M = \bar{M}$, a complete Riemannian manifold with the cost $c = \frac{1}{2}\mathrm{dist}^2$, where dist denotes the Riemannian distance function. Existence and uniqueness of optimal maps in this case is well known due to the work of Brenier [Br] and McCann [M] (and Sturm [S] and Figalli [F] for noncompact manifolds), under the condition that ρ doesn't charge lower dimensional submanifolds (see also [CNM]). Under suitable hypotheses, regularity ($C^0/C^\alpha/C^\infty$) of such optimal maps is known for Euclidean space $M = \mathbf{R}^n$ by results of Delanoë [D1], Caffarelli [Ca1] [Ca2], and Urbas [U], and for flat [Co] and near flat [D2] manifolds by Cordero-Erausquin and Delanoë. Beyond the flat case, Loeper [L2] deduced regularity on the round sphere, by combining his own breakthroughs with pioneering results of Ma, Trudinger and Wang [MTW][TW1][TW2] concerning regularity of optimal maps for general cost functions. Loeper's result is simplified in the work of present authors [KM] where we give an elementary and direct proof of a crucial maximum principle deduced by Loeper from Trudinger and Wang's theory. We referred to Loeper's maximum principle with the acronym **DASM** (see Theorem 2.7), and extended it to the manifold case. In the course of deriving new results, our method was employed and further developed by Figalli and Villani [FV], Figalli and Rifford [FR], Loeper and Villani [LV], and Villani [V2] [V3].

Our contributions in [KM] are based on a pseudo-Riemannian geometric structure h on $M \times \bar{M}$ induced by the transportation cost. Namely when $\dim M = \dim \bar{M}$ this pseudo-metric h is defined on $N \subset M \times \bar{M}$ with $c \in C^4(N)$ as the following symmetric bilinear form on $TM \oplus T\bar{M}$:

$$h := \begin{pmatrix} 0 & -\frac{1}{2}\bar{D}Dc \\ -\frac{1}{2}D\bar{D}c & 0 \end{pmatrix} \tag{1.3}$$

where D and $\bar{D}$ denote the differentials along M and $\bar{M}$, respectively. For non-degeneracy of h we assume that $D\bar{D}c$ and its adjoint $\bar{D}Dc$ are non-degenerate (Ma, Trudinger and Wang's condition **A2**: see Section 2). This is automatically true for the Riemannian distance squared cost, away from the cut-locus, since $\bar{D}Dc$ is a matrix of independent Jacobi fields. This pseudo-metric h geometrizes the regularity theory of optimal maps, by recasting the variant **A3w** [TW1] of Ma, Trudinger and Wang's [MTW] key cost hypothesis for regularity as the non-negativity of certain pseudo-Riemannian sectional curvatures of h. To explain this condition more precisely, we need

Definition 1.1 (cross-curvature). Let $(x, \bar{x}) \in N \subset M \times \bar{M}$. For each $p \oplus \bar{p} \in T_{(x,\bar{x})}N = T_x M \oplus T_{\bar{x}}\bar{M}$, *the cross-curvature* of p and $\bar{p}$ is defined as

$$\text{cross}_{(x,\bar{x})}^{(N,h)}(p, \bar{p}) = R_h((p \oplus 0) \wedge (0 \oplus \bar{p}), (p \oplus 0) \wedge (0 \oplus \bar{p})) \tag{1.4}$$

where R_h is the Riemann curvature operator of the pseudo-metric h. We drop the superscript (N, h) and the subscript $(x, \bar{x})$ when no ambiguity can occur.

Trudinger and Wang's **A3w** condition [TW1] and the strict antecedent **A3s** which they formulated with Ma [MTW] then assert

(1.5) **A3w:** $\text{cross}(p, \bar{p}) \geq 0$ for all $p \oplus \bar{p}$ with $h(p \oplus \bar{p}, p \oplus \bar{p}) = 0$;

(1.6) **A3s:** in addition, $\text{cross}(p, \bar{p}) = 0$ in (1.5) implies $p = 0$ or $\bar{p} = 0$.

A3w and **A3s** are also called *weak regularity* and *strict regularity*, respectively. If (N, h) satisfies the inequality $\text{cross}(p, \bar{p}) \geq 0$ for all $p \oplus \bar{p}$ (whether or not $h(p \oplus \bar{p}, p \oplus \bar{p})$ vanishes), then (N, h) is said to be *non-negatively cross-curved.* Loeper [L1] showed that **A3w** is necessary and **A3s** sufficient for continuity of optimal maps between suitable measures: without **A3w** there are discontinuous optimal maps between smooth measures ρ, $\bar{\rho}$ on nice domains. Trudinger and Wang [TW1] had already shown the sufficiency of **A3w** for continuity (indeed smooth differentiability) of optimal mappings, under much stronger smoothness and convexity restrictions on ρ and $\bar{\rho}$. These restrictions on ρ, $\bar{\rho}$ are relaxed in two dimensions by Figalli and Loeper [FL], where a continuity result for optimal maps was shown under **A3w**. It still remains an open question to show such continuity results in higher dimensions. However, in a separate work with Figalli [FKM1] required by [FL] to complete their argument, we show non-negative cross-curvature allows these smoothness and convexity restrictions to be relaxed without sacrificing continuity of optimal maps and without assuming **A3s**, thus obtaining a continuity theory which extends Caffarelli's $c(x, y) = \frac{1}{2}|x - y|^2$ result [Ca1] to a new class of cost functions.

For the Riemannian distance squared cost $c(x, \bar{x}) = \text{dist}^2(x, \bar{x})/2$ on a Riemannian manifold $M = \bar{M}$, an isometric copy of M is embedded totally geodesically as the diagonal of $M \times M$ with respect to the pseudo-Riemannian metric h. Along this diagonal,

$$\text{cross}_{(x,x)}(p, \bar{p}) = \frac{4}{3} R_M(p \wedge \bar{p}, p \wedge \bar{p})$$

where R_M denotes the curvature operator of M (see [KM] for details). This provides some geometric intuition motivating Loeper's result [L1] that **A3w** implies nonnegative sectional curvature of the Riemannian metric. (However, the nonnegative/positive curvature does not imply **A3w**, as shown by counterexamples in [K] and the more recent work [FRV2].) Loeper also verified **A3s** for the standard round sphere [L2] and used it to obtain $C^{1/\max\{5,4n-1\}}$ and C^∞ regularity results for optimal maps in this spherical setting. This Hölder exponent has since been improved to its sharp value by Liu [L].

1.2. **Main results.** Throughout this paper, if not specified, each Riemannian manifold M is assumed to be complete and to be equipped with the cost function $c = \frac{1}{2}\mathrm{dist}^2$ and this cost induces the pseudo metric h on $N = M \times M \setminus$ cut-locus. A Riemannian manifold M is said to be **A3w** / **A3s** / non-negatively cross-curved if (N, h) satisfies the corresponding cross-curvature condition.

Our main results provide methods of generating new examples of Riemannian manifolds which satisfy non-negative cross-curvature and thus **A3w**. As announced in [KM], we show:

Theorem 1.2. (Products and submersions, particularly of round spheres) *1) S^n with its standard round metric is non-negatively cross-curved.*
2) Let $\pi : \tilde{M} \to M$ be a Riemannian submersion (see Definition 4.1). If $\tilde{M}$ is **A3w** */* **A3s** */ non-negatively cross-curved then so is M.*
3) For product $M_+ \times M_-$ of Riemannian manifolds, if each factor $M_\pm$ is non-negatively cross-curved, then the resulting manifold $M_+ \times M_-$ is non-negatively cross-curved, thus **A3w** *holds (but never* **A3s***).*
4) Moreover, if either of the factors above fail to be non-negatively cross-curved then the product $M_+ \times M_-$ fails to be **A3w***.*

Proof. The proof of assertion 1) is a calculation given in Section 6. Assertion 2) is shown in Section 4. Assertion 3) and 4) are easy facts which are explained in Section 3 in detail. □

Remark 1.3. Following our announcement of Theorem 1.2 [KM], Figalli and Rifford [FR] gave an alternate proof of result 1) in a different form, slightly stronger than the present statement but complementary to the *almost positivity* shown in Definition 4.6 and Theorem 6.2

As a byproduct of our method, we obtain an O'Neill type inequality for cross-curvature in Riemannian submersions (see Theorem 4.5). This verifies that Riemannian submersion quotients of the round sphere all satisfy **A3s**; they have convex domains for their exponential maps by Theorem 4.9. If, in addition the convexity of these domains is strict, then optimal maps are continuous between positively bounded densities, and higher regularity follows on notable examples such as complex projective spaces $\mathbf{CP}^n$ with the Fubini-Study metric: see Section 5. These are new results which are not covered by other discrete quotient cases of Delanoë and Ge [DG] or Figalli and Rifford [FR].

As another important consequence, the Riemannian product of round spheres and Euclidean space $\mathbf{S}^{n_1} \times \cdots \times \mathbf{S}^{n_k} \times \mathbf{R}^l$ and its *Riemannian submersion quotients*, all satisfy cross-curvature non-negativity and thus **A3w**. Since this rules out the known counterexamples to regularity [L1], optimism combines with a lack of imagination to lead to the conjecture that regularity of optimal mappings also holds in such settings. Together with the perturbations [DG][LV][FR] of the round sphere and its discrete quotients discussed by Delanoë and Ge, Loeper and Villani, and Figalli and Rifford, for which the continuity [FR] or regularity [LV][DG] of optimal maps is already shown, these presently form the only examples of non-flat Riemannian manifolds on which the Riemannian distance squared is known to be **A3w**. Other functions of Riemannian distance which satisfy **A3w** in constant curvature spaces have been discovered by Lee and Li [LL], following the initial submission of the present manuscript. (In flat manifold case, **A3w** is trivial, and in fact the cross-curvature vanishes everywhere. Regularity of optimal maps in this flat case is known by Cordero-Erausquin [Co] applying Caffarelli's result [Ca1][Ca2].)

As far as we know, it remains an open challenge to show regularity of optimal maps on the new tensor product type examples $\mathbf{S}^{n_1} \times \cdots \times \mathbf{S}^{n_k} \times \mathbf{R}^l$ (when the supports of the source and target measure are the whole domain). However, Loeper's maximum principle (and a stronger convexity statement) is easily verified on these examples using the results and methods of [KM]: see Corollary 2.11 and Remark 3.4. We hope these key ingredients will make it possible to address the regularity issue in a subsequent work [FKM2].

This paper is organized as follows: Section 2 is devoted to preliminary notions and facts. Some important geometric implications of cross-curvature non-negativity (Theorem 2.10 and Corollary 2.11) are shown also. In Section 3, the tensor product construction of costs is explained. Section 4 discusses the relation between Riemannian submersion and cross-curvature. Notably, we derive an O'Neill type inequality for cross-curvature (see Theorem 4.5) and show **A3w** implies the heredity of domain convexity for the exponential map. Section 5 addresses the continuity and higher regularity of optimal maps on Riemannian submersion quotients of **A3s** manifolds, resolving these questions in examples such as the complex projective spaces $\mathbf{CP}^n$. Section 6 establishes the cross-curvature non-negativity of the standard round sphere.

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2. Preliminaries

In this section, we recall basic terminology and facts from [KM] (also [MTW] [TW1] [L1]). Theorem 2.10 and Corollary 2.11 are not stated there, but are easy consequences of the method in [KM], to which we also refer the reader for proofs of the other results summarized below. For improvements to these results, see also [V2]. Let $N \subset M \times \bar{M}$ be an open set where the cost function $c \in C^4(N)$.

Definition 2.1 (visible sets, twist condition and non-degeneracy). We define the visible sets:

$$N(\bar{x}) = \{x \in M \mid (x, \bar{x}) \in N\},$$
$$\bar{N}(x) = \{\bar{x} \in \bar{M} \mid (x, \bar{x}) \in N\}.$$

Throughout this section it is assumed as in [MTW] that for all $(x, \bar{x}) \in N$,

A1 $\quad Dc(x,\cdot) : \bar{N}(x) \to T^*_x M,\ \bar{D}c(\cdot,\bar{x}) : N(\bar{x}) \to T^*_{\bar{x}} \bar{M}$ are injective;

A2 $\quad D\bar{D}c$ is non-degenerate.

We also call **A1** the *(bi-)twist condition* and **A2** *non-degeneracy.*

We recall an important map of Ma, Trudinger & Wang [MTW], called the *cost-exponential* by Loeper [L1], which coincides with the Riemannian exponential map for $c = \frac{1}{2}\mathrm{dist}^2$.

Definition 2.2. (cost exponential) If $c \in C^2(N)$ is twisted **(A1)**, we define the *c-exponential* on

$$\begin{aligned} \mathrm{Dom}(c\text{-}\mathrm{Exp}_x) :=& - Dc(x, \bar{N}(x)) \\ =& \{p^* \in T^*_x M \mid p^* = -Dc(x,\bar{x}) \quad \text{for some } \bar{x} \in \bar{N}(x)\} \end{aligned} \tag{2.1}$$

by $c\text{-}\mathrm{Exp}_x p^* = \bar{x}$ if $p^* = -Dc(x,\bar{x})$. Non-degeneracy **(A2)** then implies the c-exponential is a diffeomorphism from $\mathrm{Dom}(c\text{-}\mathrm{Exp}_x) \subset T^*_x M$ onto $\bar{N}(x) \subset \bar{M}$.

Remark 2.3 (differential of c-Exp). Linearizing the cost exponential $\bar{x} = c\text{-}\mathrm{Exp}_x p^*$ around $p^* \in T^*_x M$ we obtain a map $c\text{-}\mathrm{Exp}_{(x,\bar{x})*} : T^*_x M \longrightarrow T_{\bar{x}} \bar{M}$ given explicitly by

$$c\text{-}\mathrm{Exp}_{(x,\bar{x}(t))*}(\dot{p}^*(t)) = \dot{\bar{x}}(t) \text{ for } p^*(t) = -Dc(x,\bar{x}(t)). \tag{2.2}$$

Equivalently $c\text{-}\mathrm{Exp}_{(x,\bar{x})*} = -\bar{D}Dc(x,\bar{x})^{-1}$, meaning the inverse tensor $\frac{1}{2}h^{-1}$ to the metric (1.3) — which gives the pseudo-Riemannian correspondence between the tangent and cotangent spaces to $N \subset M \times \bar{M}$ — also carries covectors forward through the cost-exponential.

The following lemma characterizes Ma, Trudinger & Wang's *c-segments* as geodesics of h.

Lemma 2.4. (the c-segments of [MTW] **are geodesics)** *Use a twisted (***A1***) and non-degenerate (***A2***) cost $c \in C^4(N)$ to define a pseudo-metric (1.3) on the domain $N \subset M \times \bar{M}$. Fix $x \in M$. For each line segment $(1-s)p^* + sq^* \in \mathrm{Dom}(c\text{-}Exp_x)$, $s \in [0,1]$, the curve*

$$s \in [0,1] \longrightarrow \sigma(s) := (x, c\text{-}Exp_x((1-s)p^* + sq^*))$$

is an affinely parameterized null geodesic in (N,h). Conversely, every geodesic segment in the totally geodesic submanifold $\{x\} \times \bar{N}(x)$ can be parameterized locally in this way.

To see some relevant geodesic equations in local coordinates, given $s_0 \in [0,1]$, introduce coordinates on M and $\bar{M}$ around $\sigma(s_0)$ so that nearby, the curve $\sigma(s)$ can be represented in the form $(x^1, \ldots, x^n, x^{\bar{1}}(s), \ldots, x^{\bar{n}}(s))$. Differentiating the definition of the cost exponential

$$0 = (1-s)p^*_i + sq^*_i + c_i(\sigma(s)) \tag{2.3}$$

twice with respect to s yields

$$0 = c_{i\bar{j}}\ddot{x}^{\bar{j}} + c_{i\bar{j}\bar{k}}\dot{x}^{\bar{j}}\dot{x}^{\bar{k}} \tag{2.4}$$

for each $i = 1, \ldots, n$. This equation will be used later.

Regarding curvature of the metric h, the following fact is fundamental.

Lemma 2.5. (Non-tensorial expression for curvature) *Use a non-degenerate cost $c \in C^4(N)$ to define a pseudo-metric* (1.3) *on the domain $N \subset M \times \bar{M}$. Let $(s,t) \in [-1,1]^2 \longrightarrow (x(s), \bar{x}(t)) \in N$ be a surface containing two curves $\sigma(s) = (x(s), \bar{x}(0))$ and $\tau(t) = (x(0), \bar{x}(t))$ through $(x(0), \bar{x}(0))$. Then $0 \oplus \dot{\bar{x}}(0)$ defines a parallel vector-field along $\sigma(s)$. Moreover, if $s \in [-1,1] \longrightarrow \sigma(s) \in N$ is a geodesic in (N,h) then*

$$-2\frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{s=0=t} c(x(s), \bar{x}(t)) = \operatorname{cross}_{(x(0),\bar{x}(0))}(\dot{x}(0), \dot{\bar{x}}(0)). \tag{2.5}$$

Note in this lemma that only one curve $\sigma(s)$ needs to be geodesic in (N,h). As a consequence of this result, the conditions **A3w/s** can alternately be characterized by the concavity/strong concavity for each $x \in M$ and $q_0^* \in \mathrm{Dom}(c\text{-}\mathrm{Exp}_x)$ of the function

$$q^* \in T_x^*M \longrightarrow p^i p^j c_{ij}(x, c\text{-}\mathrm{Exp}_x(q_0^* + q^*)) \tag{2.6}$$

restricted to q^* in the nullspace of $p \in T_xM$. Here *strong* concavity refers to negative-definiteness of the Hessian of this function, also called 2-uniform concavity. Cross-curvature nonnegativity asserts this concavity extends to all q^* (not necessarily in the nullspace of p) such that $q_0^* + q^* \in \mathrm{Dom}(c\text{-}\mathrm{Exp}_x)$.

Before recalling the important geometric implications of the curvature properties of h, let us define:

Definition 2.6. (Illuminated set) Given $(x, \bar{x}) \in N$, let $V(x, \bar{x}) \subset M$ denote those points $y \in N(\bar{x})$ for which there exists a geodesic curve from $(x, \bar{x})$ to $(y, \bar{x})$ in $N(\bar{x}) \times \{\bar{x}\}$.

We now state a version of Loeper's maximum principle. Although the statements below are from [KM], where a direct elementary proof is given using pseudo-Riemannian geometry, the fundamental form of the equivalence (Theorem 2.7) below was deduced by Loeper [L1] in the simpler setting $N = M \times \bar{M} \subset \mathbf{R}^n \times \mathbf{R}^n$ from results of Trudinger and Wang [TW1][TW2]. We visualize his maximum principle as asserting that the *double-mountain* $\max[f_0, f_1]$ stays *above* the *sliding mountain* $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$, hence refer to it by the acronym DASM.

Theorem 2.7 (**A3w** $\Leftrightarrow$ **local DASM**). *Let h be the pseudo-Riemannian metric on $N \subset M \times \bar{M}$ induced from the non-degenerate cost $c \in C^4(N)$ as in* (1.3). *The following are equivalent.*
1. (N,h) satisfies **A3w**.
2. **(local DASM)** *For any h-geodesic $\sigma : t \in [0,1] \to (x, \bar{x}(t)) \in N$ and sufficiently small neighbourhood $U \subset V_\sigma$ of x, where $V_\sigma := \cap_{0 \le t \le 1} V(x, \bar{x}(t))$ is from*

Definition 2.6, the sliding mountain $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$ *satisfies the maximum principle*

$$f_t(y) \le \max[f_0, f_1](y) \quad \text{for } 0 \le t \le 1.$$

Then one can prove as in [KM] the following.

Theorem 2.8 (A3w + c-convexity of domains ⇒ DASM). *Let h be the pseudo-Riemannian metric on $N \subset M \times \bar{M}$ induced by the non-degenerate cost $c \in C^4(N) \cap C(M \times \bar{M})$ as in (1.3). Suppose (N, h) is* **A3w***, and the set $V_\sigma := \cap_{0 \le t \le 1} V(x, \bar{x}(t))$ from Definition 2.6 is dense in M for some h-geodesic $\sigma : t \in [0,1] \longrightarrow (x, \bar{x}(t)) \in N$. For any y in M, the sliding mountain $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$ satisfies the maximum principle*

$$\textbf{DASM:} \qquad f_t(y) \le \max[f_0, f_1](y) \quad \text{for } 0 \le t \le 1. \tag{2.7}$$

Remark 2.9 (Relating density of the illuminated set to c-convexity of the domain). In case $N = M \times \bar{M}$, the density hypothesis of the preceding theorem holds whenever $\bar{D}c(M, \bar{x}) \subset T^*_{\bar{x}} \bar{M}$ is convex for each $\bar{x} \in \bar{M}$, since then $V_\sigma = M$. This is the case considered initially by Trudinger and Wang [TW1] and Loeper [L1]. For our argument, it is enough that $\bar{D}c(M, \bar{x}(t))$ be star-shaped around $\bar{D}c(x, \bar{x}(t))$ for each $t \in [0, 1]$. Conversely, $V(x, \bar{x}) = M$ for each $x \in M$ implies convexity of $\bar{D}c(M, \bar{x})$.

Our pseudo-Riemannian method [KM] makes it equally possible to deduce further geometric implications of the cross-curvature condition, as the next theorem and corollary show.

Theorem 2.10 (nonnegative cross-curvature ⇔ local time-convex sliding mountain). *Let h be the pseudo-Riemannian metric on $N \subset M \times \bar{M}$ induced from the non-degenerate cost $c \in C^4(N)$ as in (1.3). The following are equivalent.*
1. (N, h) is non-negatively cross-curved.
2. For each h-geodesic $\sigma : t \in [0, 1] \longrightarrow (x, \bar{x}(t)) \in N$ and sufficiently small neighborhood $U \subset V_\sigma$ of $x \in U$, where $V_\sigma := \cap_{0 \le t \le 1} V(x, \bar{x}(t))$ is from Definition 2.6, the sliding mountain $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$ is a convex function of $t \in [0, 1]$ for each $y \in U$, i.e., $\frac{\partial^2}{\partial t^2} f_t(y) \ge 0$ for $0 \le t \le 1$.

Proof: (1 implies 2). Fix an arbitrary h-geodesic $t : [0, 1] \to (x, \bar{x}(t)) \in N$. Let U be chosen open so that $x \in U \subset V_\sigma$. The existence of such a U is elementary, tedious, and independent of hypothesis 1: it requires checking — at least for δ sufficiently small depending on $T \in [0, 1]$ — that $\cap_{T-\delta \le t \le T+\delta} V(x, \bar{x}(t))$ contains a neighbourhood of x, as we now do. Choose coordinates on M near x. Since the geodesic σ is compact in the open set N, some coordinate ball satisfies $B_r(x) \times \{\bar{x}(t)\} \subset N$ for all $t \in [0, 1]$. Since the coordinate charts $x \in M \longrightarrow \bar{D}c(x, \bar{x}(t)) \in T^*_{\bar{x}(t)} \bar{M}$ are C^2 smooth functions of $(x, t) \in M \times [0, 1]$, taking $r = r(T)$ and $\delta(T) > 0$ sufficiently small ensure $\bar{D}c(B_r(x), \bar{x}(t))$ is convex for each t within $\delta(T)$ of T. This convexity implies $B_r(x) \subset V(x, \bar{x}(t))$. Extracting a finite subcover $\cup_{i=1}^N (T_i - \delta(T_i), T_i + \delta(T_i))$ of $[0, 1]$ and taking $U = \cap_{i \le N} B_{r(T_i)}(x)$ will suffice.

Now define $f_t(\cdot) := -c(\cdot, \bar{x}(t)) + c(x, \bar{x}(t))$. Fix arbitrary $t_0 \in [0,1]$ and $y \in U$. The fact $U \subset V_\sigma$ guarantees an h-geodesic $s : [0,1] \to (x(s), \bar{x}(t_0)) \in N$ with $x(0) = x$ and $x(1) = y$. Define an auxiliary function $g(s) := \frac{\partial^2}{\partial t^2}\big|_{t=t_0} f_t(x(s))$, which shall be shown to be non-negative for $s \in [0,1]$. By Lemma 2.5 and the cross-curvature non-negativity,

$$\frac{d^2 g}{ds^2} \geq 0, \text{ for } s \in [0,1]. \tag{2.8}$$

In particular, $g(s)$ is convex. It is clear that $g(0) = \frac{\partial^2}{\partial t^2}\big|_{t=t_0} f_t(x) = 0$. We also claim $g'(0) = 0$: introducing coordinates $x^1, \ldots, x^n$ around $x = x(0)$ on M and $x^{\bar{1}}, \ldots, x^{\bar{n}}$ around $\bar{x}(t_0)$ on $\bar{M}$, we compute

$$\begin{aligned} \frac{dg}{ds}\bigg|_{s=0} &= -(c_{\bar{i}k}(x(0), \bar{x}(t_0))\ddot{x}^{\bar{i}} + c_{\bar{i}\bar{j}k}(x(0), \bar{x}(t_0))\dot{x}^{\bar{i}}\dot{x}^{\bar{j}})\dot{x}^k \\ &= 0 \end{aligned} \tag{2.9}$$

by the h-geodesic equation (2.4) for $t \in [0,1] \to (x, \bar{x}(t)) \in N$. From (2.8), this shows $g(s) \geq 0$, for $0 \leq s \leq 1$, in particular at $s = 1$,

$$g(1) = \frac{\partial^2}{\partial t^2} f_t(y)\bigg|_{t=t_0} \geq 0.$$

Since we have used $U \subset V_\sigma$ but not the openness or smallness of U, we have actually deduced convexity of $t \in [0,1] \longrightarrow f_t(y)$ for all $y \in V_\sigma$. A fortiori, $1 \Longrightarrow 2$. △

Proof: (2 implies 1). Fix $(x, \bar{x}) \in N$, $p \oplus \bar{p} \in T_{(x,\bar{x})}N$. It shall be shown that $\sec_{(x,\bar{x})}((p \oplus 0) \wedge (0 \oplus \bar{p})) \geq 0$. Choose an h-geodesic $t \in [-1,1] \to (x, \bar{x}(t)) \in N$, with $\bar{x}(0) = \bar{x}$, $\dot{\bar{x}}(0) = \bar{p}$, and a curve $s \in [-1,1] \to (y(s), \bar{x}) \in N$, with $y(0) = x$, $\dot{y}(0) = p$. Let $f_t(\cdot) := -c(\cdot, \bar{x}(t)) + c(x, \bar{x}(t))$. Suppose

$$g(s) := \frac{\partial^2}{\partial t^2}\bigg|_{t=0} f_t(y(s)) \geq 0, \tag{2.10}$$

for $s \in [-1,1]$: this holds if the property 2 is assumed and the curve $y(s)$ is chosen inside the neighborhood U of x constructed at the outset. Note $g(0) = \frac{\partial^2}{\partial t^2}\big|_{t=0} f_t(x) = 0$. Thus, from (2.10), $g'(0) = 0$ and

$$\begin{aligned} 0 &\leq \frac{d^2}{ds^2}\bigg|_{s=0} g(s) \\ &= \frac{1}{2}\mathrm{cross}_{(x,\bar{x})}(\dot{y}(0), \dot{\bar{x}}(0)). \end{aligned}$$

The last equality comes from Lemma 2.5. This completes the proof $2 \Longrightarrow 1$. □

Corollary 2.11 (non-negative cross-curvature + c-convexity of domains $\Longrightarrow$ time-convex sliding mountain). *Suppose the pseudo-Riemannian metric h induced by the non-degenerate cost $c \in C^4(N) \cap C(M \times \bar{M})$ on $N \subset M \times \bar{M}$ in* (1.3) *is non-negatively cross-curved. If for some h-geodesic $\sigma : t \in [0,1] \longrightarrow (x, \bar{x}(t)) \in N$ the set $V_\sigma := \cap_{0 \leq t \leq 1} V(x, \bar{x}(t))$ from Definition 2.6 is dense in M, then for each $y \in M$, the sliding mountain $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$ satisfies*

$$f_t(y) \leq (1-t) f_0(y) + t f_1(y) \quad \text{for } 0 \leq t \leq 1. \tag{2.11}$$

Proof. For $y \in V_\sigma$, inequality (2.11) was established while deriving implication 1 $\Longrightarrow$ 2 of Theorem 2.10. The inequality extends to $y \in M$ by the density of V_σ and the continuity of c on $M \times \bar{M}$. □

Remark 2.12. The density condition of Theorem 2.8 and Corollary 2.11 can be further weakened by the works of Figalli, Loeper and Villani [FV][LV] (see [V2, Theorem 12.36]).

3. Tensor products of pseudo-Riemannian metrics

This section contains the proofs of assertions 3) and 4) in Theorem 1.2. Assertion 3) is actually a corollary of a more general theorem, which does not require the transportation cost c or manifold M to be Riemannian:

Theorem 3.1. (Products preserve non-negative cross-curvature) *Let $c_\pm \in C^4(N_\pm)$ be non-degenerate non-negatively cross-curved costs on two manifolds $N_\pm \subset M_\pm \times \bar{M}_\pm$. Then $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$ is non-degenerate and non-negatively cross-curved on $(x_+, \bar{x}_+, x_-, \bar{x}_-) \in N_+ \times N_-$, but never* **A3s**.

Proof. If the cost functions $c_\pm : N_\pm \subset M_\pm \times \bar{M}_\pm \to \mathbf{R}$ define non-degenerate metrics $h_\pm$, then so does the cost $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$ on $N = N_+ \times N_-$, since the corresponding metric h separates into block anti-diagonal components $h_\pm$ with non-vanishing determinants. In this sense the geometry (N, h) is the pseudo-Riemannian tensor product of the geometries $(N_\pm, h_\pm)$. It follows that the product $(\gamma_+(s), \gamma_-(s))$ of geodesics $s \in [-1, 1] \longrightarrow \gamma_\pm(s)$ in $N_\pm$ is a geodesic in $N = N_+ \times N_-$. Lemma 2.5 then implies

$$\mathrm{cross}(p_+ \oplus p_-, \bar{p}_+ \oplus \bar{p}_-) = \mathrm{cross}_+(p_+, \bar{p}_+) + \mathrm{cross}_-(p_-, \bar{p}_-) \tag{3.1}$$

for all $(x_+, \bar{x}_+, x_-, \bar{x}_-) \in N$ and $(p_+ \oplus p_-) \oplus (\bar{p}_+ \oplus \bar{p}_-) \in T_{(x_+, \bar{x}_+, x_-, \bar{x}_-)}N$. The proof is completed by observing that non-negativity of both summands guarantees the same for their sum. Choosing $p_- = 0$ and $\bar{p}_+ = 0$ in identity (3.1) demonstrates that the tensor product cost c cannot satisfy **A3s**. □

Proof of Theorem 1.2(3). Given two Riemannian manifolds $M_\pm = \bar{M}_\pm$, denote their geodesic distances squared by $c_\pm(x_\pm, \bar{x}_\pm) = \frac{1}{2}\mathrm{dist}^2_\pm(x_\pm, \bar{x}_\pm)$ respectively, and let $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = \frac{1}{2}\mathrm{dist}^2((x_+, x_-), (\bar{x}_+, \bar{x}_-))$ denote the geodesic distance squared on $M_+ \times M_-$ equipped with the Riemannian product metric. Then $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$, and the non-negative cross-curvature of c on $N = N_+ \times N_-$ follows from Theorem 3.1, taking $N_\pm$ to be the domains where $c_\pm$ are smooth. This concludes the proof of Theorem 1.2(3). □

The assertion 4 in Theorem 1.2 follows easily from the next lemma.

Lemma 3.2 (Certain cross-curvatures vanish in any Riemannian setting). *Define the pseudo-metric (1.3) using the cost $c = d^2/2$ on a Riemannian manifold (M, d). Each point $(x, \bar{x}) \in N = M \times M \setminus$ cut-locus with $x \neq \bar{x}$ admits a 2-plane whose cross-curvature vanishes. For example, letting $t \in [0, 1] \to \gamma(t) \in M$ be a geodesic from $x = \gamma(0)$ to $\bar{x} = \gamma(1)$ yields $\mathrm{cross}(\dot\gamma(0), \dot\gamma(1)) = 0$ but*

$$h(\dot\gamma(0) \oplus \dot\gamma(1), \dot\gamma(0) \oplus \dot\gamma(1)) = d(x, \bar{x})^2.$$

Proof. Any (affinely parameterized) Riemannian geodesic satisfies $d(\gamma(s),\gamma(t)) = |s-t|d(\gamma(0),\gamma(1))$. Defining $f(s,t) = c(\gamma(s),\gamma(t)) = |s-t|^2 d(x,\bar{x})^2/2$ allows us to compute

$$h(\dot\gamma(0)\oplus\dot\gamma(1),\dot\gamma(0)\oplus\dot\gamma(1)) = -\frac{\partial^2 f}{\partial s\partial t}\Big|_{(s,t)=(0,1)} \tag{3.2}$$

immediately. Moreover, the fact that $t\in[0,1]\longrightarrow(x,\gamma(t))$ is an h-geodesic yields the vanishing of $\mathrm{cross}(\dot\gamma(0),\dot\gamma(1)) = -2\partial^4 f/\partial s^2\partial t^2|_{(s,t)=(0,1)}$ via Lemma 2.5. □

Proof of Theorem 1.2(4). Without loss of generality, assume M_+ fails to be non-negatively cross-curved. Then $\mathrm{cross}_+(p_+,\bar p_+)<0$ for some $(p_+,\bar p_+)\in T_{(x_+,\bar x_+)}N_+$. Noting

$$h_+(p_+\oplus\pm\bar p_+, p_+\oplus\pm\bar p_+) = \pm h_+(p_+\oplus\bar p_+, p_+\oplus\bar p_+)$$

but $\mathrm{cross}_+(p_+,\pm\bar p_+) = \mathrm{cross}_+(p_+,\bar p_+)$, we may assume that $h_+(p_+\oplus\bar p_+,p_+\oplus\bar p_+)<0$. Pick any nontrivial geodesic γ_- in M_-. Noting $h_-(\dot\gamma_-(0)\oplus\dot\gamma_-(1),\dot\gamma_-(0)\oplus\dot\gamma_-(1)) = d^2(\gamma(0),\gamma(1))>0$ from (3.2), one can choose $\lambda\in\mathbf{R}$ so that

$$\begin{aligned}&h((p_+\oplus\lambda\dot\gamma_-(0))\oplus(\bar p_+\oplus\lambda\dot\gamma_-(1)),(p_+\oplus\lambda\dot\gamma_-(0))\oplus(\bar p_+\oplus\lambda\dot\gamma_-(1)))\\&=h_+(p_+\oplus\bar p_+,p_+\oplus\bar p_+)+\lambda^2 h_-(\dot\gamma_-(0)\oplus\dot\gamma_-(1),\dot\gamma_-(0)\oplus\dot\gamma_-(1))\\&=0.\end{aligned}$$

However, from (3.1) and Lemma 3.2,

$$\begin{aligned}&\mathrm{cross}(p_+\oplus\lambda\dot\gamma_-(0),\bar p_+\oplus\lambda\dot\gamma_-(1))\\&=\mathrm{cross}_+(p_+,\bar p_+)+\lambda^4\mathrm{cross}_-(\dot\gamma_-(0),\dot\gamma_-(1))\\&=\mathrm{cross}_+(p_+,\bar p_+)<0.\end{aligned}$$

This completes the proof that $(N_+\times N_-, h_+\oplus h_-)$ fails to be weakly regular **A3w** unless both $(N_\pm,h_\pm)$ are non-negatively cross-curved. □

At present, the authors do not know any Riemannian manifold which is **A3w** yet fails to be non-negatively cross-curved. On the other hand, the proof of Theorem 1.2 (4) adapts to non-Riemannian cost functions as in the next example, where it shows that the tensor product of two (or more) costs on manifolds which are not all non-negatively cross-curved is surely not **A3w**.

Example 3.3 (A3s $\times$ A3s $\not\Rightarrow$ A3w). Let $c_\pm(x,\bar x) = -\log|x-\bar x|$ on $M_\pm\times M_\pm\setminus\Delta$, $M_\pm=\mathbf{R}^n$, $\Delta := \{(x,x)\mid(x,x)\in M_\pm\times M_\pm\}$. This logarithmic cost function is known to be **A3s** [MTW] [TW1] but it does not induce non-negatively cross-curved pseudo-metric as indicated, e.g. in [KM, Example 3.5]. To see this fact one uses (2.6), whose righthand-side coincides for this logarithmic cost with

$$p^i p^j f_{ij}\Big|_{(Df)^{-1}(-q_0^*-q^*)} = 2((q_0+q)^*_i p^i)^2 - |p|^2|(q_0+q)^*|^2$$

where $f(x-\bar x) := -\log|x-\bar x|$. Here the righthand-side is strictly convex with respect to q^* along the Euclidean line parallel to p, but strictly concave along the nullspace of p. As a result $\mathrm{cross}_\pm(p_\pm, \lambda \bar p_\pm) < 0$ for $p_\pm$ parallel to $\bar D Dc(x,\bar x)\bar p_\pm$ as Euclidean vectors and $\lambda \neq 0$. On the other hand,

$$h_+(p_+ \oplus \bar p_+, p_+ \oplus \bar p_+) + h_-(p_- \oplus \lambda \bar p_-, p_- \oplus \lambda \bar p_-)$$

vanishes for some non-zero $\lambda \in \mathbf{R}$. From (3.1), the pseudo-Riemannian metric h induced by the cost $c((x,\bar x),(y,\bar y)) = c_+(x_+,\bar x_+) + c_-(x_-,\bar x_-)$ on $(M_+ \times M_-) \times (M_+ \times M_-)$ then fails to be **A3w**.

Remark 3.4 (**tensor product examples, Loeper's maximum principle, and time-convexity of the sliding mountain).** As mentioned in the introduction, Loeper's maximum principle (**DASM**) is a key property for the regularity of optimal maps. The conclusions of Theorem 2.8 and Corollary 2.11 (hence **DASM**) hold for the distance squared cost on the Riemannian product of round spheres $M = \bar M = \mathbf{S}^{n_1} \times \cdots \times \mathbf{S}^{n_k} \times \mathbf{R}^l$ — thus also on its Riemannian submersions (see Theorem 4.8). To see this, first note that by the result of present section and Section 6, M satisfies non-negative cross-curvature on $N = M \times \bar M \setminus$ cut-locus. The density condition of $\cap_{0\le t\le 1} V(x,\bar x(t))$ is easily checked since the cut locus of one point in this example is a smooth submanifold of codimension greater than or equal to 2. This new global result illustrates an advantage of our method over other approaches [TW1] [TW2] [L1], where one would require a regularity result for optimal maps (or some a priori estimates) to obtain the conclusion of Theorem 2.8. For example, to implement these other approaches for the manifolds of this tensor product example, one would need to establish that an optimal map remains *uniformly away from the cut locus*, as is currently known only for a single sphere $M = \bar M = \mathbf{S}^n$ from work of Delanoë & Loeper [DL] (alternately [L1] or Appendix of [KM]), for the case of perturbations [DG][LV][V3], and for the Riemannian submersion quotients of $\mathbf{S}^n$ discussed in Section 5, which include the complex projective space $\mathbf{CP}^n$ with Fubini-Study metric. To the best of our knowledge, no one has yet succeeded in establishing regularity results for this tensor product example, though as mentioned above we expect to resolve this in a subsequent work [FKM2].

4. Riemannian submersions and cross-curvature

In this section we prove the assertion 2) in Theorem 1.2. The key result is an O'Neill type inequality contained in Theorem 4.5 which compares cross-curvatures in Riemannian submersion. Theorem 4.8 deals with the survival of global properties such as Loeper's maximum principle (**DASM**) and time-convexity of the sliding mountain, under Riemannian submersion. In the next section these results will be applied to show the regularity of optimal maps on certain Riemannian submersion quotients of the round sphere, such as the complex projective space $\mathbf{CP}^n$ with the Fubini-Study metric (see Section 5). From now on we focus on the case of manifolds with costs given by strictly convex increasing functions of Riemannian distance (such as $c = \frac{1}{2}\mathrm{dist}^2$). In this case the c-exponential map coincides with the Riemannian exponential map: c-$\mathrm{Exp} = \exp$.

Recall the definition and basic facts of Riemannian submersion.

Definition 4.1 (Riemannian submersion). (See [CE].) A surjective differentiable map $\pi : M \to B$ from a Riemannian manifold M onto a Riemannian manifold B is said to be a *Riemannian submersion* if the following hold:

- π is a submersion, i.e. $d\pi : T_xM \to T_{\pi(x)}B$ is surjective for each $x \in M$;
- for the orthogonal decomposition $T_xM = \ker d\pi \oplus (\ker d\pi)^\perp$ for each $x \in M$, $d\pi\big|_{(\ker d\pi)^\perp}$ is an isometry.

The subspaces $\mathcal{V} := \ker d\pi$, $\mathcal{H} := (\ker d\pi)^\perp$ are called *vertical* and *horizontal* subspaces, respectively. For each $v \in T_bB$, $b \in B$, there exists its unique *horizontal lift* $\tilde{v} \in T_xM \cap \mathcal{H}$ for each $x \in \pi^{-1}(b)$ such that $d\pi(\tilde{v}) = v$. We use the metric identifications $T_b^*B = T_bB$ and $T_x^*M = T_xM$ to extend the definition of a horizontal lift to cotangent vectors. For each piecewise smooth curve $\gamma : [0,1] \to B$ and $x \in \pi^{-1}(\gamma(0))$, there exists its *horizontal lift* $\tilde{\gamma} : [0,1] \to M$ such that $\gamma(0) = x$, $\pi(\tilde{\gamma}) = \gamma$, $\dot{\tilde{\gamma}}(t) \in \mathcal{H}$. Moreover, γ is a geodesic if and only if its horizontal lift $\tilde{\gamma}$ is. If γ is minimal, then so is $\tilde{\gamma}$. This property yields, for the horizontal lifts $\tilde{v} \in T_xM \cap \mathcal{H}$ and $x \in \pi^{-1}(b)$ of $v \in T_bB$ and $b \in B$,

$$\pi(\exp_x \tilde{v}) = \exp_b v. \tag{4.1}$$

Regarding Riemannian distance,

$$\mathrm{dist}_M(x, y) \geq \mathrm{dist}_B(\pi(x), \pi(y)). \tag{4.2}$$

We call M the *total space* of the Riemannian submersion and B the *base* of the Riemannian submersion or the *Riemannian submersion quotient.*

Many examples of Riemannian submersions may be found in Cheeger & Ebin [CE], Besse [Be] or more recent book by Falcitelli, Ianus and Pastore [FIP]. Every Riemannian covering projection is obviously a Riemannian submersion. Other important examples are Hopf fibrations such as complex and quaternionic projective spaces $\mathbf{CP}^m$ and $\mathbf{HP}^m$, where the standard round sphere S^n (sectional curvature $\equiv 1$) is the total space.

Example 4.2 (Hopf fibrations). $\pi : S^{2m+1} \to \mathbf{CP}^m$, $\pi : S^{4m+3} \to \mathbf{HP}^m$ (see pages 257–258 of [Be] or 4–8 of [FIP]). The base spaces $\mathbf{CP}^m$ and $\mathbf{HP}^m$ have real dimensions $2m$, $4m$, respectively, and have non-isotropic sectional curvatures K, $1 \leq K \leq 4$.

Our goal in this section is to compare the cross-curvature of the base space with that of the total space of Riemannian submersion. For this purpose, we use the definition above to assign to each pair of points in the base space a corresponding pair of points horizontally lifted to the total space.

Definition 4.3 (horizontal lift of a pair of points). Let $\pi : M \to B$ be a Riemannian submersion. For each pair of points $(x, \bar{x}) \in N_B = B \times B \setminus$cut-locus, the unique minimal geodesic $\gamma : [0,1] \to B$ with $\gamma(0) = x$, $\gamma(1) = \bar{x}$ has its horizontal lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{x}$, $\tilde{\gamma}(1) = \tilde{\bar{x}}$. Then the pair $(\tilde{x}, \tilde{\bar{x}}) \in N_M = M \times M \setminus$ cut-locus is said to be a *horizontal lift of* $(x, \bar{x})$.

To have a bit more general conclusion in the next theorem, we observe the following (see also McCann [M, Theorem 13] for bi-twistedness and Lee and Li [LL, Proposition 2.5] for non-degeneracy):

Lemma 4.4 (non-degenerate radial costs). *[cf.* [M] [LL]*] Let $f \in C^4(\mathbf{R})$ be a strictly convex even function, so that the derivative f' has an inverse function. Let $M = \bar{M}$ be a Riemannian manifold, and $c(x,\bar{x}) = f(\mathrm{dist}(x,\bar{x}))$ a cost function defined on $M \times \bar{M}$, where* dist *denotes the Riemannian distance. By the properties of f this cost c is smooth on $N_M := M \times M \setminus$* cut-locus *and is bi-twisted* **A1***, with c-Exp and* exp *having the relation*

$$c\text{-}Exp_x(p^*) = \begin{cases} \exp_x\left(\frac{(f')^{-1}(|p|)}{|p|}p\right) & \text{for } 0 \neq p^* \in \mathrm{Dom}\, c\text{-}Exp_x, \\ x & \text{for } p^* = 0, \end{cases} \tag{4.3}$$

where the vector p and the co-vector p^* are identified by the Riemannian metric. On N_M the cost c is non-degenerate **A2** *if and only if $f''(d) > 0$ for all $0 \leq d <$* $\mathrm{Diam}(M)$*. In this case, it induces a pseudo-Riemannian metric h_M on N_M as in* (1.3)*.*

Proof. For $f(d) = d^2/2$, the formula (4.3) is well-known. For general f, this follows by the chain rule,

$$-Dc(x,\bar{x}) = -f'(\mathrm{dist}(x,\bar{x}))D\mathrm{dist}(x,\bar{x}),$$

Note that the twist condition **A1** follows from this formula, strict convexity of f (thus, f' is invertible), and the fact $f'(0) = 0$ (noting f is even). To see when non-degeneracy **A2** holds, let $b(\cdot)$ denote the function for which $f(d) = b(d^2/2)$, so that

$$\bar{D}Dc = b'\left(\frac{\mathrm{dist}^2}{2}\right)\bar{D}D\frac{\mathrm{dist}^2}{2} + b''\left(\frac{\mathrm{dist}^2}{2}\right)D\frac{\mathrm{dist}^2}{2} \otimes \bar{D}\frac{\mathrm{dist}^2}{2}. \tag{4.4}$$

For $(x,\bar{x}) \notin$ cut-locus, the mixed partial derivative $\bar{D}D\mathrm{dist}^2/2$ is non-degenerate, and gives a map from $T_{\bar{x}}M$ onto T^*_xM, or rather onto T_xM using the metric identification of tangent and cotangent space at x. With this metric identification, the map $-\bar{D}D\mathrm{dist}^2/2$ can be recognized to be the inverse map of $d\exp_x$, where d denotes the differential of $\exp_x p$ with respect to $p \in T_xM$. Thus the vectors in $T_{\bar{x}}M$ are brought by this map to those of T_pT_xM, where p is the inverse exponential image of $\bar{x}$, using the canonical identification between T_pT_xM and T_xM. If $x = \bar{x}$, we see $\bar{D}Dc$ is non-degenerate if and only if $0 \neq b'(0) = f''(0)$. If $x \neq \bar{x}$, multiply (4.4) by $d\exp_x p$ to yield a linear operator on T_xM, namely

$$-\bar{D}Dc(x,\bar{x})d\exp_x p = b'\left(\frac{\mathrm{dist}^2}{2}\right)\mathrm{Id} + b''\left(\frac{\mathrm{dist}^2}{2}\right)\mathrm{dist}^2(x,\bar{x})\hat{p} \otimes \hat{p}^*, \tag{4.5}$$

where $\hat{p}\otimes\hat{p}^*$ denotes projection onto the unit vector $\hat{p} = p/|p|$ pointing in the direction from x to $\bar{x}$; note that to obtain this second term we used the so-called Gauss lemma asserting that the Jacobi operator in the radial direction is the identity. Letting $d = \mathrm{dist}(x,\bar{x})$ and aligning $\hat{p}$ with the first coordinate axis, in Riemannian normal coordinates the right hand side of (4.5) becomes

$$b'(\frac{d^2}{2})\,\mathrm{Id} + b''(\frac{d^2}{2})d^2\,\mathrm{diag}[1, 0, 0, \cdots, 0].$$

This matrix is non-degenerate if and only if $0 \neq b'(d^2/2) + b''(d^2/2)d^2 = f''(d)$, So non-degeneracy holds if and only if $f''(d) > 0$ for all $0 \leq d < \mathrm{Diam}(M)$, meaning the convexity of f is strong. □

Now we are ready to show our main theorem in this section. We shall henceforth assume f satisfies all hypotheses of the preceding lemma, and say f is *strongly* convex if $f'' > 0$ on $\mathbf{R}$.

Theorem 4.5 (cross-curvature and Riemannian submersion). *Let $\pi : M \to B$ be a Riemannian submersion from M to B. Let $f \in C^4(\mathbf{R})$ be even and strongly convex. Let $c_M = f \circ \mathrm{dist}_M$ and $c_B = f \circ \mathrm{dist}_B$ be cost functions defined on $M \times M$ and $B \times B$, where* dist *denotes the Riemannian distance of the corresponding manifold; c_M and c_B induce the pseudo-Riemannian metrics h_M and h_B on $N_M := M \times M \setminus \text{cut-locus}$ and $N_B := B \times B \setminus \text{cut-locus}$ respectively, as in* (1.3)*. Fix $(x, \bar{x}) \in N_B$ and let $(\tilde{x}, \tilde{\bar{x}}) \in N_M$ be a horizontal lift of $(x, \bar{x})$. Given $v \oplus \bar{v} \in T_{(x,\bar{x})}N_B$ there exists $\tilde{w} \oplus \tilde{\bar{w}} \in T_{(\tilde{x},\tilde{\bar{x}})}N_M$ with $d\pi_{\tilde{x}}(\tilde{w}) = v$, $d\pi_{\tilde{\bar{x}}}(\tilde{\bar{w}}) = \bar{v}$, such that*

$$h_B(v \oplus \bar{v}, v \oplus \bar{v}) = h_M(\tilde{w} \oplus \tilde{\bar{w}}, \tilde{w} \oplus \tilde{\bar{w}}) \tag{4.6}$$

and

$$\mathrm{cross}^{(N_B,h_B)}_{(x,\bar{x})}(v, \bar{v}) \geq \mathrm{cross}^{(N_M,h_M)}_{(\tilde{x},\tilde{\bar{x}})}(\tilde{w}, \tilde{\bar{w}}). \tag{4.7}$$

For example, it suffices to take $\tilde{w}^* = -\bar{D}Dc_M(\tilde{x}, \tilde{\bar{x}})\tilde{\bar{w}} \in T^*_{\tilde{x}}M$ to be the horizontal lift of $v^* = -\bar{D}Dc_B(x, \bar{x})\bar{v} \in T^*_x B$ and $\tilde{\bar{w}}^* = -D\bar{D}c_M(\tilde{x}, \tilde{\bar{x}})\tilde{w} \in T^*_{\tilde{\bar{x}}}M$ to be the horizontal lift of $\bar{v}^* = -D\bar{D}c_B(x, \bar{x})v \in T^*_{\bar{x}}B$.

Proof of (4.7)*.* Let $(\tilde{x}, \tilde{\bar{x}})$ be a horizontal lift of $(x, \bar{x}) \in N_B$, and define q^* and $\bar{q}^*$ by c_B-$\mathrm{Exp}_x q^* = \bar{x}$ and c_B-$\mathrm{Exp}_{\bar{x}}\bar{q}^* = x$. Then $\mathrm{dist}_B(x, \bar{x}) = \mathrm{dist}_M(\tilde{x}, \tilde{\bar{x}})$ and it follows from (4.3) that the horizontal lifts $\tilde{q}^*$ of q^* and $\tilde{\bar{q}}^*$ of $\bar{q}^*$ satisfy c_M-$\mathrm{Exp}_{\tilde{x}}\tilde{q}^* = \tilde{\bar{x}}$ and c_M-$\mathrm{Exp}_{\tilde{\bar{x}}}\tilde{\bar{q}}^* = \tilde{x}$. To fixed $v \in T_x B$ and $\bar{v} \in T_{\bar{x}}B$ correspond $\bar{v}^* \in T^*_{\bar{x}}B$ and $v^* \in T^*_x B$ such that $\bar{v}^* = -D\bar{D}c_B(x, \bar{x})v$ and $v^* = -\bar{D}Dc_B(x, \bar{x})\bar{v}$. Equivalently, $v^* \oplus \bar{v}^* = 2h_B(v \oplus \bar{v}, \,\cdot\,)$.

Now let $\Sigma : (s, t) \in [-1, 1]^2 \longrightarrow (x(s), \bar{x}(t)) \in N_B$ be the surface given by $x(s) = c_B$-$\mathrm{Exp}_{\bar{x}}(\bar{q}^* + s\bar{v}^*)$ and $\bar{x}(t) = c_B$-$\mathrm{Exp}_x(q^* + tv^*)$. By Lemma 2.4, the curves $\sigma(s) = (x(s), \bar{x})$ and $\tau(t) = (x, \bar{x}(t))$ through $(x(0), \bar{x}(0)) = (x, \bar{x})$ are h_B-geodesics. Lift Σ to $\tilde{\Sigma} : (s, t) \in [-1, 1]^2 \longrightarrow (\tilde{x}(s), \tilde{\bar{x}}(t)) \in N_M$ by setting $\tilde{x}(s) = c_M$-$\mathrm{Exp}_{\tilde{\bar{x}}}(\tilde{\bar{q}}^* + s\tilde{\bar{w}}^*)$, and $\tilde{\bar{x}}(t) = c_M$-$\mathrm{Exp}_{\tilde{x}}(\tilde{q}^* + t\tilde{w}^*)$, where $\tilde{\bar{w}}^*$ and $\tilde{w}^*$ are the horizontal lifts of $\bar{v}^*$ and v^* respectively. Thus, $\tilde{\sigma}(s) = (\tilde{x}(s), \tilde{\bar{x}})$ and $\tilde{\tau}(t) = (\tilde{x}, \tilde{\bar{x}}(t))$ are h_M-geodesics, with $\tilde{\sigma}(0) = (\tilde{x}, \tilde{\bar{x}}) = \tilde{\tau}(0)$. Moreover, $\pi(\tilde{x}(s)) = x(s)$ and $\pi(\tilde{\bar{x}}(t)) = \bar{x}(t)$ from (4.3) and (4.1), so taking $\tilde{w} := \dot{\tilde{x}}(0)$ and $\tilde{\bar{w}} := \dot{\tilde{\bar{x}}}(0)$ yields $d\pi_{\tilde{x}}(\tilde{w}) = \dot{x}(0) = v$ and $d\pi_{\tilde{\bar{x}}}(\tilde{\bar{w}}) = \dot{\bar{x}}(0) = \bar{v}$. Notice that

$$-D\bar{D}c_B(x(0), \bar{x})\dot{x}(0) = \bar{v}^*, \qquad -\bar{D}Dc_B(x, \bar{x}(0))\dot{\bar{x}}(0) = v^* \;;$$

$$-D\bar{D}c_M(\tilde{x}(0), \tilde{\bar{x}})\dot{\tilde{x}}(0) = \tilde{\bar{w}}^*, \qquad -\bar{D}Dc_M(\tilde{x}, \tilde{\bar{x}}(0))\dot{\tilde{\bar{x}}}(0) = \tilde{w}^*.$$

Define an auxiliary function

$$F(s,t) := c_M(\tilde{x}(s), \tilde{\bar{x}}(t)) - c_B(x(s), \bar{x}(t)).$$

From Lemma 2.5, the inequality $\frac{\partial^4}{\partial s^2 \partial t^2} F(s,t)\big|_{(0,0)} \geq 0$ will imply (4.7). First, observe from (4.2) and the monotonicity of f that $c_B(x(s), \bar{x}(t)) \leq c_M(\tilde{x}(s), \tilde{\bar{x}}(t))$, thus,

$$F(s,t) \geq 0 \quad \text{for} \quad (s,t) \in [-1,1]^2. \tag{4.8}$$

We shall verify the desired inequalities by computing the Taylor expansion of F at $(0,0)$ to fourth order. First observe that $F(s,0) = 0 = F(0,t)$ for all $|s|, |t| \leq 1$, so

$$F(s,t) = f_{11}st + f_{21}s^2t + f_{12}st^2 + f_{31}s^3t + f_{22}s^2t^2 + f_{13}st^3 + O(|s| + |t|)^5$$

as $|s| + |t| \to 0$. Since $F(s, \pm s) \geq 0$ for $|s| \leq 1$, we deduce the vanishing of f_{11}, f_{12} and f_{21} in turn, and the inequalities $f_{31} + f_{22} + f_{13} \geq 0$ and $f_{22} - f_{31} - f_{13} \geq 0$. This implies $f_{22} \geq 0$ as desired. (Although not needed here, $f_{31} = 0$ follows from $F(s, s^2) \geq 0$, and f_{13} vanishes similarly.) Noting

$$\begin{aligned} f_{11} &= \frac{\partial^2 F}{\partial s \partial t}(0,0) \\ &= \dot{\tilde{x}}(0) \bar{D} D c_M(\tilde{x}, \tilde{\bar{x}}) \dot{\tilde{\bar{x}}}(0) - \dot{x}(0) \bar{D} D c_B(x, \bar{x}) \dot{\bar{x}}(0) \\ &= -h_M(\dot{\tilde{x}}(0) \oplus \dot{\tilde{\bar{x}}}(0), \dot{\tilde{x}}(0) \oplus \dot{\tilde{\bar{x}}}(0)) + h_B(\dot{x}(0) \oplus \dot{\bar{x}}(0), \dot{x}(0) \oplus \dot{\bar{x}}(0)) \end{aligned}$$

we have established (4.6) en passant to complete the proof. □

Before stating a corollary of this theorem, let us make a provisional definition which can serve as a strict cross-curvature condition for a Riemannian manifold. Notice from Lemma 3.2 that for each pair of points in a Riemannian manifold there are tangent vectors with zero cross-curvature.

Definition 4.6 (almost positive cross-curvature). A Riemannian manifold M with positive sectional curvature is said to be *almost positively cross-curved* if for each $(x, \bar{x}) \in N = M \times M \setminus$ cut-locus such that $x \neq \bar{x}$ and $p \oplus \bar{p} \in T_{(x,\bar{x})}N$,

$$\text{cross}(p, \bar{p}) \geq 0 \tag{4.9}$$

and the equality holds if and only if p and $\bar{p}$ are parallel to the velocity vectors $\dot{\gamma}(0)$ and $\dot{\gamma}(1)$, respectively, for the unique geodesic $t \in [0,1] \to \gamma(t) \in M$ from x to $\bar{x}$.

For example, the standard round sphere is almost positively cross-curved as shown in Section 6.

Corollary 4.7 (A3w/A3s, non-negative/almost positive cross-curvature survive Riemannian submersion). *Let $\pi : M \to B$ be a Riemannian submersion. If the cost $c_M := f \circ dist_M$ of the preceding theorem satisfies* **A3w**, **A3s**, *non-negative cross-curvature, or almost positive cross-curvature condition, then $c_B := f \circ dist_B$ satisfies the same condition, respectively.*

Proof. The relevant inequalities for the cross-curvature follow directly from (4.7). Let us consider especially the equality case of almost positive cross-curvature. Assume M is almost positively cross-curved. Suppose

$$\mathrm{cross}_{(x,\bar{x})}^{(N_B,h_B)}(p,\bar{p})=0$$

for $x\neq\bar{x}$, $(x,\bar{x})\in N_B$, $p\oplus\bar{p}\in T_{(x,\bar{x})}N_B$. Lift the unique minimizing geodesic γ linking $\gamma(0)=x$ to $\gamma(1)=\bar{x}$ to a horizontal geodesic $\tilde{\gamma}$ joining $\tilde{x}=\tilde{\gamma}(0)$ to $\tilde{\bar{x}}=\tilde{\gamma}(1)$. There is a unique choice $\tilde{p}\oplus\tilde{\bar{p}}\in T_{(\tilde{x},\tilde{\bar{x}})}N_M$ such that each component $\tilde{p}^*$ and $\tilde{\bar{p}}^*$ of $\tilde{p}^*\oplus\tilde{\bar{p}}^*=h_M(\tilde{p}\oplus\tilde{\bar{p}},\cdot)$ is the horizontal lift of the corresponding component of $p^*\oplus\bar{p}^*=h_B(p\oplus\bar{p},\cdot)$. The preceding theorem asserts $d\pi_{\tilde{x}}(\tilde{p})=p$ and $d\pi_{\tilde{\bar{x}}}(\tilde{\bar{p}})=\bar{p}$. Apply (4.7) to get $\mathrm{cross}_{(\tilde{x},\tilde{\bar{x}})}^{(N_M,h_M)}(\tilde{p},\tilde{\bar{p}})=0$. Then by almost positive cross-curvedness of M, the vectors $\tilde{p}$ and $\tilde{\bar{p}}$ are parallel to the velocities $\dot{\tilde{\gamma}}(0)$ and $\dot{\tilde{\gamma}}(1)$, respectively. This implies that the vector projections p and $\bar{p}$ are parallel to $\dot{\gamma}(0)=d\pi_{\tilde{x}}(\dot{\tilde{\gamma}}(0))$ and $\dot{\gamma}(1)=d\pi_{\tilde{\bar{x}}}(\dot{\tilde{\gamma}}(1))$, respectively, to complete the proof. □

Let us now turn to more global aspects of the distance squared cost function under Riemannian submersion. Though **local DASM/local time-convex sliding mountain** are equivalent to **A3w**/nonnegative cross-curvature, respectively (see Theorem 2.7 and Theorem 2.10), their global counterparts **DASM/time-convex sliding mountain** require additional conditions on the geometry of the domain (see Theorem 2.8 and Corollary 2.11). The following theorem, uses a simple comparison of distance to give a direct proof that both Loeper's maximum principle and time-convexity of the sliding mountain survive Riemannian submersion even in the absence of restrictions on domain geometry.

Theorem 4.8 (**Loeper's maximum principle and time-convexity of the sliding mountain survive Riemannian submersion).** *Let $\pi:M\longrightarrow B$ be a Riemannian submersion. Let $f\in C^4(\mathbf{R})$ be even and strongly convex. Compose it with the Riemannian distance on M to define a cost function $c_M=f\circ\mathrm{dist}_M$. It induces a pseudo-Riemannian metric h_M on $N_M:=M\times M\setminus$ cut-locus as in (1.3) which is* **A2** *non-degenerate and* **A1** *bi-twisted. Similarly $c_B=f\circ\mathrm{dist}_B$ defines a non-degenerate and bi-twisted pseudo-metric h_B on $N_B:=B\times B\setminus$ cut-locus. Suppose that for each h_M-geodesic of the form $t\in[0,1]\longrightarrow(\tilde{x},\tilde{\bar{x}}(t))$ in N_M, $\tilde{f}_t(\tilde{y})=-c_M(\tilde{y},\tilde{\bar{x}}(t))+c_M(\tilde{x},\tilde{\bar{x}}(t))$ satisfies (2.7) (or (2.11) respectively) for each $\tilde{y}\in M$. Then for each h_B-geodesic $t\in[0,1]\longrightarrow(x,\bar{x}(t))\in N_B$, $f_t(y)=-c_B(y,\bar{x}(t))+c_B(x,\bar{x}(t))$ satisfies the same inequality for each $y\in B$.*

Proof. Let $t\in[0,1]\longrightarrow\sigma(t)=(x,\bar{x}(t))\in N_B$ be an h_B-geodesic and set $\bar{x}:=\bar{x}(0)$. Define the sliding mountain $f_t(\cdot):=-c_B(\cdot,\bar{x}(t))+c_B(x,\bar{x}(t))$ on B. Identify tangent vectors with co-tangent vectors by the Riemannian metric. By Lemma 2.4 there exists $p,q\in T_xB$ such that $\bar{x}(t)=c\text{-}\mathrm{Exp}_x(p+tq)$. Lift p,q to horizontal vectors $\tilde{p},\tilde{q}$ at $\tilde{x}\in\pi^{-1}(x)$. Let $\tilde{\bar{x}}(t)=c\text{-}\mathrm{Exp}_{\tilde{x}}(\tilde{p}+t\tilde{q})$. From the Riemannian submersion property and Lemma 4.4,

$$\begin{aligned} c_B(x, \bar{x}(t)) &= f\Big((f')^{-1}(|p + tq|)\Big) \\ &= f\Big((f')^{-1}(|\tilde{p} + t\tilde{q}|)\Big) \\ &= c_M(\tilde{x}, \tilde{\bar{x}}(t)) \text{ for all } t \in [0,1]. \end{aligned}$$

Moreover, $(\tilde{x}, \tilde{\bar{x}}(t)) \in N_M$ for $t \in [0,1]$. This last property comes from the fact that Riemannian submersions lift the minimal geodesic from x to $\bar{x}(t)$ to the minimal geodesic from $\tilde{x}$ to $\tilde{\bar{x}}(t)$. Thus, $t \in [0,1] \to (\tilde{x}, \tilde{\bar{x}}(t)) \in N_M$ gives an h_M-geodesic. Define $\tilde{f}_t(\cdot) = -c_M(\cdot, \tilde{\bar{x}}(t)) + c_M(\tilde{x}, \tilde{\bar{x}}(t))$. Fix $t \in [0,1]$, $y \in B$. Let $\gamma : [0,1] \to B$ be a geodesic from $\gamma(0) = \bar{x}(t)$ to $\gamma(1) = y$. Let $\tilde{\gamma}$ be the horizontal lift of γ such that $\tilde{\gamma}(0) = \tilde{\bar{x}}(t)$. Let $\tilde{y} := \tilde{\gamma}(1) \in \pi^{-1}(y)$. Notice that

$$c_B(y, \bar{x}(t)) = c_M(\tilde{y}, \tilde{\bar{x}}(t)); \quad c_B(y, \bar{x}(s)) \leq c_M(\tilde{y}, \tilde{\bar{x}}(s)) \text{ for all } s \in [0,1].$$

The last inequality is from (4.2) and the monotonicity of f. Therefore,

$$f_t(y) = \tilde{f}_t(\tilde{y}) \ ; \quad \tilde{f}_s(\tilde{y}) \leq f_s(y) \text{ for all } s \in [0,1]. \tag{4.10}$$

Now, assume $s \in [0,1] \longrightarrow \tilde{f}_s(\tilde{y})$ is convex. Choosing $s = 0, 1$, from (4.10),

$$\begin{aligned} f_t(y) = \tilde{f}_t(\tilde{y}) &\leq (1-t)\tilde{f}_0(\tilde{y}) + t\tilde{f}_1(\tilde{y}) \\ &\leq (1-t) f_0(y) + t f_1(y). \end{aligned}$$

Since $t \in [0,1]$ was arbitrary, the same convexity holds for $t \in [0,1] \longrightarrow f_t(y)$. The survival of Loeper's maximum principle (**DASM**) follows by a similar argument. □

For the regularity of optimal maps, the so-called *c-convexity* of domains is crucial. In the Riemannian setting this condition corresponds to convexity of domain of the exponential maps. The following theorem addresses the heredity of this condition under Riemannian submersion.

Theorem 4.9 (**c-convexity survives Riemannian submersion**)**.** *Let π, M, B, N_M, N_B, f, c_M, c_B from Theorem 4.8 satisfy Loeper's maximum principal* (2.7)*. Suppose* Dom(*c-Exp*$_{\tilde{x}}$) *is convex for each $\tilde{x} \in M$. Then,* Dom(*c-Exp*$_x$) *is convex for each $x \in B$.*

Proof. To show that convexity is inherited by the submersion, fix a point $x \in B$ and two distinct vectors $p_0, p_1 \in T_xB$ in the boundary $\partial\,\mathrm{Dom}(c\text{-Exp}_x)$. Let $t \in]0,1[$ and $p_t = (1-t)p_0 + tp_1$. Note the relation between c-Exp and exp given in (4.3). Thus, it is enough to show

$$c_B(x, c\text{-Exp}_x p_t) = f\Big((f')^{-1}(|p_t|\Big).$$

Suppose to the contrary $c_B(x, c\text{-Exp}_x p_t) < f\Big((f')^{-1}(|p_t|\Big)$. Then, there exists $p' \neq p_t \in T_xB$ such that $c\text{-Exp}_x p' = c\text{-Exp}_x p_t$ and

$$f\Big((f')^{-1}(|p'|)\Big) = c_B(x, c\text{-Exp}_x p_t).$$

Choose a point $\tilde{x} \in \pi^{-1}(x)$. Let $\tilde{p}_0, \tilde{p}_1$ and $\tilde{p}_t$ be the horizontal lifts to M at $\tilde{x}$ of p_0, p_1 and p_t respectively. The convexity of Dom(c-Exp$_{\tilde{x}}$) in $T_{\tilde{x}}M$ implies $c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}}\tilde{p}_t) = f\Big((f')^{-1}(|p_t|)\Big)$. Consider the curve $s \in [0,1] \mapsto c\text{-Exp}_x sp'$. Lift it horizontally to M at c-Exp$_{\tilde{x}}\tilde{p}_t$. Let $\tilde{x}'$ be the other end of this lifted curve so that $\tilde{x}' \in \pi^{-1}(x)$. Therefore,

$$c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}}\tilde{p}_i) \geq c_B(x, c\text{-Exp}_x p_i), \qquad i = 0, 1.$$

Moreover, the choice of p' implies

$$c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}}\tilde{p}_t) = f\Big((f')^{-1}(|p'|)\Big) = c_B(x, c\text{-Exp}_x p_t).$$

Now, use Loeper's maximum principle **DASM** to see

$$\begin{aligned}
-c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}}\tilde{p}_t) + c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}}\tilde{p}_t) &\leq \max_{i=0,1}[-c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}}\tilde{p}_i) + c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}}\tilde{p}_i)] \\
&\leq \max_{i=0,1}[-c_B(x, c\text{-Exp}_x p_i) + c_B(x, c\text{-Exp}_x p_i)]] \\
&= 0.
\end{aligned} \tag{4.11}$$

Thus, from the left-hand side of the inequality we see

$$f\Big((f')^{-1}(|p_t|\Big) \leq c_B(x, c\text{-Exp}_x p_t), \tag{4.12}$$

a contradiction. This finishes the proof for the survival of convexity under Riemannian submersion. □

5. Regularity of optimal maps on Riemannian submersion quotients of the round sphere

This section discusses the continuity and higher regularity of optimal maps between positively bounded densities on Riemannian submersion quotients B of manifolds equipped with **A3s** costs, such as distance squared on the round sphere. To obtain continuity,we shall need to assume the convexity of Dom(c-Exp$_x$) $\subset T^*_x B$ proved in the preceding theorem is *strict*. To obtain higher regularity, we require the additional hypothesis that B is *not purely focal*, meaning at least two (distinct) minimal geodesics link each pair of points in the cut locus of B. Equivalently, B not purely focal means the subset of B^2 where dist^2_B fails to be differentiable is closed. The additional assumptions play an explicit role in the following theorem, which — at least for Riemannian distance squared — has become part of the recent folklore in the subject. Its proof is pieced together from [L1] [L2] [V2] and the arXiv version of [KM]; see also [FR] [LV].

The specific submersion quotients of the round sphere from Example 4.2 satisfy all the hypotheses of this theorem, as discussed following its proof. Indeed, our theorems 4.5 and 6.2 show $c = \mathrm{dist}^2_B$ satisfies **A3s** on any submersion quotient B of the sphere, but whether or not each such submersion quotient has a strictly convex tangent injectivity locus or is not purely focal remains unknown to us. Loeper and Villani have conjectured that **A3s** manifolds always enjoy strict convexity of

Dom($\exp_x$), and they proved this under additional technical assumptions [LV]; see also [FR] [FRV1].

Theorem 5.1 (Regularity of optimal maps in the Riemannian setting)**.** *Let* dist_B *denote the Riemannian distance on a compact manifold* B*, and suppose the strongly convex even function* $f \in C^4(\mathbf{R})$ *defines an* **A3s** *cost function* $c = f \circ \mathrm{dist}_B$*. Assume* $\mathrm{Dom}\, c\text{-}Exp_x$ *is strictly convex for each* $x \in B$*, and let* $U(x;r)$ *denote the ball of radius* $r > 0$ *around* $x \in B$*. Given probability measures* ρ *and* $\bar\rho$ *on* $M = \bar M = B$ *satisfying the bounds*

$$\lim_{r\to 0} \sup_{x\in B} \frac{\rho(U(x;r))}{r^{n-1}} = 0 \quad \text{and} \quad \inf_{x\in B} \liminf_{r\to 0} \frac{\bar\rho(U(x;r))}{r^n} > 0 \tag{5.1}$$

we assert continuity of the map $F : B \longrightarrow B$ *minimizing* (1.1) *among all Borel maps* (1.2) *pushing* ρ *forward to* $\bar\rho$*. Moreover* d_B^2 *is differentiable at each point* $(x, F(x))$*. If in addition* B *is not purely focal, this implies* $F(x)$ *is never a cut point of* $x \in B$*; in this case* F *will be smooth provided both* ρ *and* $\bar\rho$ *are smooth.*

Proof. By hypothesis, the domain of exponential map Dom(c-Exp$_x$) — which coincides with the special case of (2.1) with $c_B = \frac{1}{2}\mathrm{dist}_B^2$ — is strictly convex. We also identify vectors and co-vectors using the Riemannian metric. Loeper's maximum principal **DASM** holds by e.g. Example 12.34 and Theorem 12.36 of Villani [V2]. Exactly the same method as in Theorem E.1 of the arXiv version of [KM], which gives a refinement of Loeper's argument [L1], shows the continuity of optimal maps for the source and target measures ρ, $\bar\rho$ on B, or equivalently differentiability of the corresponding c-convex potential. Differentiability of c at $(x, F(x))$ follows exactly as in Corollary E.2 of the arXiv version of [KM]. Unless B has purely focal points, this prevents the graph of F from intersecting the cut locus, so these disjoint compact sets remain a positive distance apart. Smoothness of ρ and $\bar\rho$ can then be used to deduce smoothness of F on a not purely focal manifold using the method of Ma, Trudinger and Wang [MTW] as Loeper did on the round sphere [L2], or alternately by applying the interior regularity theory of Liu, Trudinger and Wang [LTW] — which requires the continuity just proved. □

Example 5.2 (**Hopf fibrations).** The Hopf fibrations $\pi : S^{2m+1} \to \mathbf{CP}^m$ and $\pi : S^{4m+3} \to \mathbf{HP}^m$ of the round sphere discussed in Example 4.2 lead to a cost function $c = \mathrm{dist}_B^2$ on $B = \mathbf{CP}^m$ (or $B = \mathbf{HP}^m$) which satisfies all the hypotheses (and hence all the conclusions) of the preceding theorem. The cost function is **A3s** according to Corollary 4.7 and Theorem 6.2, while the domain of the exponential map is the ball of radius $\frac{\pi}{2}$ in the tangent space, and the conjugate locus coincides with the cut locus. Thus, these manifolds are focal (but, not purely focal), in contrast to the nonfocal manifolds analyzed by Loeper and Villani in the preprint version of [LV].[1] For example in the case of $\mathbf{CP}^m$, the exponential map gives a

[1] Remark added in revision: Shortly after we communicated the present manuscript to Loeper and Villani, we learned they had revised [LV] to address **A3s** manifolds which are not purely focal. Our results of the preceding section establish $\mathbf{CP}^n$ and $\mathbf{HP}^n$ to be **A3s**, hence examples of such manifolds.

submersion of the sphere of radius $\pi/2$ in $T_x\mathbf{CP}^m$ onto the cutlocus $\mathbf{CP}^{m-1}$ of x, in which the fibre over each point in the base is a great circle.

As an immediate corollary of the theorem and example above, we obtain smoothness of optimal maps between smooth positive source and target densities for the distance squared cost corresponding to the Fubini-Study metric on $\mathbf{CP}^m$. Intermediate results such as Hölder continuity of the map F can be deduced as in or [L] [L1] or Appendix E of the arXiv version of [KM] without assuming continuity of ρ and $\bar\rho$ if we replace (5.1) by the density bounds

$$\sup_{x\in B}\limsup_{r\to 0}\frac{\rho(U(x;r))}{r^n}<\infty \quad \text{and} \quad \inf_{x\in B}\liminf_{r\to 0}\frac{\bar\rho(U(x;r))}{r^n}>0. \tag{5.2}$$

See also [L] [LTW].

For completeness let us mention that for *covering maps* of the round sphere (i.e. Riemannian submersions with discrete fibers), it is known that lifting the measures on B to the total space $M = S^n$ can be applied to show the regularity of optimal maps using established regularity results [L2] on the round sphere. This was discovered independently by Delanoë and Ge in Appendix C of [DG]. An alternative approach (in the same spirit to our discussion above) to this covering case has also been given by Figalli and Rifford [FR].

6. Sphere is almost positively cross-curved

In this section we show our final result, namely that the standard round sphere is almost positively cross-curved. This represents a significant advance over Loeper's discovery that the round sphere satisfies **A3s**. Its proof will require the following elementary lemma.

Lemma 6.1 (Calculus fact)**.** *For $0 \le \rho \le \pi$, the function*

$$a(\rho) := \sin^2\rho + \rho\sin\rho - \rho^2(1+\cos\rho)$$

satisfies $a(\rho) \ge 0$. Moreover $a(\rho) = 0$ if and only if $\rho = 0, \pi$.

Proof. Reparameterize $\rho := \pi/2 + \arcsin(\lambda)$ by $|\lambda| < 1$. Then

$$\begin{aligned} &a(\pi/2+\arcsin(\lambda)) \\ &= 1-\lambda^2 + (\pi/2+\arcsin(\lambda))\sqrt{1-\lambda^2} - (\pi/2+\arcsin(\lambda))^2(1-\lambda). \end{aligned}$$

Define

$$b(\lambda) := \frac{a(\pi/2+\arcsin(\lambda))}{1-\lambda}.$$

The assertion holds if $b(\lambda) > 0$, for $|\lambda| < 1$. From

$$(1-\lambda)b'(\lambda) = 2-\lambda+\Big(\frac{\pi}{2}+\arcsin\lambda\Big)(2\lambda-1)/(1-\lambda^2)^{1/2}$$

one can check $b(-1) = 0$, $b'(-1) = 0$, and $b'(\lambda) \ge 1$ if $\frac{1}{2} \le \lambda \le 1$. Moreover,

$$\sqrt{1-\lambda^2}(1-2\lambda)^2\frac{d}{d\lambda}\Big(\frac{(1-\lambda)\sqrt{1-\lambda^2}}{1-2\lambda}\frac{db}{d\lambda}\Big) = 2(1-\lambda)(1+\lambda)^2 > 0$$

for $-1 < \lambda < \frac{1}{2}$, which shows $b'(\lambda)$ increases monotonically in this range. Thus $b'(\lambda)$ and $b(\lambda)$ both remain positive throughout $|\lambda| < 1$, completing the proof. □

Theorem 6.2 (**Sphere is almost positively cross-curved.**). *The n-dimensional sphere $M = S^n$ with the standard round metric (i.e., sectional curvature $K \equiv 1$) is almost positively cross-curved* (4.9)*, a fortiori non-negatively cross-curved.*

Proof. This theorem follows from the following nontrivial (and tedious) calculations. Let us first set up the geometric configuration we are going to analyze. Let x be a point in the round sphere S^n of diameter π. Fix two unit tangent vectors $q, w \in T_xS^n$, $|q|, |w| > 0$. For $t \in \mathbf{R}$, with $|t|$ sufficiently small, let $r(t)$ be a line in T_xS^n with $\dot r(t) = q$, $\ddot r(t) = 0$, $|r(t)| < \pi$, where $\dot f$, $\ddot f$ denote the time derivatives $\frac{d}{dt} f(t)$, $\frac{d^2}{dt^2} f(t)$ of a function f. Let $\bar x(t)$ be the c-segment $\bar x(t) := \exp_x r(t)$. Denote $\bar x = \bar x(0)$, $\rho = |r(t)|$ (thus $0 \le \rho < \pi$), and $\hat r = \frac{r(t)}{|r(t)|}$. Let $\langle \cdot, \cdot \rangle$ denote the Riemannian inner product.

To apply Lemma 2.5, define

$$H := \text{Hess}\left(\frac{\text{dist}(\cdot, \bar x(t))^2}{2}\right)\Big|_x (w, w). \tag{6.1}$$

To prove (4.9) we will first show $-\ddot H = -\frac{d^2}{dt^2} H \ge 0$, and then the equality case shall be determined. By continuity we may assume $0 < \rho < \pi$ without loss of generality. From a standard Riemannian geometry calculation (for example see [DL][L2]), one can show that

$$H = |w|^2 - I\, G, \tag{6.2}$$

where

$$I := |w|^2 - \langle \hat r, w\rangle^2, \qquad G := 1 - \frac{\rho \cos \rho}{\sin \rho}.$$

Step 1: reduction to 2-dimensional case. One of the key points of the proof is to rearrange the expression of $-\ddot H$ in a clever way to enable further analysis. Before differentiating H, let us list some preliminary computations in the order of complexity. Define a function

$$g(u) = -\frac{u \cos u}{\sin u}, \qquad u \in]0, \pi[.$$

Then,

$$g'(u)\Big|_{u=\rho} = \frac{1}{\sin \rho} B, \qquad g''(u)\Big|_{u=\rho} = \frac{\rho}{\sin^3 \rho} A.$$

where

$$A := \frac{2(\sin \rho - \rho \cos \rho)}{\rho}, \qquad B := \frac{\rho - \cos \rho \sin \rho}{\sin \rho}.$$

Here one can check that $A, B > 0$ for $0 < \rho < \pi$ and this will be important later. We use these and the identities

$$\dot \rho = \langle \hat r, q\rangle, \qquad \ddot \rho = \frac{1}{\rho}(|q|^2 - \langle \hat r, q\rangle^2)$$

to do the following differentiations and rearrangements:

$$
\begin{aligned}
(6.3)\qquad G &= \frac{\rho}{2\sin\rho}A,\\
\dot{G} &= \frac{1}{\sin\rho}B\langle\hat{r},q\rangle,\\
\ddot{G} &= \frac{\rho}{\sin^3\rho}A\langle\hat{r},q\rangle^2 + \frac{1}{\rho\sin\rho}B(|q|^2-\langle\hat{r},q\rangle^2),\\
\dot{I} &= \frac{2}{\rho}(-\langle\hat{r},w\rangle\langle q,w\rangle + \langle\hat{r},w\rangle^2\langle\hat{r},q\rangle),\\
\ddot{I} &= \frac{2}{\rho^2}(4\langle\hat{r},q\rangle\langle\hat{r},w\rangle\langle q,w\rangle - 4\langle\hat{r},w\rangle^2\langle\hat{r},q\rangle^2 - \langle q,w\rangle^2 + \langle\hat{r},w\rangle^2|q|^2).
\end{aligned}
$$

Key observations here are first,

$$\ddot{G} > 0$$

and second, the quantities $\dot{I}$, $\ddot{I}$ are independent of the normal component $w^\perp$ of w to the plane $\Sigma \subset T_xS^n$ generated by $\hat{r}$ and q. Let $w_1 = w - w^\perp$ be the projection of w to Σ. By separating $I = |w^\perp|^2 + |w_1|^2 - \langle\hat{r},w\rangle^2$, one sees

$$
\begin{aligned}
(6.4)\qquad -\ddot{H} &= \ddot{G}I + 2\dot{G}\dot{I} + G\ddot{I}\\
&= \ddot{G}|w^\perp|^2 - \ddot{H}_1\\
&\geq -\ddot{H}_1
\end{aligned}
$$

where H_1 is the quantity defined by replacing w in (6.1) with w_1, thus independent of $w^\perp$. Notice that the quantity $-\ddot{H}_1$ becomes identical to $-\ddot{H}$ of r, q, w_1 viewed as tangent vectors of the 2-dimensional round sphere that is the exponential image of Σ in the original sphere. This reduces the consideration to two dimensions.

We will need the following key expression obtained by (6.3) and rearrangement:

$$
\begin{aligned}
(6.5)\qquad -\ddot{H}_1 = \frac{1}{\rho\sin\rho}\Big\{&\big[A\frac{\rho^2}{\sin^2\rho}\langle\hat{r},q\rangle^2 + B(|q|^2-\langle\hat{r},q\rangle^2)\big](|w_1|^2-\langle\hat{r},w_1\rangle^2)\\
&+4(B-A)\big(\langle\hat{r},q\rangle^2\langle\hat{r},w_1\rangle^2 - \langle\hat{r},q\rangle\langle\hat{r},w_1\rangle\langle q,w_1\rangle\big)\\
&+A\big(\langle\hat{r},w_1\rangle^2|q|^2 - \langle q,w_1\rangle^2\big)\Big\}.
\end{aligned}
$$

Note that

$$(6.6)\qquad B - A = \frac{\rho^2 + \rho\sin\rho\cos\rho - 2\sin^2\rho}{\rho\sin\rho} > 0 \quad \text{for } 0 < \rho < \pi,$$

as can be checked by taking the fourth-order derivative of the numerator. At this point, Loeper's result in [L2] that S^n is **A3s** can be obtained by substituting $\langle q,w\rangle = \langle q,w_1\rangle = 0$ into the expression (6.5) and using the second line of (6.4).

Step 2: two dimensional case. From (6.4) it suffices to show $-\ddot{H}_1 \geq 0$. From now on, we assume without loss of generality the dimension is two, and let $\hat{r} = (0,1)$, $q = (\cos\theta,\sin\theta)$, $w_1 = (\cos\psi,\sin\psi)$ in $\mathbf{R}^2 \cong T_xS^2$, with $0 \leq \theta,\psi \leq 2\pi$. Let

$T := \tan\theta$, $S := \tan\psi$, $-\infty \leq T, S \leq +\infty$. One checks from (6.5),

$$-\ddot{H}_1 = \frac{\cos^2\theta\cos^2\psi}{\rho\sin\rho}P, \tag{6.7}$$

where

$$P = A\,S^2 - 2(2B-A)T\,S + A\frac{\rho^2}{\sin^2\rho}T^2 + B - A.$$

Thus it suffices to show $P > 0$. P is a convex (since $A > 0$) quadratic polynomial in S with discriminant

$$\begin{aligned} D :=& 4(2B-A)^2T^2 - 4\Big(A\frac{\rho^2}{\sin^2\rho}T^2 + B - A\Big)A \\ =& 4\Big\{\Big((2B-A)^2 - A^2\frac{\rho^2}{\sin^2\rho}\Big)T^2 - A(B-A)\Big\}. \end{aligned}$$

We show $D < 0$ (regardless of T), which implies $P > 0$. Since $A(B-A) > 0$, D is always negative if

$$\begin{aligned} 0 &\geq (2B-A)^2 - A^2\frac{\rho^2}{\sin^2\rho} \\ &= \Big(2B - A + A\frac{\rho}{\sin\rho}\Big)\Big(2B - A - A\frac{\rho}{\sin\rho}\Big). \end{aligned}$$

The first factor is positive, and the second factor is negative since

$$2B - A - A\frac{\rho}{\sin\rho} = -\frac{2}{\rho\sin\rho}a(\rho),$$

where

$$a(\rho) := \sin^2\rho + \rho\sin\rho - \rho^2(1+\cos\rho)$$

which is positive from Lemma 6.1 (since $0 < \rho < \pi$). This establishes the desired inequality (4.9).

Step 4: equality case. Let us analyze the cases of equality, to conclude the almost positive cross-curvature (4.9) of S^n. We only need to show for $0 < \rho < \pi$ that $-\ddot{H} = 0$ holds if and only if the three vectors q, w, $\hat{r}$ at T_xS^n are all parallel. The necessity is easy to verify. For sufficiency, suppose $-\ddot{H} = 0$. From (6.4), $w^\perp = 0$. Thus it reduces to two dimensional case as in Step 2. Now, from (6.7) and $P > 0$, $\cos\theta\cos\psi = 0$. Thus either q or w is parallel to $\hat{r}$. In either case examining with (6.5) shows the other vector is also parallel to $\hat{r}$. This establishes almost positivity of the cross-curvature of S^n. □

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