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Equilibrium Shapes for Planar Crystals in an External Field

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Dedicated to the memory of Frederick Justin Almgren, Jr.

Abstract: The equilibrium shape of a two-dimensional crystal in a convex background potential $g(\mathbf{x})$ is analyzed. For $g = 0$ the shape of minimum energy may be deduced from surface tension via the Wulff construction, but if g is not constant, little is known beyond the case of a crystal sitting in a uniform field. Only an unpublished result of Okikiolu shows each connected component of the equilibrium crystal to be convex. Here it will be shown that any such component minimizes energy *uniquely* among convex sets of its area. If the Wulff shape and $g(\mathbf{x})$ are symmetric under $\mathbf{x} \leftrightarrow -\mathbf{x}$, it follows that the equilibrium crystal is unique, convex and connected. This last result leads to a new proof that convex crystals away from equilibrium remain convex as they evolve by curvature-driven flow. Subsequent work with Felix Otto shows – without assuming symmetry – that no equilibrium crystal has more than *two* convex components.

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Crystals in an External Field

The purpose of this article is to provide a characterization for the equilibrium shape of a crystal in an external field. When the field-strength is negligible, the crystal will form a convex set given by Wulff's construction [66] of 1901. Effects introduced by a

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(uniform) gravitational field have long been studied in the context of, e.g., the sessile or pendant liquid drop – see Finn's book [21] for a review – while some of the analogous problems for anisotropic solids have been addressed by Avron, Taylor and Zia [4] as well as Taylor and Almgren [60]. Concerning the effects of non-uniform fields, much less is known. Even when the field is the (negative) gradient of a convex potential, the equilibrium crystal is not known to be connected, much less convex or unique. This problem, formulated as a variational minimization in d dimensions, was suggested to us by Almgren. Here it is addressed in the plane $d = 2$, where an unpublished result of Okikiolu [48] shows that any energy minimizing solution consists of a countable disjoint union of closed convex sets. Our main result – also limited to the plane – states that each convex set in this union is the *unique* energy minimizer among *convex sets* of its area. If the system is reflection symmetric under $\mathbf{x} \leftrightarrow -\mathbf{x}$, it follows that the crystal formed must be unique, convex and connected. In the context of curvature-driven flow, this last result leads to a new proof that a non-equilibrium crystal $K = -K$, initially convex, will remain convex and balanced as it melts or relaxes. In a subsequent article with Felix Otto, we go beyond these results to conclude $\frac{1}{m}$ without assuming reflection symmetry – that no equilibrium crystal has more than *two* convex components.

A crystal in contact with its vapor or melt may be modeled as a subset $U \subset \mathbf{R}^2$ having area determined by the ambient thermodynamic variables. The variational principle formulated by Gibbs [30] and Curie [13] asserts that the equilibrium shape of such a crystal must minimize its free energy – in our case an interfacial surface energy plus a bulk potential energy measuring interaction with the external field – among all sets of equal area; see alternately Herring [36] or Cahn and Hoffman [11]. The interfacial energy will typically be anisotropic: an initial quantity of the condensate breaks the symmetry of the underlying space, establishing preferred directions. Though microscopic in origin, this anisotropy is represented macroscopically by a *surface tension* $f : \mathbf{S}^1 \longrightarrow (0, \infty)$ which depends continuously on the outward unit normal $\hat{\mathbf{n}}_U(\mathbf{x})$ at each boundary point $\mathbf{x}$ of the set U. Presuming f to be known from experiment or theory, the surface energy $F(U)$ of any crystal shape may be calculated as the integral

$$F(U) := \int_{\partial_* U} f(\hat{\mathbf{n}}_U(\mathbf{x}))\, d\mathcal{H}^1(\mathbf{x}) \tag{1}$$

of the surface tension with respect to Hausdorff one-dimensional measure $\mathcal{H}^1$ on the *measure theoretic boundary* $\partial_* U$ (Definition A.1) of U. Here $U \subset \mathbf{R}^2$ is assumed to be bounded, Borel, and have *finite perimeter* $\mathcal{H}^1(\partial_* U) < +\infty$ so that its surface energy is finite; the collection of such sets is denoted by $\mathcal{F}(\mathbf{R}^2)$. To each set $U \in \mathcal{F}(\mathbf{R}^2)$, a construction of geometric measure theory (Definition A.3) associates a Borel map $\hat{\mathbf{n}}_U : \partial_* U \longrightarrow \mathbf{S}^1$ defining the outward unit normal $\hat{\mathbf{n}}_U(\mathbf{x})$ for $\mathcal{H}^1$-almost every $\mathbf{x}$ in $\partial_* U$; of course, $\hat{\mathbf{n}}_U(\mathbf{x})$ coincides with the Gauss map and $\partial_* U = \partial U$ when U has a smooth boundary.

If U is dilated by $\lambda \geq 0$, its surface energy scales: $F(\lambda U) = \lambda F(U)$. Among sets of a fixed area, F is uniquely minimized by the *Wulff shape* $W \subset \mathbf{R}^2$ and its translates. In the isotropic case $f(\hat{\mathbf{n}}_U(\mathbf{x})) := 1$, the surface energy measures the boundary of U so the Wulff shape is a ball in view of the isoperimetric inequality; this corresponds to the shape of a small drop of liquid in equilibrium with its vapour. In the anisotropic case, the set constructed by Wulff is obtained by Legendre transforming the surface tension, after extending f from the unit circle $\mathbf{S}^1 = \{|\mathbf{x}| = 1\} \subset \mathbf{R}^2$ to the plane by taking it to be *positively homogeneous*: $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$ whenever $\mathbf{x} \in \mathbf{R}^2$ and $\lambda \geq 0$. A convex dual function to f, also positively homogeneous, is defined by

$$f^*(\mathbf{y}) := \sup_{\mathbf{x}\in\mathbf{R}^2} \langle \mathbf{y}, \mathbf{x}\rangle / f(\mathbf{x}); \tag{2}$$

the Wulff shape is the compact convex set $W := \{f^* \leq 1\}$. This construction extends immediately to higher dimensions, and is interpreted geometrically in, e.g., [35, 36] and [56, 58].

Working under the assumption that W was a polyhedron (a polygon in dimension $d = 2$), what Wulff had attempted to show was that the surface energy attains a local minimum at W among nearby polyhedra having parallel faces and the same size. Apparently his argument was flawed [36], but correct proofs were given by Hilton [39], Liebmann [42] and Laue [63]. However, it was only later that Dinghas would show W to represent the *global* minimum of the surface energy among all convex polyhedral shapes of its size [15]. His idea exploited the Brunn-Minkowski inequality, and was subsequently extended to prove that – polyhedral or not – W minimizes $F(U)$ uniquely among a much larger class of sets U: the extension to sets with piecewise smooth boundary was accomplished by Herring [36] and Taylor [56]; to bounded sets of finite perimeter $\mathcal{F}(\mathbf{R}^2)$ by Taylor [57, 58]; and to all sets of finite perimeter by Fonseca and Müller [23]. A completely different proof of the Wulff theorem was found by Dacorogna and Pfister [14], but is restricted to sets in the plane ($d = 2$).

If the crystalline material interacts with an external potential g on $\mathbf{R}^2$, then – in addition to its surface energy – $U \in \mathcal{F}(\mathbf{R}^2)$ carries potential energy

$$G(U) := \int_U g(\mathbf{x})\, d\mathcal{H}^2(\mathbf{x}). \tag{3}$$

Here $\mathcal{H}^2$ denotes two-dimensional Lebesgue measure. At equilibrium, the shape of the crystal in the field $-\nabla g$ will minimize the energy

$$E(U) := F(U) + G(U), \tag{4}$$

subject to the constraint of fixed area $\mathcal{H}^2(U) = m$. The competition between surface and bulk energy makes this variational problem more complicated than the Wulff problem, so that – apart from special cases – one no longer expects exact solutions. One none the less wants to know that with the area constraint, the energy (4) attains its minimum on $\mathcal{F}(\mathbf{R}^2)$, and to deduce what properties one can about the minimimizing shape $U_0 \subset \mathbf{R}^2$.

One special case which has received some attention corresponds to the sessile crystal in a uniform gravitational field:

$$g(\mathbf{x}) := \begin{cases} \langle \mathbf{g}, \mathbf{x}\rangle & \text{where } \langle \mathbf{g}, \mathbf{x}\rangle \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\mathbf{g} \in \mathbf{R}^2$ specifies the strength of the field and its orientation relative to the crystal. In two dimensions and subject to a wetting condition (substrate prefers crystal to melt), Avron, Taylor and Zia [4] obtain the equilibrium shape by quadrature, showing it to be convex and unique. They also explore conditions for a large enough field to induce roughened facetting in the direction normal to $\mathbf{g}$. In higher dimensions $d \geq 3$, neither convexity nor uniqueness is known for the equilibrium shape, except in the isotropic case $f(\hat{\mathbf{n}}) := 1$ (the sessile drop) studied by Finn [20] and Wente [64].

In this paper we study the minimization problem for potentials $g(\mathbf{x})$ which are merely convex. The surface tension also is presumed to extend to a convex function $f(\mathbf{x}) = |\mathbf{x}| f(\hat{\mathbf{x}})$ on the plane, though this restriction costs no generality. Convexity of $f(\mathbf{x})$ means merely that the most economical interface connecting two points will be a straight

line; see e.g. Lemma 2.1. When this condition fails, there are interfacial directions so unfavorable that surfaces will develop microscopic zig-zags to avoid them. In this case a minimizing solution may not exist except in the varifold sense. However, the variational problem can always be solved after replacing the surface tension with its convex hull f^{**}, (which by duality of the Legendre transform shares the same Wulff shape as f). A prescription for recovering the varifold structure of the minimizing set $U_0 \subset \mathbf{R}^2$ is then given by Taylor in [58, Theorem 3.3].

With the convexity of f and g, existence of an energy minimizer $U_0 \subset \mathbf{B}_R(\mathbf{0})$ with given area follows from a standard continuity-compactness argument summarized in Appendix A. Kate Okikiolu's theorem is also stated there: apart from sets of measure zero (which, being irrelevant to the energy, are ignored throughout), any minimizer U_0 can be expressed as a countable disjoint union of closed convex sets. Finally, the appendix includes two a priori bounds needed for the existence and uniqueness arguments concerning energy minimizing crystals in $\mathcal{F}(\mathbf{R}^2)$. These bounds control the radius of U_0 and the number of convex components in terms of f, g, and the area constraint.

The main result of the present manuscript applies to this equilibrium crystal U_0. Formulated precisely in the next section, it asserts that each connected component of U_0 is the *unique* energy minimizer among convex sets of its area (Theorems 1.1 and 1.3). The underlying idea is evident in the uniqueness proof for the more restricted (and simpler) problem of minimizing energy among convex sets $K \subset \mathbf{R}^2$ having area $\mathcal{H}^2(K) = m$. The possibility of two minimizers for the restricted problem is precluded by defining a family of convex sets $K(t) \subset \mathbf{R}^2$ which interpolate between any pair of sets $K(0) = K$ and $K(1) = K'$ while satisfying energy estimates of the form

$$E\Big(K(t)\Big) \le (1-t)\,E(K) + t\,E(K'). \tag{5}$$

Here $K(t) := [(1-t)K + tK']_m$ is the convex (Minkowski) combination of K and K', intersected with the smallest level set of $g(\mathbf{x})$ for which $\mathcal{H}^2\Big(K(t)\Big) \ge m$. Apart from exceptional cases (which occur when g is not strictly convex and are analyzed separately), $\mathcal{H}^2\Big(K(t)\Big) = m$ and (5) holds strictly for $t \in (0,1)$ unless the minimizer $K = K'$ is unique. The proof of the central estimate (5) is divided into a series of convexity and monotonicity lemmas concerning the surface and potential energies. Section 2 contains the results concerning surface energy, while Section 3 relies crucially on the interpolation technique introduced in [45, 47] to address potential energy. A final section shows how convexity of the minimizer U_0 for our statical problem can be used to deduce that convexity is preserved by the dynamical process of curvature-driven flow [1]. It is followed by an appendix with background material and some technical propositions.

Since crystal formation and evolution are of the most physical interest in dimension $d = 3$, it is regretable that two of the key estimates do not extend beyond the plane. Proposition 2.4, which controls the surface energy $F((1-t)K + tK')$ for convex sets $K, K' \subset \mathbf{R}^2$, is false in higher dimensions. And Corollary 2.8, which asserts that the surface energy of a connected open set dominates that of its convex hull, also fails when $d > 2$. The latter is used in the proof of Okikiolu's theorem. Counterexamples to both are easily constructed in the isotropic case $f(\mathbf{x}) = |\mathbf{x}|$.

Regarding the physical applicability of this theory, we caution that equilibrium is attained only with great difficulty in the laboratory – and then only for small crystals (typically with sizes measured in microns) or for exotic systems such as solid helium in contact with the superfluid. Examples of the former include the metal experiments of

Heyraud and Métois [37, 38], and the studies of Pavlovska and Nenow on negative crystals (inclusions) in organic materials [50, 49, 51]. The solid-liquid interface of helium is described in reviews by Keshishev, Parshin and Shal'nikov [41], Balibar and Caisting [5] and Maris and Andreev [44]. On the theoretical side, a statistical mechanical derivation of the surface tension and justification for the variational approach may be provided by monograph of Dobrushin, Kotecky and Shlosman and the references quoted therein [16].

1. Overview of Methods and Results

Our first results concern the minimimization of the energy $E(K)$ when restricted to *convex* sets $K \subset \mathbf{R}^2$ of fixed area. This restriction is not physical, but simplifies the variational problem considerably. Moreover, it turns out to provide the key to the structure of the energy minimizing crystal for the physical problem in which all sets of finite perimeter compete. Therefore, let $\mathcal{K}(\mathbf{R}^2)$ consist of all compact convex sets in the plane, and define $\mathcal{K}_m(\mathbf{R}^2) := \{K \in \mathcal{K}(\mathbf{R}^2) \mid \mathcal{H}^2(K) = m\}$. The first theorem asserts that the shape of the energy minimizing set in $\mathcal{K}_m(\mathbf{R}^2)$ is unique. Whether or not its position is also uniquely determined depends on the external potential: strict convexity of $g(\mathbf{x})$ will select a position uniquely, while if the convexity of g is not strict then a convex set of translations may be allowed.

The proof of this theorem relies on the monotonicity and convexity estimates established in the following sections for surface and potential energies. Like all results in this paper, these estimates are obtained under the assumption that the energy (4) be defined through a pair of convex integrands $f, g : \mathbf{R}^2 \longrightarrow [0, \infty)$ satisfying the normalization conventions:

(E1) f is convex with $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$ for $\lambda > 0$; $f > 0$ unless $\mathbf{x} = \mathbf{0}$;
(E2) g is convex, and the set where it vanishes is bounded but non-empty.

The positivity restrictions on f and on g are assumed for convenience only: any linear function $\langle \mathbf{v}, \mathbf{x} \rangle$ can be added to $f(\mathbf{x})$ without changing closed surface energies; similarly, adding constants to g merely shifts the total energy. The results could also be extended to the case in which the potential $g(\mathbf{x})$ takes the value $+\infty$ on the complement of a convex set $C \subset \mathbf{R}^2$; then finite energy $E < +\infty$ must be added as a hypothesis, while the proof of Proposition A.9 changes slightly. This models the case in which crystal and melt are constrained by a container C in the complete non-wetting regime (where separation of the crystal from the container walls by a layer of fluid is favored energetically), and was used by Schonmann and Shlosman to study boundary effects in the Ising model [54]. The "summertop" trick [67] of exchanging crystal with melt can be used in the opposite regime (complete wetting), where container walls will be covered by crystal condensate. The intermediate regime (partial wetting) can sometimes be modelled by modifying the surface tension f appropriately: e.g. if C is a convex cone and the problem shares its symmetry, see Zia, Avron and Taylor [67, 4] and Winterbottom [65].

Taken together, the estimates of Sects. 2 and 3 show the total energy to be "convex" in a suitable sense. More specifically, given $\lambda > 0$ and two sets $K, K' \subset \mathbf{R}^2$, define $\lambda K := \{\lambda\mathbf{x} \mid \mathbf{x} \in K\}$ and $K + K' := \{\mathbf{x} + \mathbf{x}' \mid \mathbf{x} \in K, \mathbf{x} \in K'\}$ in the usual way. If a Borel set $U \subset \mathbf{R}^2$ has area $\mathcal{H}^2(U) \geq m$, then its *truncation* $[U]_m$ *to size* m is defined by intersection with level sets of g: use $U_\lambda := \{\mathbf{x} \in U \mid g(\mathbf{x}) \leq \lambda\}$ to set

$$[U]_m := \bigcap_{\{\lambda \in \mathbf{R} \mid \mathcal{H}^2(U_\lambda) \geq m\}} U_\lambda. \tag{6}$$

The continuous dependence of $\mathcal{H}^2(U_\lambda)$ on λ shows the set (6) to have area m unless $g(\mathbf{x})$ vanishes throughout $[U]_m$. Finally, given $K, K' \in \mathcal{K}_m(\mathbf{R}^2)$, the "convexity" proved for the energy is with respect to the interpolation $K(t) := [(1-t)K + tK']_m$. Thus $t \in [0,1]$ implies (5), with conditions for strict inequality yielding uniqueness in the theorem below. Existence of a minimizer among convex sets is well-known: it follows from the compactness and lower semi-continuity results of Appendix A since $\mathcal{K}_m(\mathbf{R}^2)$ embeds as a closed subset of $L^1(\mathbf{R}^2)$ when the set K is identified with its characteristic function

$$\chi_K(\mathbf{x}) := \begin{cases} 1 \text{ if } \mathbf{x} \in K \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 1.1 (*Uniqueness of Energy Minimizer Among Convex Sets***).** *Fix $0 < m < +\infty$. Among closed convex sets $K \subset \mathbf{R}^2$ with area $\mathcal{H}^2(K) = m$, the energy $E(K)$ is minimized by a set $K_0(m)$ which is unique up to possible translations.*

Proof. Uniqueness will be established first. Suppose two convex sets K and K' in $\mathcal{K}_m(\mathbf{R}^2)$ share the same energy: $E(K) = E(K')$. Unless K' is a translate of K, Theorem 3.5 provides a path $K(t) \in \mathcal{K}_m(\mathbf{R}^2)$ joining K to K' along which inequality (5) is strict for $t \in (0,1)$. But then $E(K(t)) < E(K)$, which precludes the possibility that both K and K' minimize energy in $\mathcal{K}_m(\mathbf{R}^2)$.

To address existence of the minimizing set $K_0(m)$, let $K \in \mathcal{K}_m(\mathbf{R}^2)$ be an arbitrary convex set with energy $E(K) \le \lambda < +\infty$. The potential energy G is non-negative, so the surface energy $F(K) \le \lambda$ bounds the diameter of K by (37). Thus K is contained in a ball $\mathbf{B}_r(\mathbf{y})$ whose radius depends only on f and λ. This ball cannot be centered too far from the origin since $g(\mathbf{x})$, being convex, grows linearly away from the bounded set where $g = 0$, yet

$$\inf_{|\mathbf{x}-\mathbf{y}| \le r} g(\mathbf{x}) \le m^{-1} G(K) \le m^{-1} \lambda.$$

Thus K also lies in some larger ball $\mathbf{B}_R(\mathbf{0})$ depending only on λ, m, f and g. Now recall that the subsets of $\mathbf{B}_R(\mathbf{0})$ with bounded perimeter embed compactly in $L^1(\mathbf{R}^2)$ (Theorem A.2). Any energy minimizing sequence in $\mathcal{K}_m(\mathbf{R}^2)$ has bounded perimeter $F(K_i) \le \lambda$ and must eventually lie in $\mathbf{B}_R(\mathbf{0})$, hence admits an $L^1(\mathbf{R}^2)$ convergent subsequence whose limit is (the characteristic function of) a convex set K_0 with area m since $\mathcal{K}_m(\mathbf{R}^2)$ is also known to be closed [53, Theorem 1.8.5 and Note 1.8.8]. Theorem A.4 couples with the obvious lower semi-continuity of G to ensure that K_0 has minimum energy on $\mathcal{K}_m(\mathbf{R}^2)$. □

Remark 1.2. If the position of the energy minimizing set $K_0(m)$ fails to be unique, then there is a closed convex set $V \subset \mathbf{R}^2$ of possible translations such that $K_0(m) + \mathbf{v}$ has minimum energy if and only if $\mathbf{v} \in V$. Convexity of V follows by an easy estimate (or Theorem 3.5), while V is closed because of lower semi-continuity of the energy (Theorem A.4). The size of V is controlled by g.

If the surface tension is *smooth* – say $f \in C^2(\mathbf{R}^2 \setminus \{0\})$ – and *uniformly elliptic* – meaning $trD^2 f(\hat{\mathbf{n}}) > 0$, or equivalently $\{\mathbf{x} \mid f(\mathbf{x}) \le 1\}$ is uniformly convex – then it is possible to prove that the energy minimizer U_0 among sets in $\mathcal{F}(\mathbf{R}^2)$ or $\mathcal{K}(\mathbf{R}^2)$ with area m has a C^2-smooth boundary; cf. Almgren, Taylor, Wang [1, Theorem 3.10]. The curvature $\kappa(\mathbf{x})$ along this boundary satisfies the Euler-Lagrange equation

$$\kappa(\mathbf{x})\, D^2_{tt} f(\hat{\mathbf{n}}_{U_0}(\mathbf{x})) + g(\mathbf{x}) = \lambda, \tag{7}$$

where $\mathbf{x} \in \partial U_0$, $D^2_{tt} f := trD^2 f$, and λ is a Lagrange multiplier conjugate to the area constraint. Okikiolu [48] has argued from (7) that each connected component of U_0 is convex: $\kappa(\mathbf{x}) \geq 0$. Indeed, convexity of g's level sets forces $\kappa(\mathbf{x}_0) \geq 0$ at that point $\mathbf{x}_0$ on the boundary of U_0 which has maximum potential energy $g(\mathbf{x}_0) \geq \sup_{\partial U_0} g(\mathbf{x})$. Using (7) to compare $\mathbf{x}_0$ with any other point $\mathbf{x} \in \partial U_0$ yields $\kappa(\mathbf{x}) D^2_{tt} f(\hat{\mathbf{n}}_{U_0}(\mathbf{x})) \geq 0$.

In a subsequent article with Felix Otto, we shall prove that any convex domain $U_0 \subset \mathbf{R}^2$ whose C^2-smooth boundary satisfies (7) has minimum energy in $\mathcal{K}_m(\mathbf{R}^2)$. Thus the only connected solutions to the Euler-Lagrange equation are the sets $K_0(m)$ of Theorem 1.1. This observation motivates the theorem to follow – which holds even in the absence of smoothness and uniform ellipticity of f.

Theorem 1.3 (*Classification of Connected Crystal Components***).** *Assume (E1–E2), and suppose U_0 minimizes $E(U)$ among sets $U \in \mathcal{F}(\mathbf{R}^2)$ of finite perimeter and unit area. Then U_0 is a finite disjoint union of closed convex sets $K_0(m)$, each with distinct area m and minimizing $E(K)$ among convex sets of area m.*

Proof. Okikiolu's result is adapted in Theorem A.7 to show that U_0 consists of a countable disjoint union of closed convex components. Proposition A.9 bounds the number of such components. Let K be a convex component of U_0, and let $K' := K_0(m)$ denote the set with minimum energy among convex sets with area $m := \mathcal{H}^2(K)$. We first prove that $E(K) = E(K')$. Theorem 3.5 defines a curve $K(t) \subset (1-t)K + tK'$ in $\mathcal{K}_m(\mathbf{R}^2)$ joining K to K' along which the energy satisfies (5) for $t \in (0,1)$. Thus $E(K(t)) < E(K)$ holds unless K has the same energy as K'. Since K is one of finitely many compact convex components of U_0, it enjoys a neighbourhood which is disjoint from $U_0 \setminus K$. Compactness of K' ensures that for $t > 0$ small enough, $K(t) \subset (1-t)K + tK'$ must lie in this neighbourhood; it can be substituted for K without disturbing the remainder of U_0. The energy of U_0 would be lowered and its area unchanged, contradicting the fact that U_0 is a minimizer and establishing the main assertion.

The uniqueness result of Theorem 1.1 shows that $K = K_0(m) + \mathbf{v}$. The proof is concluded by showing that even if translates of the minimizer $K_0(m)$ share its energy, no two such translates K and K' can occur as components in U_0. Otherwise, K may be translated toward K' using $K(t) := (1-t)K + tK'$; for $t \in [0,1]$ the energy $E(K(t))$ remains constant by Remark 1.2. As long as $K(t)$ remains disjoint from $U_0 \setminus K$, the set

$$U := K(t) \cup (U_0 \setminus K) \tag{8}$$

is a minimizer sharing the area and energy of U_0. As soon as $K(t)$ touches K' or some other component of U_0, a contradiction is reached: either these two components share an edge, in which case the surface energy has been reduced and $E(U) < E(U_0)$, or else they meet at a point, in which case U has a non-convex component violating Theorem A.7. □

A corollary shows the equilibrium crystal to consist of a single convex component whenever the surface and potential energy integrands $f(\hat{\mathbf{n}}) = f(-\hat{\mathbf{n}})$ and $g(\mathbf{x}) = g(-\mathbf{x})$ are even. When the position of each energy minimizer $K_0(m)$ among convex sets is unique, this follows immediately from Theorems 1.1 and 1.3: each $K_0(m)$ must be convex and balanced, hence contain the origin; no two of these sets are disjoint.

Corollary 1.4 (*Convex Equilibrium Crystals for Even Energies***).** *Assume reflection symmetry $E(U) = E(-U)$ as well as (E1–E2). Among sets $U \in \mathcal{F}(\mathbf{R}^2)$ of finite perimeter and area m, the minimizer U_0 of $E(U)$ is convex: it is unique up to translation, and may be taken to be balanced $U_0 = -U_0$.*

Proof. If U_0 is convex, the first claim is proved. If not, choose two convex components K and K' of U_0; each is the energy minimizer among convex sets of its area by Theorem 1.3. Since $-K$ has the same energy and area as K, Theorem 1.1 forces them to be translates: $-K = K + \mathbf{v}$. Similarly, $-K' = K' + \mathbf{v}'$. Define $K(t) := K + t\,\mathbf{v}$ and $K'(t) := K' + t\mathbf{v}'$ for $t \in [0, 1]$. The energy $E(K(t))$ is independent of t by Remark 1.2. As in the proof of the preceding theorem, a contradiction would be reached if $K(t)$ and $K'(t)$ intersected each other or $U_0 \setminus (K \cup K')$ at any $t \in (0, 1)$. But these sets cannot remain disjoint up to $t = 1/2$, for $K(1/2)$ and $K'(1/2)$ are both convex and balanced and their intersection includes the origin. Thus U_0 consists of a single convex component K.

Theorem 1.1 then asserts $U_0 = K$ to be unique apart from translations. The translate $K(1/2)$ defined as above also minimizes $E(U)$ and is balanced. □

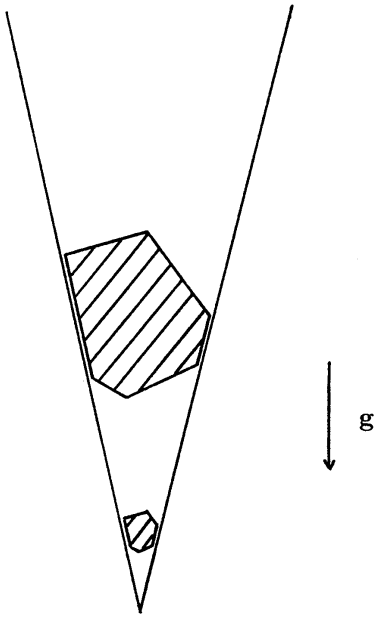

Fig. 1. Can these two crystals be in equilibrium?

Even without additional symmetry, the minimizer U_0 of Theorem 1.3 can be shown to consist of at most *two* convex components. Obtained in collaboration with Felix Otto, this result will be published separately. The heuristic idea underlying its proof is that the physics divides the problem into two different regimes: for small crystals surface tension plays the dominant role in determining equilibrium shape, while for large crystals the dominant influence is potential energy. As a result, there is some critical size m_{cr} such that the energy $E_0(m) := E(K_0(m))$ is concave on the interval $m \leq m_{cr}$ but convex on the interval $m \geq m_{cr}$. If two convex crystals $K_0(m_1)$ and $K_0(m_2)$ coexist in equilibrium, then the Lagrange multipliers $\lambda(m_1) = \lambda(m_2)$ in (7) must agree. But $\lambda(m) = dE_0/dm$ decreases to the left of m_{cr} while increasing to its right. Thus $m_1 < m_{cr}$ selects $m_2 > m_{cr}$ uniquely, and m_2 selects m_1 conversely.

Of course, this begs the question of whether two crystalline components can coexist at all (without getting in each other's way). One situation where we anticipate this may occur is for a crystal and its melt inside a conical container – Fig. 1. Assuming the

container wall prefers melt to solid, surface energetics will prevent the crystal from penetrating all the way to the tip of the cone [67]; thus the crystal rests above a bubble of fluid even in the presence of gravity. Under suitable conditions, we predict a tiny, second crystal can be formed inside this bubble in stable equilibrium with the fluid and the first crystal. Though surface energetics favor expansion of the larger crystal at the expense of the smaller one, this would be penalized by an increase in gravitational potential energy. The criticality argument shows these two effects can balance exactly, but fails to address disjointness of $K_0(m_1)$ and $K_0(m_2)$ or the energy comparison $E_0(m_1 + m_2) < E_0(m_1) + E_0(m_2)$ required for stability. Such questions remain a challenge, especially tantalizing in simple settings like the cone with gravity, where disconnected equilibria – should they exist – might be accessible experimentally.

2. Surface Energy and Isoperimetry

This section lays out the basic convexity and monotonicity properties of the surface energy. In particular, the surface energy $F(U)$ is shown not to increase when $U \in \mathcal{F}(\mathbf{R}^2)$ is replaced by (i) its intersection $U \cap K$ with any convex set, or, in the case of a connected domain U with $\mathcal{H}^2(\partial U) = 0$, (ii) its convex hull conv $[U]$. Moreover, the surface energy of $(1 - t)K + tK'$ proves to be convex as a function of $t \in [0, 1]$ for convex sets $K, K' \subset \mathbf{R}^2$. These estimates suggest a proof of the isoperimetric inequality in which uniqueness is shown before the minimizer is known to be round. Regretably, the second and third estimates rely crucially on the one-dimensional nature of domain boundaries in the plane.

A continuous curve $\sigma : [a, b] \longrightarrow \mathbf{R}^2$ is said to be *rectifiable* if has finite *arc length* $\mathcal{L}(\sigma)$. The latter is defined by polygonal approximation using a supremum over finite partitions $\Pi = \{s_0 < s_1 < \cdots < s_n\} \subset [a, b]$:

$$\mathcal{L}(\sigma) := \sup_{\Pi \subset [a,b]} \sum_{i=1}^{n} |\sigma(s_i) - \sigma(s_{i-1})|. \tag{9}$$

In its arc length reparameterization $\tau : [0, \mathcal{L}(\sigma)] \longrightarrow \mathbf{R}^2$, the rectifiable curve σ is seen to be Lipschitz. As a consequence, the tangent $\tau'(s)$ exists for almost all s, and it is natural to define the surface energy $F(\sigma)$ associated with the interface σ by

$$F(\sigma) := \int_0^{\mathcal{L}(\sigma)} \tilde{f}(\tau'(s))ds, \tag{10}$$

where $\tilde{f}(x, y) := f(y, -x)$ employs a rotation by 90° to express the surface tension as a function of the oriented tangent vector rather than the outward normal. If the measure theoretic boundary $\partial_* U$ of a set $U \in \mathcal{F}(\mathbf{R}^2)$ coincides (up to sets of $\mathcal{H}^1$ measure zero) with a rectifiable simple closed curve σ oriented positively, then definitions (1) and (10) coincide; this is certainly the case when U is a convex domain. $F(\sigma)$ may also be computed by polygonal approximation in a manner analogous to the arc length (9):

$$F(\sigma) := \sup_{\Pi \subset [a,b]} \sum_{i=1}^{n} \tilde{f}\Big(\sigma(s_i) - \sigma(s_{i-1})\Big). \tag{11}$$

Expression (11) is manifestly invariant under reparameterization; in the arc length parameterization it can be seen to coincide with (10) by the dominated convergence theorem.

Note that finiteness of (11) is equivalent to the rectifiability of σ: the surface tension $\tilde{f}(\mathbf{x})$ is positively homogeneous, bounded away from zero and infinity on the unit circle. Thus (11) extends the definition of interfacial energy to all continuous curves in the plane. The first lemma it yields is well-known:

Lemma 2.1 (*Triangle Inequality***).** *Among all continuous curves $\sigma:[0,1] \longrightarrow \mathbf{R}^2$ with endpoints $\sigma(0) = \mathbf{x}$ and $\sigma(1) = \mathbf{y}$, the interfacial energy $F(\sigma)$ is minimized by the line segment $\sigma(s) = (1-s)\mathbf{x} + s\mathbf{y}$.*

Proof. Let $\sigma : [0,1] \longrightarrow \mathbf{R}^2$ be a rectifiable curve joining $\sigma(0) = \mathbf{x}$ to $\sigma(1) = \mathbf{y}$; otherwise $F(\sigma) = +\infty$. Choosing any partition $0 = s_0 < s_1 < \ldots < s_n = 1$ of the interval, convexity of the surface tension yields

$$\tilde{f}\left(\frac{\mathbf{y}-\mathbf{x}}{n}\right) \le \frac{1}{n}\sum_{i=1}^{n} \tilde{f}(\sigma(s_i) - \sigma(s_{i-1})). \tag{12}$$

Multiplying (12) by n, the positive homogeneity (E1) of $\tilde{f}$ shows the energy $\tilde{f}(\mathbf{y}-\mathbf{x})$ of the line segment to be less than the energy $F(\sigma)$ of the curve (11). □

Remark 2.2. When the surface tension is *elliptic*, meaning $\{\mathbf{x} \in \mathbf{R}^2 \mid f(\mathbf{x}) \le 1\}$ is strictly convex, strict inequality in (12) shows the line segment to minimize surface energy uniquely (up to reparameterization) among curves connecting its endpoints.

Lemma 2.3 (*Convexity for Curves***).** *Let σ and $\sigma' : [a,b] \longrightarrow \mathbf{R}^2$ be continuous curves, and define $\sigma_t(s) := (1-t)\,\sigma(s) + t\,\sigma'(s)$ on $[a,b]$. Then the interfacial energy $F(\sigma_t)$ is convex as a function of $t \in [0,1]$.*

Proof. From the definition (11) of the interfacial energy,

$$\begin{aligned} F(\sigma_t) &= \sup_{\Pi\subset[a,b]} \sum_i \tilde{f}\Big(\sigma_t(s_i) - \sigma_t(s_{i-1})\Big) \\ &= \sup_{\Pi\subset[a,b]} \sum_i \tilde{f}\Big((1-t)\,\big(\sigma_0(s_i) - \sigma_0(s_{i-1})\big) + t\,\big(\sigma_1(s_i) - \sigma_1(s_{i-1})\big)\Big) \\ &\le (1-t)\,F(\sigma_0) + t\,F(\sigma_1). \end{aligned}$$

The final inequality was obtained using the convexity of the surface tension $\tilde{f}$. □

Proposition 2.4 (*Convexity for Sets***).** *Let $K, K' \in \mathcal{K}(\mathbf{R}^2)$ be closures of two convex domains. Then the surface energy $F((1-t)K + tK')$ is convex as a function of $t \in [0,1]$.*

Proof. First assume K and $K' \in \mathcal{K}(\mathbf{R}^2)$ to be strictly convex, so that both are compact sets with non-empty interiors. Given $\hat{\mathbf{n}} \in \{\mathbf{x} \in \mathbf{R}^2 \mid |\mathbf{x}| = 1\}$, strict convexity ensures the existence of a unique point $\sigma(\hat{\mathbf{n}}) \in \partial K$, where K is supported by a half-plane with outward unit normal $\hat{\mathbf{n}}$. Let $\sigma'(\hat{\mathbf{n}})$ parameterize the boundary of K' in a similar way. Then the curves σ and σ' are continuous on the unit circle $\mathbf{S}^1$. Now observe that $\sigma_t(\hat{\mathbf{n}}) := (1-t)\sigma(\hat{\mathbf{n}}) + t\sigma'(\hat{\mathbf{n}})$ must be the unique point at which $(1-t)K + tK'$ has outward unit normal $\hat{\mathbf{n}}$. Thus σ_t gives a positively oriented parameterization for the boundary of $(1-t)K + tK'$. Convexity of $F((1-t)K + tK')$ follows from Lemma 2.3, and the equivalence between our definitions of surface energy for sets (1) and for curves (11).

Should K or K' (or both) fail to be strictly convex, the argument must be modified since σ and σ' will no longer be continuous. However, strict convexity of K and K' can

fail in at most countably many directions $\hat{\mathbf{n}}_i \in \mathbf{S}^1$. By inserting an interval I_i of length 2^{-i} into the unit circle at $\hat{\mathbf{n}}_i$, and extending σ (and σ') linearly to I_i, we parameterize ∂K and $\partial K'$ continuously and consistently in the sense that the outward unit normals agree: $\hat{\mathbf{n}}_K(\sigma(s)) = \hat{\mathbf{n}}_{K'}(\sigma'(s))$. The preceding argument can then be applied. □

The spirit of our approach is illustrated by the following proof of the isoperimetric inequality. Like several other proofs, it relies on the Brunn-Minkowski theorem [34]; what is unusual is that uniqueness is shown before the minimizer is known to be round.

Remark 2.5 (Isoperimetric Inequality). The shortest Jordan curve enclosing unit area is unique (up to translation and parameterization), and therefore must be a circle.

Proof. Setting $f(\mathbf{x}) = |\mathbf{x}|$, Lemma 2.1 implies that only the curves bounding convex regions need be considered: any simple closed curve may be replaced by its convex hull, which is not longer but encloses greater area. This makes it easy to deduce existence of a shortest closed curve from standard compactness and continuity results. We prove only that this convex curve is unique.

Suppose two convex sets K and K' in $\mathcal{K}_1(\mathbf{R}^2)$ both have minimum perimeter. Proposition 2.4 shows the perimeter $(K + K')/2$ to be no longer than that of K. The Brunn-Minkowski theorem (or Proposition 3.1 and Lemma 3.2) shows $(K + K')/2$ to have area $m > 1$ unless K' is a translate of K. Thus K and K' must be translates: otherwise the convex set $(K + K')/2m^{1/2}$ would have unit area but perimeter less than $\mathcal{H}^1(\partial K) = \mathcal{H}^1(\partial K')$. This contradiction establishes uniqueness up to translation.

Now there is a unique convex set $K \in \mathcal{K}_1(\mathbf{R}^2)$ with shortest perimeter having the origin as its center of mass. This set is invariant under rotations, therefore a disk. The shortest curve enclosing unit area must parameterize its boundary. □

The next lemma extends a result of Vogel [62] to anisotropic surface energies. For elliptic f, the inequality is strict unless $U \subset K$ (up to sets of measure zero).

Lemma 2.6 (*Convex Intersections***).** *If $U \in \mathcal{F}(\mathbf{R}^2)$ has finite perimeter and $K \subset \mathbf{R}^2$ is convex, then the surface energy $F(U \cap K) \leq F(U)$.*

Proof. If K is a half-space $H = \{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{y}\rangle \leq \lambda\}$ with $y \in \mathbf{R}^2$ and $\lambda \in \mathbf{R}$, the desired inequality is well-known [1, §3.1.9]; when the boundary of U is a simple closed curve, the result also follows directly from Lemma 2.1.

Now consider general convex $K \subset \mathbf{R}^2$. Assume K to be closed, since the difference is a set of measure zero and therefore irrelevant. Choose a countable dense set of points x_i from ∂K, so that K is the intersection of the supporting half-spaces $H_i \supset K$ at these points. Define $U_{i+1} := H_i \cap U_i$ inductively starting with $U_1 := U$. The preceding paragraph asserts $F(U_i)$ to be non-increasing. Moreover, $\chi_{U_i} \to \chi_{U \cap K}$ in $L^1(\mathbf{R}^2)$. The lower semi-continuity of F from Theorem A.4 then yields $F(U \cap K) \leq \lim_i F(U_i) \leq F(U)$. □

Proposition 2.7 (*Obstacles in the Plane***).** *Assume f is elliptic and fix a bounded domain $\Omega \subset \mathbf{R}^2$ whose boundary has measure zero: $\mathcal{H}^2(\partial\Omega) = 0$. If $U \supset \Omega$ has the least surface energy $F(U)$ among all sets in $\mathcal{F}(\mathbf{R}^2)$ containing Ω, then $\mathcal{H}^2(\partial U) = 0$ and U is given by a union of convex domains $K_i \subset \mathbf{R}^2$ whose closures $\overline{K_i}$ are disjoint.*

Proof. Modifying U on a set of measure zero, one may take $\partial U = \overline{\partial_* U}$ without loss [31, Theorem 4.4]. The first step will be to prove $\mathcal{H}^2(\partial U) = 0$, so that U can be assumed to be the interior of its closure $\overline{U}$. By hypothesis, $\mathcal{H}^2(\partial U \cap \partial\Omega) = 0$, so it remains to

control the boundary of U outside $\overline{\Omega}$. A standard estimate derived from the comparison $F(U) \le F(U \setminus \mathbf{B}_r(\mathbf{x}))$ yields the lower density ratio bound

$$\mathcal{H}^2(U \cap \mathbf{B}_r(\mathbf{x})) \ge \frac{\pi r^2}{(1 + f_{max}/f_{min})^2} \tag{13}$$

for every $\mathbf{x} \in \overline{U}$ and $r < \operatorname{dist}(\mathbf{x}, \Omega)$; the argument parallels Proposition A.6, which follows Giusti [31, Proposition 5.14] except that the ratio f_{max}/f_{min} of the maximum and minimum values of the surface tension $f(\hat{\mathbf{n}})$ appear because of the anisotropy. On the other hand, $\mathbf{R}^2 \setminus U$ minimizes the surface energy $F(-U)$ among sets avoiding the obstacle Ω, so it enjoys a similar bound (33) outside of $\overline{\Omega}$. This impies an upper density ratio bound for U along ∂U, forcing $\partial U \setminus \overline{\Omega} \subset \partial_* U$ from Definition A.1. Since U has finite perimeter, $\mathcal{H}^1(\partial_* U) < +\infty$ concludes the proof that $\mathcal{H}^2(\partial U) = 0$.

The next step is to prove convexity for each connected component K_i of the open set U. Should K_i fail to be convex, one can find a pair of points $\mathbf{a}, \mathbf{c} \in K_i$ such that $\mathbf{b} := (1-t)\mathbf{a} + t\mathbf{c}$ lies a positive distance from K_i for some $0 < t < 1$. The points $\mathbf{a}$ and $\mathbf{c}$ can be connected by a smooth Jordan arc σ in K_i. By choosing $\mathbf{a}$ and $\mathbf{c}$ closer to $\mathbf{b}$ if necessary, the arc σ may be assumed not to intersect the interior of the line segment $[\mathbf{a}, \mathbf{c}]$. Then the arc σ together with the segment $[\mathbf{a}, \mathbf{c}]$ forms a simple closed curve enclosing some region V. A contradiction will be obtained by showing $F(U \cup V) < F(U)$. Clearly $\mathbf{b} \notin \overline{K_i}$ forces $W := V \setminus U$ to have positive area, and

$$F(U \cup W) = F(U) + \int_{\partial^* W \setminus \partial^* U} f(\hat{\mathbf{n}}_W(\mathbf{x}))\, d\mathcal{H}^1(\mathbf{x}) - \int_{\partial^* W \cap \partial^* U} f(\hat{\mathbf{n}}_U(\mathbf{x}))\, f\mathcal{H}^1(\mathbf{x}). \tag{14}$$

Now $\hat{\mathbf{n}}_W(\mathbf{x}) = \hat{\mathbf{n}}_\perp$ is constant on $\mathbf{x} \in \partial^* W \setminus \partial^* U$, where it is perpendicular to $\mathbf{c} - \mathbf{a}$. On the other hand, $\partial^* W \cap \partial^* U$ has positive $\mathcal{H}^1$ measure, and $\hat{\mathbf{n}}_U(\mathbf{x}) = -\hat{\mathbf{n}}_W(\mathbf{x})$ cannot be constant there. Stokes' theorem $\int_{\partial^* W} \hat{\mathbf{n}}_W = 0$ couples with Jensen's inequality, (F1) and ellipticity of f to yield a strict inequality $F(U \cup W) < F(U)$ from (14), as in Almgren, Taylor and Wang [1, §3.1.9]. This contradiction proves that K_i is convex.

Finally, it remains to establish that $\overline{K_i}$ and $\overline{K_j}$ are disjoint for $i \neq j$. Since these two convex sets have disjoint interiors, their intersection can be at most a line segment or a point. They cannot intersect along a line segment since U was chosen as the interior of $\overline{U}$. Nor can they intersect in a single point $\mathbf{x}$, for then it would be possible to find $\mathbf{a} \in K_i$ and $\mathbf{c} \in K_j$ with $\mathbf{b} = (1-t)\mathbf{a} + t\mathbf{c}$ not in $\overline{K_i} \cup \overline{K_j}$. The Jordan arc $[\mathbf{a}, \mathbf{x}] \cup [\mathbf{x}, \mathbf{c}]$ in $\overline{K_i} \cup \overline{K_j}$ would connect $\mathbf{a}$ to $\mathbf{c}$, so the preceding argument again yields a contradiction. This shows $\overline{K_i} \cap \overline{K_j} = \emptyset$ to finish the proof. □

A corollary will be used to extend Okikiolu's result to non-smooth, degenerate elliptic surface tensions in Theorem A.7. The same observation has been exploited by Gage [26, Corollary 2.6] in a slightly less technical setting.

Corollary 2.8 (*Convex Hulls***).** *Assume f satisfies (E1), and let $\Omega \subset \mathbf{R}^2$ be a bounded, connected domain with $\mathcal{H}^2(\partial\Omega) = 0$. Its convex hull $K := \operatorname{conv}[\Omega]$ has the least surface energy $F(K)$ among all sets in $\mathcal{F}(\mathbf{R}^2)$ containing Ω. The same result holds if $\Omega = \Omega_1 \cup \Omega_2$ is a union of two convex sets with connected closure $\overline{\Omega}$.*

Proof. First assume that f is elliptic, and define $U \supset \Omega$ to be a surface energy minimizer among all sets $U \in \mathcal{F}(\mathbf{R}^2)$ containing Ω. Proposition 2.7 shows $U = \bigcup_i K_i$ to consist of a countable union of convex domains whose closures $\overline{K_i}$ are disjoint. Whether Ω is connected or consists of two connected components whose closures intersect, it is clear

that Ω must lie in a single connected component of U. Thus $U \supset \text{conv}\,[\Omega] = K$. Lemma 2.6 shows that $K = U \cap K$ must also minimize surface energy among all sets containing Ω.

If the surface tension f is not elliptic but satisfies (F1), it may be expressed as a pointwise limit $f_i(\mathbf{x}) \to f(\mathbf{x})$ of a sequence of elliptic f_i. (These are obtained as the gauge functions for a sequence of strictly convex approximants to the unit ball $\{\mathbf{x} \mid f(\mathbf{x}) \leq 1\}$.) Since $F_i(K) \leq F_i(U)$ holds for each $U \supset \Omega$, the limit yields $F(K) \leq F(U)$. Thus K again minimizes surface energy, though in contrast to the elliptic case it need no longer be unique. □

3. Convexity and Monotonicity of the Potential Energy

Choose two convex sets K and K' from $\mathcal{K}_m(\mathbf{R}^2)$. The primary purpose of the present section is to establish the convexity estimate

$$G\Big([(1-t)K + tK']_m\Big) \leq (1-t)\,G(K) + t\,G(K'), \tag{15}$$

together with conditions for strict inequality which imply the uniqueness results outlined earlier. Here $[K]_m$ denotes the truncation (6) of K to size m, without which (15) would typically fail. This estimate, together with those of the previous section, is summarized in Theorem 3.5.

The argument will be presented in two steps. First the idea introduced in [45, 47] is recalled, which defines an interpolant $\rho_t \in L^1(\mathbf{R}^2)$ between $\rho_0 = \chi_K$ and $\rho_1 = \chi_{K'}$ for $t \in [0,1]$. The salient features of this *displacement interpolation* are that $0 \leq \rho_t \leq \chi_{(1-t)K+tK'}$ has total mass m, while its potential energy $G(\rho_t)$ is convex as a function of t, when the definition (3) of potential energy is extended to $\rho \in L^1(\mathbf{R}^2)$ by

$$G(\rho) := \int_{\mathbf{R}^2} g(\mathbf{y})\,\rho(\mathbf{y})\,d\mathcal{H}^2(\mathbf{y}). \tag{16}$$

The second step is to replace ρ_t by a convex set $K(t) \subset \mathbf{R}^2$ having area m. Truncating the support of ρ_t using level sets of $g(\mathbf{x})$ assures $G\big(K(t)\big) \leq G(\rho_t)$.

The displacement interpolant is defined in [45, 47] using a result of Brenier [6] which guarantees the existence of a convex function $\psi : K \longrightarrow \mathbf{R}$ whose gradient $\nabla\psi$ gives an area-preserving map between K and K'; see alternately Brenier [7], Gangbo [29] or McCann [46]. The function ψ is unique up to additive constant. When the sets are convex, Caffarelli has shown $\nabla\psi$ to be a homeomorphism from K to K': it is a smooth diffeomorphism of their interiors and Hölder continuous up to the boundary [8, 9, 10]. Viewing $d\rho_0(\mathbf{x}) := \chi_K(\mathbf{x})d\mathbf{x}$ as a Borel measure on $\mathbf{R}^2$, the displacement interpolant ρ_t is defined for $t \in [0,1]$ to be the density of this measure pushed forward through the mapping $\mathbf{x} \longrightarrow (1-t)\mathbf{x} + t\,\nabla\psi(\mathbf{x})$. Thus for $\mathbf{x}$ from the interior of K, the value of ρ_t at $(1-t)\mathbf{x} + t\,\nabla\psi(\mathbf{x})$ is given by the Jacobian determinant

$$\rho_t\Big((1-t)\mathbf{x} + t\,\nabla\psi(\mathbf{x})\Big) = \det\,[(1-t)I + t\,\nabla^2\psi(\mathbf{x})]^{-1}, \tag{17}$$

while $\rho_t(\mathbf{y})$ vanishes elsewhere; here I is the identity matrix while $\nabla^2\psi(\mathbf{x})$ is the derivative of $\nabla\psi$ at $\mathbf{x}$ – and non-negative definite by convexity of ψ. Unlike $\rho_0 = \chi_K$ and $\rho_1 = \chi_{K'}$, ρ_t will not generally be the characteristic function of any set, though the first proposition shows $0 \leq \rho_t(\mathbf{y}) \leq \chi_{(1-t)K+tK'}(\mathbf{y})$. This proposition may also be deduced

from [47, Theorems 2.2–2.3], but the two-dimensional setting and smoothness of ψ facilitate a direct proof in the present context. The notation $\rho_0 \xrightarrow{t} \rho_1$ is used instead of ρ_t to emphasize explicit dependence on the endpoints ρ_0 and ρ_1.

Proposition 3.1 (*Displacement Interpolation***).** *Given $K, K' \in \mathcal{K}_m(\mathbf{R}^2)$ convex, let $\rho_t := \chi_K \xrightarrow{t} \chi_{K'}$ be the displacement interpolant between them. Then $0 \le \rho_t \le \chi_{(1-t)K+tK'}$. Unless K and K' are translates, $0 < \rho_t(\mathbf{x}) < 1$ holds on a set of positive area whenever $t \in (0,1)$.*

Proof. Let $\psi : K \longrightarrow \mathbf{R}$ be the convex function whose gradient is an area-preserving diffeomorphism between the interiors of K and K'. Define $\psi_t(\mathbf{x}) := (1-t)\,\mathbf{x}^2/2 + t\,\psi(\mathbf{x})$ and $\mathbf{y}_t(\mathbf{x}) := \nabla\psi_t(\mathbf{x})$ on the interior K^0 of K. For $t \in (0,1)$, the uniform convexity of ψ_t ensures that $\mathbf{y}_t$ is injective: the inverse map $\mathbf{y}_t^{-1}$ satisfies a Lipschitz condition with constant $(1-t)^{-1}$. Thus ρ_t is well-defined by (17), together with the condition that ρ_t vanish outside the image of $\mathbf{y}_t$. Now fix $\mathbf{x} \in K^0$ and set $\Lambda = \nabla^2\psi(\mathbf{x})$. Then

$$h(t) := \det\left[(1-t)I + t\Lambda\right] \tag{18}$$

$$= 1 + t\,\mathrm{tr}\left[\Lambda - I\right] + t^2\left(1 + \det\left[\Lambda\right] - \mathrm{tr}\left[\Lambda\right]\right) \tag{19}$$

varies continuously between $h(0) = 1$ and $h(1) = \det[\Lambda] = 1$. Moreover, $h(t)$ is concave on $[0,1]$: this follows from (19) since Λ has positive eigenvalues and $\mathrm{tr}\,[\Lambda]/2 \ge \sqrt{\det[\Lambda]} = 1$. Unless $\Lambda = I$, the two eigenvalues are distinct, the arithmetic mean dominates the geometric mean strictly, and $h(t)$ is strictly concave. In any case, $h(t) \ge 1$ on $[0,1]$ and hence $0 < \rho_t(\mathbf{y}_t(\mathbf{x})) = h(t)^{-1} \le 1$. Since $\mathbf{y}_t(\mathbf{x}) \in (1-t)K + tK'$ this establishes $0 \le \rho_t \le \chi_{(1-t)K+tK'}$.

Next choose $\mathbf{x}_0 \in K^0$ for which $\nabla^2\psi(\mathbf{x}_0) \neq I$; if no such point exists then $\nabla\psi(\mathbf{x}) = \mathbf{x} + \mathbf{v}$ reduces to translation by some $\mathbf{v} \in \mathbf{R}^2$ and $K' = K + \mathbf{v}$. Otherwise $\nabla^2\psi(\mathbf{x}) \neq I$ holds on a neighbourhood of $\mathbf{x}_0$. For $t < 1$ the image of this neighbourhood under the diffeomorphism $\mathbf{y}_t$ has positive measure, and $0 < \rho_t(\mathbf{y}) < 1$ there. □

Lemma 3.2 (*Displacement Convexity of the Potential Energy***).** *Let $K, K' \in \mathcal{K}_m(\mathbf{R}^2)$ be two convex sets, and define the displacement interpolant $\rho_t := \chi_K \xrightarrow{t} \chi_{K'}$ between them. Then $\int \rho_t = m$, while the potential energy $G(\rho_t)$ is convex as a function of t on $[0,1]$.*

Proof. Let $\psi : K \longrightarrow \mathbf{R}$ be the convex function whose gradient is an area-preserving diffeomorphism between the interiors of K and K'. Define $\psi_t(\mathbf{x}) := (1-t)\,\mathbf{x}^2/2 + t\,\psi(\mathbf{x})$ and $\mathbf{y}_t(\mathbf{x}) := \nabla\psi_t(\mathbf{x})$ on the interior of K as before. Then the change of variables $\mathbf{y} = \mathbf{y}_t(\mathbf{x})$ combines with (17) to express the potential energy (16) by

$$G(\rho_t) = \int_K g\Big((1-t)\,\mathbf{x} + t\,\nabla\psi(\mathbf{x})\Big)\,d\mathcal{H}^2(\mathbf{x}). \tag{20}$$

The integrand is manifestly convex as a function of t, so the integral must be as well. Setting $g(\mathbf{x}) = 1$ in (20) yields conservation of mass: $\int \rho_t = \mathcal{H}^2(K) = m$. □

Remark 3.3. The fact that the endpoints ρ_0 and ρ_1 were characteristic functions of convex sets played no role in the preceding proof. The displacement interpolant ρ_t may in fact be defined between any pair of probability measures on $\mathbf{R}^d$ given by densities ρ_0 and ρ_1 with respect to Lebesgue, and in any dimension [45, 47]. *Displacement convexity* of G – convexity of $G(\rho_t)$ as a function of t – holds true in this generality.

Lemma 3.4 (*Monotonicity of the Potential Energy***).** *Let $\rho \in L^1(\mathbf{R}^2)$ and a Borel set $U \subset \mathbf{R}^2$ satisfy $0 \le \rho \le \chi_U$. Fix $m \le \|\rho\|_1$. Then $G([U]_m) \le G(\rho)$, and strict inequality holds unless $G(\rho) = 0$ or $\rho = \chi_{[U]_m}$.*

Proof. Recall that $[U]_m = \{\mathbf{x} \in U \mid g(\mathbf{x}) \le \lambda\}$, where $\lambda \ge 0$ is chosen (6) so that either (i) $\mathcal{H}^2([U]_m) = m$ or (ii) $\lambda = 0$. In either case,

$$G\Big(\rho\Big) - G\Big([U]_m\Big) \ge \int_U \Big(g(\mathbf{x}) - \lambda\Big)\Big(\rho(\mathbf{x}) - \chi_{[U]_m}(\mathbf{x})\Big)\, d\mathcal{H}^2(\mathbf{x}). \tag{21}$$

Moreover, the integrand of (21) is non-negative for all $\mathbf{x} \in U$ so the desired inequality is established. The integral will be strictly positive unless $\rho(\mathbf{x})$ vanishes a.e. outside $[U]_m$; in the first case (i) above $\mathcal{H}^2([U]_m) \le \|\rho\|_1$ then implies $\rho = \chi_{[U]_m}$, while in the second case (ii) $\lambda = 0$ ($= \inf_{\mathbf{x}} g(\mathbf{x})$) couples with $\rho \le \chi_{[U]_m}$ to imply $G(\rho) = 0$. These are the only exceptions to strict inequality in the lemma. □

Theorem 3.5 (*Interpolation Between Convex Crystals***).** *Let $K, K' \in \mathcal{K}_m(\mathbf{R}^2)$ be convex sets. Then there is a curve $K(t) \subseteq [(1-t)K + tK']_m$ in $\mathcal{K}_m(\mathbf{R}^2)$ joining $K(0) = K$ to $K(1) = K'$ along which the energy satisfies*

$$E\Big(K(t)\Big) \le (1-t)\,E(K) + t\,E(K'). \tag{22}$$

Strict inequality holds for $t \in (0,1)$ unless K' is a translate of K.

Proof. Given $t \in [0,1]$, let $\rho_t := \chi_K \xrightarrow{t} \chi_{K'}$ denote the displacement interpolant (17) between χ_K and $\chi_{K'}$. Proposition 3.1 and the conservation of mass shown in Lemma 3.2 make it clear that the set $U = (1-t)K + tK'$ has area no smaller than $m = \int \rho_t$; we denote its truncation to size m by $K(t) := [(1-t)K + tK']_m$. The surface energy of $K(t)$ is controlled by the convexity estimate of Proposition 2.4 and the monotonicity of F proved in Lemma 2.6 for intersections with convex sets, and more particularly with level sets (6) of the potential $g(\mathbf{x})$:

$$\begin{aligned} F\Big(K(t)\Big) &\le F(\,(1-t)K + tK\,) \\ &\le (1-t)\,F(K) + t\,F(K'). \end{aligned} \tag{23}$$

At the same time, the convexity $G(\rho_t) \le (1-t)\,G(\chi_K) + t\,G(\chi_{K'})$ of Lemma 3.2 combines with the monotonicity of Lemma 3.4 to control the potential energy

$$\begin{aligned} G\Big(K(t)\Big) &\le G(\rho_t) \\ &\le (1-t)\,G(K) + t\,G(K'), \end{aligned} \tag{24}$$

since $0 \le \rho_t \le \chi_U$ by Proposition 3.1. Summing the energy estimates for F and for G yields (22). Moreover, Lemma 3.4 gives a strict inequality $G(K(t)) < G(\rho_t)$ in (24) and (22), unless (i) $\rho_t = \chi_{K(t)}$ or (ii) $G(\rho_t) = 0$. If K is a translate of K' then the proof is complete. If not, the inequality $0 < \rho_t < \chi_{(1-t)K+tK'}$ of Proposition 3.1 holds strictly on a subset with positive measure, precluding possibility (i). Should (22) fail to be strict, meaning (ii) has occurred, then $K(t) \supset \{\mathbf{x} \in U \mid g(\mathbf{x}) = 0\}$ contains a set $\{\mathbf{x} \mid \rho_t > 0\}$ whose area exceeds $m = \int \rho_t$. Then $K(t)$ can be replaced by a contraction $\lambda K(t)$ towards its center of mass with $\lambda = 1/\mathcal{H}^2(K(t)) < 1$ chosen to yield area m. This contraction does not increase potential energy, yet it lowers the surface energy $F(\lambda K(t)) = \lambda F(K(t))$ to make inequality (22) strict. Even when (ii) does not occur, it

may happen that $K(t)$ has area greater than m if $G(K(t)) = 0$; in these cases also $K(t)$ is replaced by $\lambda K(t)$. Either way, the new set $K(t) \subseteq [(1-t)\,K + t\,K']_m$ is manifestly convex and has area m, completing the proof of the theorem. □

We close by remarking that estimate (22) should really be viewed as a statement of sublinearity rather than convexity: the theorem does not assert that the curve joining, e.g., $K(1/3)$ to $K(2/3)$, coincides with $K((1-t)/3 + 2t/3)$.

4. Curvature-Driven Flows Preserve Convexity

This final section gives an application of our results to the dynamical process of *curvature-driven flow*. Curvature-driven flow, or *motion by weighted mean curvature*, is a geometrical model for the time evolution of a non-equilibrium crystal under the influence of its surface tension. In this model, the normal velocity v of each point $\mathbf{x}$ on the crystalline interface is presumed proportional to the local change of surface energy with volume, or rather, area in $\mathbf{R}^2$. Thus it depends on the curvature κ of the interface at $\mathbf{x}$, and the normal direction $\hat{\mathbf{n}}$ via tangential derivatives of the surface tension f:

$$v = -\kappa\, D^2_{tt} f(\hat{\mathbf{n}}); \tag{25}$$

a further discussion and references are given by Taylor, Cahn and Handwerker in the review articles [61, 59].

If the interface and its evolution are both smooth, the flow may be referred to as *classical*. For curves in the plane, the resulting motion – and its generalization (28) – have been studied by Angenent and Gurtin. Here a connection will be established between one of their results [2, §7.3, 3, §10] and the statical problem that we have analyzed: Corollary 1.4 will be used to provide an alternative proof that a convex crystal $K_0 \in \mathcal{K}(\mathbf{R}^2)$ away from equilibrium, remains convex for all time under curvature-driven flow. This result was proved first in the isotropic case $f(\mathbf{x}) = |\mathbf{x}|$ by Gage and Hamilton [24, 27] for curves in the plane, and by Huisken [40] for surfaces in higher dimensions. Evans and Spruck provided a different proof [18] using the viscosity solutions which they introduced simultaneously with Chen, Giga and Goto [12] to the level set approach. Furthermore, in the plane it is true that non-convex curves eventually become convex, as shown by Grayson [32] for isotropic motion and by Angenent and Gurtin [2, 3] and Gage [25] in the anisotropic case. The asymptotic shape of the vanishing curve has also been determined by Gage [26] in joint work with Li [28], as well as by Soner [55], Angenent and Gurtin [3] and Gurtin, Soner and Souganidas [33] in non-smooth formulations with varying degrees of generality. Despite this plethora of techniques and results, our approach – currently limited to the case $K_0 = -K_0$ – may be of interest since it requires neither smoothness nor uniform ellipticity of f or K_0, while reducing the anisotropic question in higher dimensions to the study of a statical problem.

The relationship between the dynamical and statical problems was established by Almgren, Taylor, Wang [1] and Luckhaus and Sturzenhecker [43], who showed that a curvature-driven flow starting from $K_0 \in \mathcal{F}(\mathbf{R}^2)$ can be approximated by a discrete time flow in which the evolved crystal K after time Δt is the minimizer on $\mathcal{F}(\mathbf{R}^2)$ of a functional (4). The potential $g(\mathbf{x})$, which need not be convex, represents the tendency of the flow to remain near its initial condition K_0 for short times: it is proportional to the signed distance to the boundary $\partial^* K_0$ and decays with elapsed time Δt:

$$g(\mathbf{x})\,\Delta t := \mathrm{dist}_{\pm}(\mathbf{x}, K_0) := \begin{cases} \mathrm{dist}(\mathbf{x}, \partial^* K_0) \text{ if } \mathbf{x} \notin K_0 \\ -\mathrm{dist}(\mathbf{x}, \partial^* K_0) \text{ if } \mathbf{x} \in K_0. \end{cases} \tag{26}$$

A discrete evolution is generated by repeated minimization, replacing K_0 by K at each step. A continuous time flow, dubbed *flat F curvature flow*, is extracted in the limit $\Delta t \to 0$. Under additional restrictions this flat flow coincides with the classical curvature flow (25) provided the latter exists.

If the initial configuration K_0 is a convex set, Lemma 4.2 shows that the corresponding potential (26) is convex. If K_0 is balanced and $f(\hat{\mathbf{n}}) = f(-\hat{\mathbf{n}})$ is even, an application of Corollary 1.4 then shows that the crystal remains balanced and convex at all subsequent times. It is of interest to note that the distance function appearing in (26) need not be Euclidean distance: it is sufficient that

$$\text{dist}(\mathbf{x}, \partial^* K) := \inf_{\mathbf{k} \in \partial^* K} \text{M}(\mathbf{x} - \mathbf{k}) \tag{27}$$

for any norm $\text{M}(\mathbf{x})$ on $\mathbf{R}^2$. Non-Euclidean norms – or rather their duals (2) – model certain non-isotropic *mobilities*: direction dependent responses of the crystalline interface to applied force [1, §2.13, 2, 61]. The potential $g(\mathbf{x})$ may also be shifted by a constant $\omega \in \mathbf{R}$ representing the difference in bulk energy between the crystal and its melt. Heuristically at least, (25) then becomes

$$v = -M^*(\hat{\mathbf{n}})\Big(\omega + \kappa\, D^2_{tt} f(\hat{\mathbf{n}})\Big). \tag{28}$$

Theorem 4.1 (*Flat F Curvature Flows Preserve Balanced Convex Sets***).** *Let $K(t) \in \mathcal{F}(\mathbf{R}^2)$ be a flat F curvature flow [1] on some interval $t \in [0, T]$. If the initial condition $K(0) = -K(0)$ is convex and the surface tension $f(\hat{\mathbf{n}}) = f(-\hat{\mathbf{n}})$ is even, then the crystal $K(t)$ will be convex at each subsequent time. $K(t)$ will also have reflection symmetry through some point $\mathbf{x}_t \in \mathbf{R}^2$: $K(t) - \mathbf{x}(t) = \mathbf{x}(t) - K(t)$.*

Proof. Since $K_0 = K(0)$ is convex, balanced and bounded, Lemma 4.2 shows the potential $g(\mathbf{x}) + \omega$ from (26–27) to be convex and balanced. Since $g(\mathbf{x})$ is positive except in K_0, it assumes its minimum on a subset of the bounded set K_0. Let K minimize $E(U)$ over $\mathcal{F}(\mathbf{R}^2)$, i.e. among sets of all areas. K exists by results of Appendix A, though it may coincide with the null set; it is a fortiori an energy minimizer among sets of its area. Although g takes both signs, a constant may be added to yield $g \geq 0$ without affecting the constrained minimization of Corollary 1.4. Thus K will be convex, and if not balanced, then reflection symmetric through some other point $\mathbf{x} \in \mathbf{R}^2$. K is an approximant to $K(\Delta t)$. The approximant to $K(2\Delta t)$ is obtained by repeating the procedure, starting from K instead of K_0. Since the problem is translation invariant, and a translate of K satisfies the hypotheses on K_0, the approximants to $K(n\Delta t)$ must all be convex and symmetric for $n > 1$. The flat F curvature flow $\chi_{K(t)}$ at time t is obtained [1, 2.6] as a limit of such approximants in $L^1(\mathbf{R}^2)$. $K(t)$ is convex since the characteristic functions of convex sets form a closed subset of $L^1(\mathbf{R}^2)$; it has a balanced translate for a similar reason. □

Lemma 4.2 (*Convexity of Signed Distance to a Convex Set***).** *Let $K \subset \mathbf{R}^d$ be a convex set, and $\text{M}(\mathbf{x})$ a norm on $\mathbf{R}^d$. The signed distance $\text{dist}_\pm(\mathbf{x}, K)$ of (26–27) is a convex function of $\mathbf{x}$ on $\mathbf{R}^d$. If K is balanced then $\text{dist}_\pm(x, K) = \text{dist}_\pm(-x, K)$.*

Proof. Choose any supporting hyperplane to K, and let $H \supset K$ be the corresponding half-space. The first observation

$$\text{dist}_\pm(\mathbf{x}, K) \geq \text{dist}_\pm(\mathbf{x}, H) \tag{29}$$

is seen from three cases: if $\mathbf{x} \notin H$, the boundary of H lies M-closer to $\mathbf{x}$ than the boundary of K; if $\mathbf{x} \in K$ the situation is reversed, and (29) holds since both distances are negated; if $\mathbf{x} \in H \setminus K$, (29) holds on the basis of sign.

Now fix $\mathbf{x} \in \mathbf{R}^d$. There is some $\mathbf{k}$ in the boundary of K such that dist$(\mathbf{x}, \partial^* K)$ = $M(\mathbf{x} - \mathbf{k})$. A supporting half-space $H \supset K$ exists with $\mathbf{k} \in \partial H$ and with dist$(\mathbf{x}, H)$ = $M(\mathbf{x} - \mathbf{k})$: if $\mathbf{x} \in K$ this is obvious, while if $\mathbf{x} \notin K$ the hyperplane ∂H must be slipped between the convex sets K and $\{\mathbf{y} \mid \mathrm{M}(\mathbf{x}-\mathbf{y}) < \mathrm{dist}(\mathbf{x}, K)\}$. Thus (29) will be saturated for this H, and

$$\mathrm{dist}_\pm(\mathbf{x}, K) = \sup_{H \supset K} \mathrm{dist}_\pm(\mathbf{x}, H),$$

where the supremum is over half-spaces $H \supset K$. Convexity of dist$_\pm(\mathbf{x}, K)$ is manifest since dist$_\pm(\mathbf{x}, H)$ is linear (or at least affine). dist$_\pm(\mathbf{x}, K)$ is obviously even when $K = -K$. □

A. Convexity, Size and Number of Components

This appendix begins by recalling a few definitions concerning sets of finite perimeter (see Federer [19], Giusti [31], Fonseca [22] or Evans and Gariepy [17]) with related compactness and continuity results. Further background material is then developed from the literature, including a proof of the existence of an energy minimizing crystal U_0 among $\mathcal{F}(\mathbf{R}^2)$ sets having area m, and of Okikiolu's theorem expressing U_0 as a countable disjoint union of closed convex sets. The appendix concludes with two a priori estimates: the first constrains the support of U_0, while the second provides a lower bound for the area of its convex components, implying an upper bound on their number.

A Borel set $U \subset \mathbf{R}^2$ is said to have *finite perimeter* if $\mathcal{H}^1(\partial_* U) < +\infty$, where:

Definition A.1. *The* measure theoretic boundary $\partial_* U$ *of a (Borel) set* $U \subset \mathbf{R}^2$ *consists of those points* $\mathbf{x} \in \mathbf{R}^2$ *such that*

$$\limsup_{r \to 0} \fint_{\mathbf{B}_r(\mathbf{x})} \chi_U > 0 \quad \text{while} \quad \liminf_{r \to 0} \fint_{\mathbf{B}_r(\mathbf{x})} \chi_U < 1. \tag{30}$$

The bar denotes an average over the given domain (in this case the ball $\mathbf{B}_r(\mathbf{x})$) with respect to Lebesgue measure $\mathcal{H}^2$.

The sets of finite perimeter coincide with those sets $U \subset \mathbf{R}^2$ whose characteristic functions have *bounded variation:* $\|\chi_U\|_{BV} = \mathcal{H}^2(U)+\mathcal{H}^1(\partial_* U)$. The basic compactness result for these sets is stated in Giusti [31, Theorem 1.19]:

Theorem A.2 (*Compactness***).** *Fix* $R < +\infty$*, and consider the collection of sets* $U \subset \mathbf{B}_R(\mathbf{0})$ *with uniformly bounded perimeters* $\mathcal{H}^1(\partial_* U) < R$*. Their characteristic functions* χ_U *form a compact subset of* $L^1(\mathbf{R}^2)$*.*

Definition A.3. *The unit vector* $\hat{\mathbf{n}} \in \mathbf{R}^2$ *is said to be a* measure theoretic outward normal *to* $U \subset \mathbf{R}^2$ *at* $\mathbf{x} \in \partial_* U$ *if setting* $H^+ := \{\mathbf{y} \in \mathbf{R}^2 \mid \langle \hat{\mathbf{n}}, \mathbf{y} - \mathbf{x} \rangle \geq 0\}$ *yields*

$$\lim_{r \to 0} \fint_{\mathbf{B}_r(\mathbf{x}) \cap H^+} \chi_U = 0 \quad \text{and} \quad \lim_{r \to 0} \fint_{\mathbf{B}_r(\mathbf{x}) \setminus H^+} \chi_U = 1. \tag{31}$$

The measure theoretic outward normal $\hat{\mathbf{n}}_U(\mathbf{x})$ can be defined at each point $\mathbf{x}$ in a subset $\partial^* U$ of $\partial_* U$, called the *reduced boundary*. However, when U has finite perimeter, the difference between $\partial^* U$ and $\partial_* U$ has $\mathcal{H}^1$ measure zero and the Gauss map $\hat{\mathbf{n}}_U : \partial^* U \longrightarrow \mathbf{S}^1$ is Borel. Then the surface energy $F(U)$ given by (1) is well-defined, and will satisfy the inclusion-exclusion estimate [1, §3.1.4]:

$$F(U \cup V) + F(U \cap V) \leq F(U) + F(V). \tag{32}$$

Moreover, a theorem of Reshetnyak [52, Theorem 2] shows $F(U)$ to be lower semi-continuous since f is convex and positively homogeneous (E1):

Theorem A.4 (*Lower Semi-continuity of the Surface Energy***).** *Let* $\chi_{U_i} \rightarrow \chi_U$ *in* $L^1(\mathbf{R}^2)$ *for a sequence of sets* $U_i \in \mathcal{F}(\mathbf{R}^2)$. *Then* $F(U) \leq \liminf_{i \to \infty} F(U_i)$.

Taken together with the obvious lower semi-continuity of G, these results imply existence of a minimizer for $E(U)$ among sets in the ball of radius $R < +\infty$: $\mathcal{F}_m(\mathbf{B}_R) := \{U \in \mathcal{F}(\mathbf{R}^2) \mid U \subset \mathbf{B}_R(\mathbf{0}) \text{ and } \mathcal{H}^2(U) = m\}$.

Corollary A.5 (*Existence of Energy Minimizing Crystals in the Disk***).** *Among sets in* $\mathcal{F}_m(\mathbf{B}_R)$, *the energy* $E(U)$ *attains its minimum.*

Proof. Choose a sequence $U_i \in \mathcal{F}_m(\mathbf{B}_R)$ whose energy $E(U_i) = F(U_i) + G(U_i)$ tends to a minimum on $\mathcal{F}_m(\mathbf{B}_R)$. This infimum is presumed to be finite and non-negative since $E \geq 0$. Since $G \geq 0$ and the sequence is minimizing, the surface energies $F(U_i) \geq f_{min}\mathcal{H}^1(\partial_* U_i)$ must be bounded above. The perimeters are also bounded in terms of the minimum value $f_{min} > 0$ for the surface tension $f(\hat{\mathbf{n}})$ on $|\hat{\mathbf{n}}| = 1$. Thus Theorem A.2 provides a convergent subsequence, also denoted U_i, with limit $\chi_{U_i} \rightarrow \chi_U$ in $L^1(\mathbf{R}^2)$. Here $U \subset \mathbf{B}_R(\mathbf{0})$ is a set of finite perimeter, and clearly $\mathcal{H}^2(U) = m$. Moreover, Theorem A.4 yields $F(U) \leq \liminf F(U_i)$, and a similar inequality holds for E since G is lower semi-continuous as well. Thus $U \in \mathcal{F}_m(\mathbf{B}_R)$ is the desired energy minimizing set. □

The next proposition provides an upper density ratio bound for the minimizing crystal U_0 along its boundary. It is based on Giusti [31, Proposition 5.14].

Proposition A.6 (*Upper Density Ratio Bounds***).** *Let* $U_0 \subset \mathbf{R}^2$ *minimize the energy* $E(U)$ *among sets in* $\mathcal{F}_m(\mathbf{B}_R)$. *For each* $\mathbf{x} \notin U_0$ *and* $0 < r < R - |\mathbf{x}|$ *one has*

$$\mathcal{H}^2(\mathbf{B}_r(\mathbf{x}) \setminus U_0) \geq \frac{\pi r^2}{\left(1 + f_{max}/f_{min}\right)^2}. \tag{33}$$

Here f_{max} *and* f_{min} *denote the maximum and minimum values of* $f(\hat{\mathbf{n}})$ *on* $|\hat{\mathbf{n}}| = 1$.

Proof. The first step is to show that for $s < r$, the surface energy of $U = U_0 \cup \mathbf{B}_s(x)$ exceeds that of U_0 alone. To produce a contradiction, assume $F(U) < F(U_0)$. Although U may have area greater than m, Lemma 2.6 shows that intersection with the convex level sets of g can only reduce its surface energy: $F([U]_m) \leq F(U)$. The potential energy is controlled by applying Lemma 3.4 to $\rho = \chi_{U0}$ to conclude $G([U]_m) \leq G(U_0)$. These estimates combine to yield $E([U]_m) < E(U_0)$ and contradict the hypothesis that U_0 minimize energy. Our construction ensures $[U]_m \subset \mathbf{B}_R(\mathbf{0})$ has area m (or greater); if strictly greater, a contraction of $[U]_m$ toward its center of mass yields the correct area while lowering its total energy.

Having accomplished step one, let $L \subset \mathbf{R}^2$ denote the Lebesgue points of χ_{U_0} in the plane. Since this set has full measure, Fubini's theorem guarantees $\mathcal{H}^1(\partial B_s(\mathbf{x}) \setminus L) = 0$ for almost every $s \in (0, r)$. Fix such an s and observe L is disjoint from $\partial_* U_0$ to conclude

$$F(U) - F(U_0) = \int_{\partial \mathbf{B}_s(\mathbf{x}) \setminus U_0} f(\hat{\mathbf{n}}_{\mathbf{B}_s(\mathbf{x})}(\mathbf{y}))\, d\mathcal{H}^1(\mathbf{y}) - \int_{\mathbf{B}_s(\mathbf{x}) \cap \partial_* U_0} f(\hat{\mathbf{n}}_{U_0}(\mathbf{y}))\, d\mathcal{H}^1(\mathbf{y}).$$

The inequality $F(U_0) \le F(U)$ established above then yields

$$f_{min}\, \mathcal{H}^1\Big(\mathbf{B}_s(\mathbf{x}) \cap \partial_* U_0\Big) \le f_{max}\, \mathcal{H}^1\Big(\partial \mathbf{B}_s(\mathbf{x}) \setminus U_0\Big).$$

The remainder of the argument to control $a(s) := \mathcal{H}^2(\mathbf{B}_s(\mathbf{x}) \setminus U_0)$ now follows Giusti [31, Proposition 5.14]. By Fubini's theorem, the function $a(s)$ is Lipschitz continuous. Its derivative is given almost everywhere on $(0, r)$ by

$$a'(s) = \mathcal{H}^1(\partial \mathbf{B}_s(\mathbf{x}) \setminus U_0).$$

For almost all $s \in (0, r)$

$$\begin{aligned} \mathcal{H}^1(\partial_*[\mathbf{B}_s(\mathbf{x}) \setminus U_0]) &= \mathcal{H}^1(\partial \mathbf{B}_s(x) \setminus U_0) + \mathcal{H}^1(\mathbf{B}_s(\mathbf{x}) \cap \partial_* U_0) \\ &\le (1 + f_{max}/f_{min})\, \mathcal{H}^1(\partial \mathbf{B}_s(\mathbf{x}) \setminus U_0), \end{aligned}$$

so one may estimate $a'(s)$ by the isoperimetric inequality:

$$a'(s) \ge 2(1 + f_{max}/f_{min})^{-1} \sqrt{\pi a(s)}. \tag{34}$$

Integrating (34) from $s = 0$ to r yields the desired inequality (33) for $a(r)$. □

Okikiolu's theorem, which she proved for $f(\hat{\mathbf{n}})$ smooth and uniformly elliptic [48], may now be extended to all convex surface tensions f:

Theorem A.7 (*Energy Minimizing Crystals have Convex Components* **).** *If U_0 minimizes the energy $E(U)$ among sets in $\mathcal{F}_m(\mathbf{B}_R)$, then U_0 coincides (up to sets of zero area) with a countable disjoint union of closed convex sets $K_i \subset \mathbf{R}^2$. Moreover, $\partial^* U_0$ coincides with $\bigcup_{i=1}^{\infty} \partial^* K_i$, up to a set of $\mathcal{H}^1$ measure zero.*

Proof. Modifying U_0 on a set of measure zero, one may take $\partial U_0 = \overline{\partial^* U_0}$ without loss [31, Theorem 4.4]. Moreover, lower density ratio bounds can be proved for energy minimizers as in Almgren, Taylor and Wang [1, §3.4]; these force the topological boundary of U_0 to coincide with its reduced boundary up to a set of $\mathcal{H}^1$ measure zero: $\mathcal{H}^1(\partial U_0 \setminus \partial^* U_0) = 0$. Thus U_0 may be assumed to be the interior of its closure $\overline{U_0}$.

Decompose the open set U_0 into connected components $K_i \subset \mathbf{B}_R(\mathbf{0})$. Since $\chi_{K_1 \cup \cdots \cup K_n} \to \chi_{U_0}$ in $L^1(\mathbf{R}^2)$, applying the lower semi-continuity of Theorem A.4 with $f(\hat{\mathbf{n}}) = 1$ yields

$$\begin{aligned} \mathcal{H}^1(\partial^* U_0) &\le \liminf_{n \to \infty} \mathcal{H}^1(\partial^*[K_1 \cup \cdots \cup K_n]) \\ &\le \sum_{i=1}^{\infty} \mathcal{H}^1(\partial^* K_i). \end{aligned} \tag{35}$$

Here the second inequality follows from (32) or the fact that $\| \cdot \|_{BV}$ is a norm.

To obtain the reverse inequality, we shall need to establish that $\partial^* K_i$ and $\partial^* K_j$ are disjoint when $i \neq j$. From Definition A.3, $\hat{\mathbf{n}}_{K_i}(\mathbf{x}) = -\hat{\mathbf{n}}_{K_j}(\mathbf{x})$ and U_0 has Lebesgue density one at $\mathbf{x} \in \partial^* K_i \cap \partial^* K_j$. Since $\mathbf{x} \in \partial K_i$ must lie outside of U_0, this contradicts Proposition A.6. Thus $\partial^* K_i$ and $\partial^* K_j$ are disjoint. Since $\partial^* K_i \subset \partial K_i \subset \partial U_0$, it then follows from $\mathcal{H}^1(\partial U_0 \setminus \partial^* U_0) = 0$ that equality holds in (35). Apart from a set of $\mathcal{H}^1$ measure zero, we have shown $\partial^* U_0 = \bigcup_{i=1}^{\infty} \partial^* K_i$.

The next step is to establish convexity of K_i. Should convexity fail, Corollary 2.8 shows that K_i can be replaced by its convex hull $\mathrm{conv}\,[K_i] \subset \mathbf{B}_R(\mathbf{0})$ without increasing its surface energy. Observe that $U = U_0 \cup \mathrm{conv}\,[K_i]$ will have area greater than m in this case. The inclusion-exclusion estimate (32) and disjointness of the $\partial^* K_j$ yield $F(U) \leq F(U_0 \setminus K_i) + F(\mathrm{conv}\,[K_i]) \leq F(U_0 \setminus K_i) + F(K_i) = F(U_0)$. Furthermore, $F([U]_m) \leq F(U)$ by Lemma 2.6 while $G([U]_m) \leq G(U_0)$ by Lemma 3.4. Thus $E([U]_m) \leq E(U_0)$, the last two inequalities being strict unless (i) $G(U_0) = 0$ or (ii) $U_0 = [U]_m$. Since U_0 is an energy minimizer, we must be in case (i) or case (ii). In the first case both U_0 and hence $U \subset \mathrm{conv}\,[U_0]$ lie in the convex set $\{g = 0\}$. If U had area greater than m, contracting U by a factor $\lambda = \sqrt{m/\mathcal{H}^2(U)}$ toward its center of mass would violate minimality: $E(\lambda U) = \lambda E(U) < E(U_0)$. In case (ii), $U_0 = [U]_m = U \cap \{g \leq \lambda\}$ and hence $U \subset \mathrm{conv}\,[U_0]$ both lie in the convex set $\{g \leq \lambda\}$. Either way, $U = U_0$, in which case K_i was convex.

Finally, if each pair of convex components K_i and K_j are separated by positive distance, the proof of the theorem will be complete. Since $\partial^* K_i$ and $\partial^* K_j$ are disjoint, the two convex sets cannot share any edges. Being convex, the intersection of their closures can consist of no more than a point. If it is a point, then Corollary 2.8 demonstrates that replacing $K_i \cup K_j$ by its convex hull increases area without increasing surface energy. Comparing the energy of $U = U_0 \cup \mathrm{conv}\,[K_i \cup K_j]$ with U_0 would then yield the same contradiction as before. Thus $\overline{K_i}$ and $\overline{K_j}$ are disjoint. □

The next proposition implies that $E(U)$ attains its minimum among all sets in $\mathcal{F}(\mathbf{R}^2)$ having area m, and that the minimizers all lie in a single large ball of radius R_0. To prove this, it is useful to have scaled copies of the Wulff shape W available for comparison. Recall that $W := \{f^* \leq 1\}$ minimizes $F(U)$ for its area, and is given through the dual function (2) to the surface tension. Given $m > 0$, define W_m to be the dilate of W having $\mathcal{H}^2(W_m) = m$. Then if $U \in \mathcal{F}(\mathbf{R}^2)$,

$$F(U) \geq F(W_1)\sqrt{\mathcal{H}^2(U)}. \tag{36}$$

In two dimensions, a simple estimate (following, e.g., from Corollary 2.8 and Lemma 2.6) controls the diameter of a connected open set $U \in \mathcal{F}(\mathbf{R}^2)$ in terms of its surface energy:

$$\mathrm{diam}\, U := \sup_{\mathbf{x},\mathbf{y} \in U} |\mathbf{x} - \mathbf{y}| \leq (2 f_{min})^{-1} F(U); \tag{37}$$

here again, $f_{min} > 0$ is the minimum of the surface tension $f(\hat{\mathbf{n}})$ on $|\hat{\mathbf{n}}| = 1$.

Proposition A.8 (*Bound on the Location of Energy Minimizing Crystals* **).** *There is some radius $R_0 < +\infty$ – given by (38) – depending only on f, g and m, such that $U_0 \subset \mathbf{B}_{R_0}(\mathbf{0})$ whenever U_0 minimizes $E(U)$ on $\mathcal{F}_m(\mathbf{B}_R)$ for some $R < +\infty$.*

Proof. Assume the Wulff shape W_m is translated so that $G(W_m) \leq G(W_m + \mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$, and define $\lambda_m := \sup\{g(\mathbf{x}) \mid \mathbf{x} \in W_m\}$ and $V_m := \{\mathbf{x} \mid g(\mathbf{x}) \leq \lambda_m\}$. This set is bounded since the convex potential g attains its minimum on a bounded set $\{g = 0\}$.

Taking R_0 from (38) ensures $W_m \subset \mathbf{B}_{R_0}(\mathbf{0})$, hence the energy $E(U_0) \le E(W_m)$ is controlled in all cases of interest: $R \ge R_0$. The surface energy $F(U_0)$ is controlled a fortiori, leading through (37) to a bound $r := (2f_{min})^{-1}E(W_m)$ on the diameter of each of the convex components which make up U_0 according to Theorem A.7. Enlarging V_m by radius r, the result will be proved by showing that unless $U_0 \subset V_m + \mathbf{B}_r(\mathbf{0})$, some $U \in \mathcal{F}_m(\mathbf{B}_R)$ has lower energy. It will therefore be sufficient to take

$$R_0 := (2f_{min})^{-1}E(W_m) + \sup_{\mathbf{x}\in V_m} |\mathbf{x}|. \tag{38}$$

Suppose that a connected component K of U_0 fails to be contained in $V_m + \mathbf{B}_r(\mathbf{0})$; it must then be disjoint from V_m. This observation, together with (36), shows that the energy gained by removing K from U_0 to leave $U' := U_0 \setminus K$ is at least

$$\begin{aligned} E(U_0) - E(U') &= F(K) + G(K) \\ &> F(W_1)\sqrt{\mathcal{H}^2(K)} + \lambda_m\, \mathcal{H}^2(K), \end{aligned} \tag{39}$$

where the equality follows from (32) and Theorem A.7. U' will not satisfy the area constraint, but there is room inside $W_m \setminus U'$ to restore the excess mass $\mathcal{H}^2(K)$ since $\mathcal{H}^2(U') + \mathcal{H}^2(K) = m = \mathcal{H}^2(W_m)$. Because $W_m \subset V_m$, the potential energy cost for introducing this mass will be

$$G(U) - G(U') \le \lambda_m \mathcal{H}^2(K).$$

If U can be formed without paying too great a price in surface energy, the gain (39) will dominate. Choose a scaled copy of the Wulff shape $W_\mu \subset W_m$ for which $\mathcal{H}^2(W_\mu \setminus U') = \mathcal{H}^2(K)$, and define $U := W_\mu \cup U'$. Certainly $U \in \mathcal{F}(\mathbf{B}_R)$ if $R \ge R_0$. The surface energy of U is controlled by (32), (36) and $\mu \ge \mathcal{H}^2(K)$:

$$\begin{aligned} F(U) - F(U') &\le F(W_\mu) - F(W_\mu \cap U') \\ &\le F(W_1)\left(\mu^{1/2} - \sqrt{\mu - \mathcal{H}^2(K)}\right) \\ &\le F(W_1)\sqrt{\mathcal{H}^2(K)}. \end{aligned}$$

The three preceding inequalities yield $E(U) < E(U_0)$, contradicting the fact that U_0 is an energy minimizer. We must therefore conclude that $K \subset \mathbf{B}_{R_0}(\mathbf{0})$. □

Proposition A.9 (*Bound on the Smallness of Convex Components* **).** *Let U_0 minimize the energy $E(U)$ among sets in $\mathcal{F}(\mathbf{R}^2)$ having area m. If K is one of the disjoint convex components of U_0 then $\mathcal{H}^2(K) \ge \mu$, the bound $\mu > 0$ depending only on m and the integrands f and g.*

Proof. Choose the origin of $\mathbf{R}^2$ to lie somewhere in K. Since K is convex, it may be contracted by a factor $0 < \eta < 1$ without intersecting $U := U_0 \setminus K$ or indeed any dilation of U by factor $\lambda > 1$. Moreover, η and λ may be chosen to depend on each other in such a way that $\eta K \cup \lambda U$ has area m. Then the energy of this configuration is bounded below by $E(U_0)$, which will lead to a lower bound on $\mathcal{H}^2(K)$.

Before the origin was shifted, U_0 was contained in the ball $\mathbf{B}_{R_0}(\mathbf{0})$ by Proposition A.8; R_0 depended only on f, g and m. It will still be true that U_0 is contained in a ball of radius $2R_0$ about the new origin. The infinitesimal increase in $E(\lambda U) = \lambda F(U) + \lambda^2 \int_U g(\lambda \mathbf{x})\, d\mathcal{H}^2(\mathbf{x})$ with λ is given by

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1} E(\lambda U) = F(U) + \int_U (2g + \langle \mathbf{x},\, \nabla g\rangle)\, d\mathcal{H}^2(\mathbf{x}); \tag{40}$$

being convex, $g(\mathbf{x})$ is Lipschitz on $|\mathbf{x}| < 2R_0$ and the dominated convergence theorem has been applied. Since $F(U) \leq E(U_0)$ (by Theorem A.7) the cost (40) of dilating U can be controlled by a constant depending only on f, g and m. On the other hand, both the surface and potential energy of K will decrease as it is contracted . . . the latter because $\eta K \subset K$. The proposition is proved by the next estimate, which shows that unless $\mathcal{H}^2(K)$ is bounded below, the gain in $F(\eta K)$ with a small change in λ outweighs the cost (40); this would be inconsistent with $E(U_0)$ a global minimum. The estimate relies on (36) and the area constraint $\eta^2\mathcal{H}^2(K) + \lambda^2(m - \mathcal{H}^2(K)) = m$; when $\mathcal{H}^2(K)$ is small, a slight change in λ results in a huge change in η. Thus

$$-\left.\frac{d}{d\lambda}\right|_{\lambda=1} F(\eta K) = -F(K)\left.\frac{d\eta}{d\lambda}\right|_{\eta=\lambda=1} \tag{41}$$

$$\geq \mathcal{H}^2(K)^{-1/2}\,(m - \mathcal{H}^2(K))\; F(W_1) \tag{42}$$

diverges with $\mathcal{H}^2(K) \to 0$. The proposition is concluded by choosing μ small enough so $\mathcal{H}^2(K) < \mu$ implies the gain (42) in surface energy alone outweighs the U_0 independent bound on the cost (40). □

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