Arch. Rational Mech. Anal. (2004)
Digital Object Identifier (DOI) 10.1007/s00205-004-0336-3

Fast Diffusion to Self-Similarity: Complete Spectrum, Long-Time Asymptotics, and Numerology

JOCHEN DENZLER & ROBERT J. MCCANN

Dedicated to Elliott H. Lieb on the occasion of his 70th birthday.

Communicated by F. OTTO

Abstract

The complete spectrum is determined for the operator $\mathbf{H} = -m\rho^{m-1}\Delta + \boldsymbol{x}\cdot\nabla$ on the Sobolev space $W^{1,2}_\rho(\mathbf{R}^n)$ formed by closing the smooth functions of compact support with respect to the norm $\|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} := \int_{\mathbf{R}^n} |\nabla\Psi|^2 \rho\, d\boldsymbol{x}$. Here the Barenblatt profile ρ is the stationary attractor of the rescaled diffusion equation $\frac{\partial u}{\partial t} = \Delta(u^m) + \mathrm{div}(\boldsymbol{x}u)$ in the fast, supercritical regime $m \in \,]\frac{n-2}{n}, 1[$. For $m \geqq \frac{n}{n+2}$, the same diffusion dynamics represent the steepest descent down an entropy $E(u)$ on probability measures with respect to the Wasserstein distance d_2. Formally, the operator $\mathbf{H} = \mathrm{Hess}_\rho E$ is the Hessian of this entropy at its minimum ρ, so the spectral gap $\mathbf{H} \geqq \alpha := 2 - n(1-m)$ found below suggests the sharp rate of asymptotic convergence:

$$\lim_{t\to\infty} \frac{\log d_2(u(t), \rho)}{t} \leqq -\alpha < 0$$

from any centered initial data $0 \leqq u(0, \boldsymbol{x}) \in L^1(\mathbf{R}^n)$ with second moments. This bound improves various results in the literature, and suggests the conjecture that the self-similar solution $u(t, \boldsymbol{x}) = R(t)^{-n}\rho(\boldsymbol{x}/R(t))$ is always slowest to converge. The higher eigenfunctions – which are polynomials with hypergeometric radial parts – and the presence of continuous spectrum yield additional insight into the relations between symmetries of $\mathbf{R}^n$ and the flow. Thus the rate of convergence can be improved if we are willing to replace the distance to ρ with the distance to its nearest mass-preserving dilation (or still better, affine image). The strange numerology of the spectrum is explained in terms of the number of moments of ρ.

Contents

1. Introduction

This manuscript concerns the long-time behavior of solutions $v(\tau, \boldsymbol{y}) \geqq 0$ on $[0, \infty[\times \mathbf{R}^n$ to the fast diffusion equation

$$\frac{\partial v}{\partial \tau} = (v^m) \tag{1.1}$$

with supercritical nonlinearity $m \in]\frac{n-2}{n}, 1[$. The same equation goes by different names and models different phenomena according to the degree of the nonlinearity: when $m > 1$ it is called the *porous medium equation* and models a process in which the speed of diffusion mv^{m-1} increases with the concentration (or temperature), v, such as thermal conductivity in a hot plasma [46] or fluid penetrating a rock [8, 28]; when $m = 1$ it is the ordinary heat equation, in which the speed of diffusion is constant; when $m < 1$ it is called the *fast diffusion* (or *singular diffusion*) equation, since the speed of diffusion diverges as v vanishes.

For suitable initial data $0 \leqq v(0, \boldsymbol{y}) \in L^1(\mathbf{R}^n)$, unique solutions exist for all time in the case $0 < m < 1$, and become positive everywhere immediately. Indeed, HERRERO & PIERRE [22] have shown existence and uniqueness for strong solutions and any $0 < m < 1$ with merely L^1_{loc} initial data. Furthermore, L^1 initial data remain in L^1 under the evolution; for the *critical* and *supercritical* case $1 - \frac{2}{n} \leqq m < 1$, their mass is actually preserved. For *subcritical* nonlinearities, $0 < m < 1 - \frac{2}{n}$, low concentrations diffuse so quickly that a flux at infinity causes extinction in finite time; see BÉNILAN & CRANDALL [6, Proposition 10].

Supercritical diffusion will be our exclusive concern hereafter: for such nonlinearities, FRIEDMAN & KAMIN [19] (see VÁZQUEZ [41] for further references and a useful review) showed that all solutions become small over time, acquiring a characteristic shape known as the Barenblatt profile (1.4) as they dwindle away to nothing; this profile is Gaussian for the heat equation $m = 1$. The question addressed below is the precise rate of convergence to this characteristic profile. This rate is often measured in $L^1(\mathbf{R}^n)$, since the higher norms L^p with $p > 1$ capture only the rate of disappearance. For all $m > \frac{n-1}{n}$ with $m > \frac{1}{3}$, a uniform and global-in-time lower bound on the L^1 rate was derived simultaneously and independently by DOLBEAULT & DEL PINO [17] and OTTO [31], and, for $m > 1$, by CARRILLO & TOSCANI [10]. In a certain sense, their bound is sharp, though we shall presently see how to improve it. In the very fast regime, $m \in]\frac{n-2}{n}, \frac{n-1}{n}[$, rates were completely unknown until a preprint by CARRILLO, LEDERMAN, MARKOWICH & TOSCANI [14] used a nonlinear methodology to provide eigenvalue lower bounds for the spectral gap in a linearized problem. Below we shall find the complete spectrum for the linearized problem (though linearized in rather different variables than those chosen by Carrillo et al.), thus giving the sharp rates of asymptotic convergence to the Barenblatt profile – an improvement on known bounds. Also, the complete spectrum and eigenfunctions provide much information not only about the slowest mode to converge, but the geometry of all other modes as well; in particular, we glimpse the role played by the affine symmetries of $\mathbf{R}^n$ – dilations, translations, rotations, shears – in determining the solutions of (1.1). For $m < 1$ the appearance of continuous spectrum limits improvement in rate of convergence that may be achieved by quotienting out such symmetries; this is in sharp contradistinction to the case $m = 1$ of the heat equation, whose asymptotics to all orders are well known. It is also quite different from the porous medium equation $m \geqq 1$, where asymptotics to all orders were established by ANGENENT [3] in one dimension $n = 1$, following the spectral calculation of ZEL'DOVICH & BARENBLATT [44]. The chief delicacy in that calculation is the free boundary associated with compactly supported solutions; in the fast-diffusion regime the analogous difficulty is the finite number of moments possessed by the Barenblatt profile. Indeed, we shall see the mysterious numerology associated with different values of m can be explained in terms of the precise number of moments $p = 2(1 - m)^{-1} - n$. Although our analysis of the spectral problem is rigorous, we caution the reader that the linearization of (1.1) is a formal calculation, so that – as in Carrillo, Lederman, Markowich and Toscani – any conclusions about the nonlinear evolution must be treated as conjectures which, though formally justified, are not rigorously established here. See however, the nonlinear results of CARRILLO & VÁZQUEZ [11] described in the epilog below.

Let us note that for the critical exponent $m = \frac{n-2}{n}$ with $n \geqq 3$, the Barenblatt profile continues to exist as a solution, but does not have finite mass any more. The asymptotics in this case have been studied by GALAKTIONOV, PELETIER & VÁZQUEZ [20]. The subcritical case – although not discussed here – is also of interest: the exponent $p = 1 - \frac{n}{2}$ of DEL PINO & SAEZ arises in differential geometry [34], where (1.1) gives the evolution of the conformally flat metric $ds^2 = v^{4/(n+2)} \sum_{i=1}^{n} dx_i^2$ under scalar curvature (or Yamabe) flow; the Ricci flow analog is discussed by VÁZQUEZ, ESTEBAN & RODRIGUEZ in the critical planar case $n = 2$ [42]. The equation has also been used (but with $n = 1$ and $m < 0$) for modelling avalanches in sandpiles by CARLSON, CHAYES, GRANNAN & SWINDLE [9] and CHAYES, OSHER & RALSTON [15].

1.1. Sharp rates of contraction via time-dependent rescaling of space

To quantify the foregoing discussion, we make the customary change of variables,

$$u(t,\boldsymbol{x}) = e^{nt} v\left(\frac{e^{\alpha t}-1}{\alpha}, e^t \boldsymbol{x}\right),$$
$$v(\tau,\boldsymbol{y}) = \frac{1}{(1+\alpha\tau)^{n/\alpha}} u\left(\frac{1}{\alpha}\log(1+\alpha\tau), \frac{\boldsymbol{y}}{(1+\alpha\tau)^{1/\alpha}}\right), \tag{1.2}$$

where $\alpha := 2 - n(1-m)$; this corresponds to a logarithmic rescaling of time, coupled with a time-dependent contraction of space chosen at rate $1/\alpha$ just large enough to prevent the mass of u from spreading out very much. The new density $u(t,\boldsymbol{x})$ satisfies the *confined* fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta(u^m) + \operatorname{div}(\boldsymbol{x}u) \tag{1.3}$$

if and only if $v(\tau,\boldsymbol{y})$ satisfies the original equation (1.1), with the same initial condition $u(0,\boldsymbol{x}) = v(0,\boldsymbol{x})$. The advantage of this reformulation is that it shifts the fixed point of the dynamics from infinity to a finite-mass *Barenblatt profile* $u(t,\boldsymbol{x}) = u_\infty(\boldsymbol{x}) = \rho(|\boldsymbol{x}|)$ about which the evolution can be linearized. This profile possesses moments up to but excluding order

$$p := \frac{2}{1-m} - n\ ;$$

it is given explicitly by

$$\rho(r) := \left(\frac{r^2+C}{p+n-2}\right)^{-(p+n)/2} \quad \text{with} \quad C > 0 \text{ chosen so } \int_{\mathbf{R}^n} \rho(r)\,d\boldsymbol{x} = 1, \tag{1.4}$$

i.e.,

$$C^p := \pi^n\,(p+n-2)^{p+n}\left(\frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{p+n}{2})}\right)^2. \tag{1.5}$$

Here $r = |\boldsymbol{x}|$ and $\Gamma(z)$ is Euler's Gamma function (5.1). Notice that the supercritical range of fast diffusion exponents $m \in]\frac{n-2}{n}, 1[$ corresponds to a range of maximal moments $p \in]0,\infty[$. In the critical case $p = 0$, ρ has infinite mass and cannot be normalized. For the linear heat equation, $\rho(r) = (2\pi)^{-n/2} e^{-r^2/2}$ has moments of all orders, so $p = +\infty$. We shall often find it convenient to work with p rather than $m = 1 - \frac{2}{p+n}$. We also tacitly assume $n \geqq 2$ throughout the manuscript, except for passages where the case $n = 1$ is explicitly addressed.

We now indicate some of the rates of convergence obtained by CARRILLO & TOSCANI, DOLBEAULT & DEL PINO, and OTTO for $m \in [\frac{n-1}{n}, \infty[$, or equivalently $|p| \geqq n$ [10, 17, 31]. Following OTTO [31], we state these in terms of the *Wasserstein metric* d_2 defined in (2.3): the confined evolution (1.3) acts as a global contraction on the space of probability measures with finite second moments. Moreover, this contraction has a uniform rate independent of m: any two solutions $u(t)$ and $\tilde{u}(t)$ satisfy

$$d_2(u(t),\tilde{u}(t)) \leqq e^{-t} d_2(u(0),\tilde{u}(0)), \quad |p| \geqq n, \quad (p \notin [0,2]). \tag{1.6}$$

(Actually, Otto assumed $\tilde{u}(t, \boldsymbol{x}) = \rho(\boldsymbol{x})$, a restriction lifted in CARRILLO, MCCANN & VILLANI [12], and independently in forthcoming work by LYTCHAK, STURM & VON RENESSE [25].) This rate is sharp in the sense that it is attained for the Barenblatt solution $\tilde{u}(t, \boldsymbol{x}) = \rho(\boldsymbol{x})$ and its translates $u(t, \boldsymbol{x}) = \rho(\boldsymbol{x} - e^{-t} z_0)$. Although Dolbeault and del Pino use more traditional quantities like entropy or L^1 norm – instead of Wasserstein distance – to measure the deviation of $u(t)$ from ρ, their results are morally equivalent to Otto's (and can be deduced from his); again the rates they obtain are saturated by convergence of the translates $u(t, \boldsymbol{x}) = \rho(\boldsymbol{x} - e^{-t} z_0)$ to ρ. There is now a rapidly emerging literature exploring the relationships between these various notions of convergence, summarized in Villani's book and references there [43]; since in many cases of interest it is well understood how to deduce rates of convergence in a strong norm (such as L^1) from a weak metric (such as d_2), we shall not pursue alternative notions of convergence any further. Let us mention however that the restriction on nonlinearities $p \geqq n$ turns out to reflect the presence of a *phase transition* at $p = n$ from a *translation-governed* to a *dilation-governed* regime; this corresponds to a level crossing in the eigenvalue (3.7) found below; cf. Figs. 1–3 and our announcement [16]. Notice that the significance of translations can be explained as an artifact of the rescaling (1.2). The original fast diffusion equation (1.1) is translation invariant, so the effects of translation can be accounted for in the rescaled evolution, and it costs no generality to assume $v(0, \boldsymbol{x}) = u(0, \boldsymbol{x})$ to have its center of mass at the origin *a priori*. This will eventually permit us to improve the asymptotic rate of convergence – at least formally – from unity to

$$\lim_{t \to \infty} \frac{\log d_2(u(t), \rho)}{t} \leqq -\frac{2p}{p+n} = -2 + n(1-m) = -\alpha, \quad 2 < p \leqq \infty, \tag{1.7}$$

at the same time extending the range of allowable nonlinearities into the dilation-governed (very fast) regime $p < n$.

1.2. Source-type solutions, faster diffusions, and linearization

To provide motivation for the preceding formula, we start by observing that for very fast diffusion, $p \in \,]0, n[$, the nonlinear evolution is no longer a global contraction, so we cannot hope to derive estimates like (1.6). The reason is simple. Consider the source-type solutions

$$u_{R(t)}(\boldsymbol{x}) := R(t)^{-n} \rho\left(\boldsymbol{x}/R(t)\right) \tag{1.8}$$

of ZEL'DOVICH & KOMPANEETS [45], BARENBLATT [5], and PATTLE [33]. These are given by dilations of the Barenblatt profile whose radius

$$R(t) := (1 - e^{-\alpha t})^{1/\alpha}, \qquad \alpha = 2p/(p+n), \tag{1.9}$$

increases from zero to one over time. Although $R(t)$ is eventually concave, its initial convexity depends on the size of α relative to 1. Thus for very fast diffusion,

$p < n$, $R(t)$ is convex for small t, and the distance between two source solutions started at slightly different times 0 and $\delta \ll 1$,

$$d_2(u_{R(t+\delta)}, u_{R(t)}) = |R(t+\delta) - R(t)|\sqrt{\int_{\mathbf{R}^n} \boldsymbol{x}^2 \rho(\boldsymbol{x}) d^n \boldsymbol{x}}\ ,$$

will increase before it decreases. Worse yet, taking $\delta \to +\infty$, $R(t+\delta) \to 1$ and $u_1 = \rho$ in the same example shows $d_2(u_{R(t)}, \rho) \leqq e^{-\lambda t} d_2(u_{R(0)}, \rho)$ cannot hold unless $\lambda \leqq 0$; otherwise $dR/dt|_{t=0^+} = 0$ produces the contradiction:

$$0 = \left.\frac{d}{dt}\right|_{t=0^+} d_2(u_{R(t)}, \rho) \leqq -\lambda\, d_2(u_{R(0)}, \rho).$$

So relinquishing hope for a global estimate, we settle for asymptotic convergence in the long-time limit. Notice that the dilating source-type solution $u(t) = u_{R(t)}$ shows the rate constant given by (1.7) is the best possible, which is why we call this range of parameters a *dilation-governed* regime.

To study long-time asymptotics, it is natural to linearize problem (1.3) around its attracting fixed point $u(\infty, \boldsymbol{x}) = \rho(|\boldsymbol{x}|)$. This strategy was recently explored by CARRILLO, LEDERMAN, MARKOWICH & TOSCANI [14], who used it to derive a linear evolution equation, which they then analyzed directly via entropy methods and a nonlinear analog of the BAKRY & EMERY semigroup approach [4]. Their results, although stated in terms of decay of entropy rather than Wasserstein distance, translate to

$$\lim_{t\to\infty} \frac{\log d_2(u(t), \rho)}{t} \leqq -\frac{p}{p+n} = \frac{n(1-m)}{2} - 1 = -\frac{\alpha}{2}\ , \qquad 0 < p \leqq \infty, \tag{1.10}$$

formally. In case the decay starts from spherically symmetric initial conditions $u(0, \boldsymbol{x}) = u_0(|\boldsymbol{x}|)$, they improve this rate to

$$\lim_{t\to\infty} \frac{\log d_2(u(t), \rho)}{t} \leqq -\frac{p+1}{p+n} = \frac{(n-1)(1-m)}{2} - 1\ , \qquad 0 < p \leqq \infty. \tag{1.11}$$

In Fig. 1, both bounds are compared with the sharp spectral gap that we find. Although neither one of these rates is as sharp as (1.7), it is startling to see them asserted in the near-critical range of diffusion parameters $0 < p \leqq 2$. Here the Barenblatt profile no longer has second moments, so both the Wasserstein distance (2.3) and entropy (2.2) diverge. However, developing an idea used by LEDERMAN & MARKOWICH [24] to relax the restriction $p > 2$, Carrillo et al. assert that the problem can be renormalized by carefully subtracting infinities, provided the tails of $u(0, \boldsymbol{x})$ are sufficiently similar to those of the Barenblatt profile; this renormalization is further exploited in the nonlinear context by CARRILLO & VÁZQUEZ [11]. Since the same procedure applies in our case, the spectral analysis carried out in the sections below leads us to conjecture

$$\lim_{t\to\infty} \frac{\log d_2(u(t), \rho)}{t} \leqq -\frac{(\frac{p}{2}+1)^2}{p+n}, \qquad 0 < p \leqq 2, \tag{1.12}$$

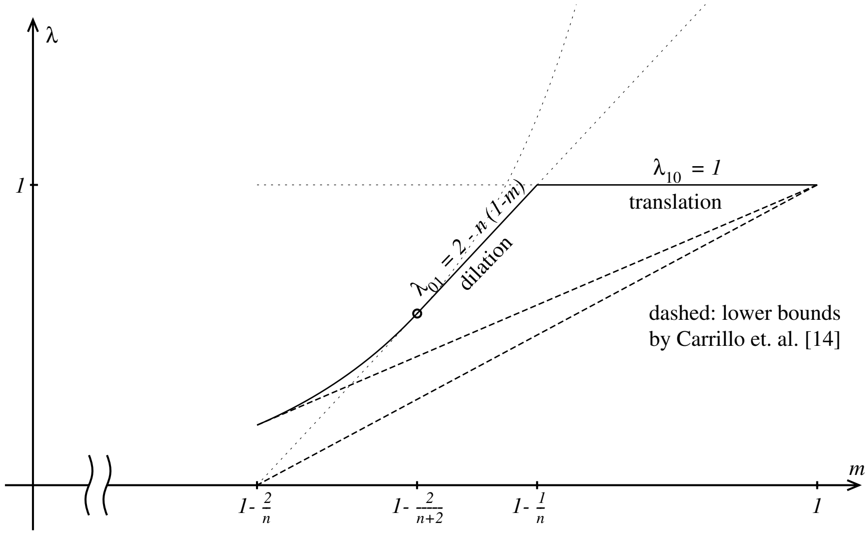

Fig. 1. The spectral gap $\Lambda_0(m)$ from (3.7) as a function of m; phase transitions at $p = 2$; and from dilation- to translation-governed dynamics at $p = n$, compared with bounds by CARRILLO *et al.* [14]. Dimension $n = 5$ was chosen for this precise graph.

and furthermore that this estimate is sharp among solutions $u(t)$ which start with the same tail behavior as ρ – meaning similar enough for $d_2(u(0), \rho) < +\infty$ in (2.3). (Neither spherical symmetry nor fixed center of mass is assumed in this conjecture; indeed, for $p \leqq 1$ the center of mass will not be well defined.) However, the sharp rate is not much more satisfactory than the bounds (1.10), (1.11) for the following reason. In the range $0 < p \leqq 2$, the restriction $d(u(0), \rho) < +\infty$ – like Carrillo et al.'s assumption about the tails of $u(0)$ – becomes an unrealistically severe constraint on the initial data. Indeed, both predict rates (1.11), (1.12), faster than (1.7) because the source type solution is no longer allowed to compete: its tail mass is so spread out that $d_2(u_{R(t)}, \rho) = +\infty$. We therefore advance a more interesting conjecture concerning dilation-persistence: namely, that the source solution (1.8) will continue to be slowest to converge once the right class of competitors and measure of convergence have been identified for $0 < p \leqq 2$. (For a framework in which to explore this *dilation-persistence conjecture* and more supporting evidence, see the manuscript of CARRILLO & VÁZQUEZ [11] discussed in Section 1.3.)

Since our analysis yields the complete spectrum – and not just the ground state – it is possible to get better rates of convergence by identifying and quotienting out the slow modes. For large p, the slowest modes turn out to correspond to translations, dilations, and affine symmetries of $\mathbf{R}^n$; (as Figs. 2 and 3 below show, there are level crossings at smaller values of p). For each invertible $n \times n$ matrix $\mathbf{A}$, define the affine image $\mathbf{A}_{\#}\rho$ of ρ by

$$\mathbf{A}_{\#}\rho(\boldsymbol{x}) = \frac{\rho(\mathbf{A}^{-1}\boldsymbol{x})}{|\det(\mathbf{A})|}. \tag{1.13}$$

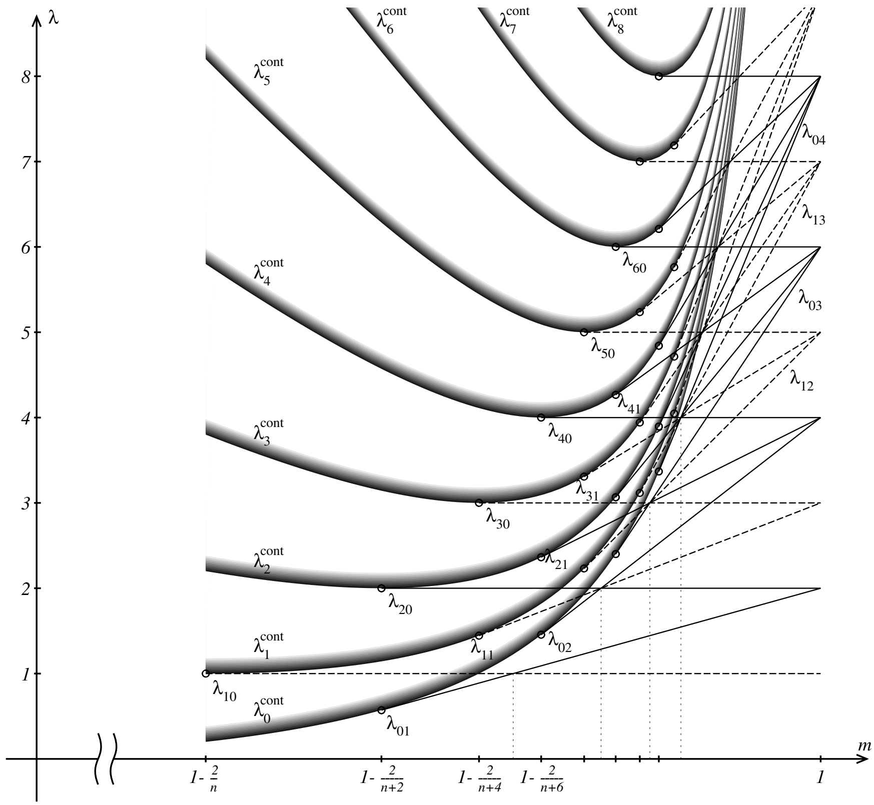

Fig. 2. The spectrum in dimension $n = 5$ as a function of nonlinearity m.

For a dilation $\mathbf{A} = R\,\mathbf{I}$ with $R > 0$ we write $R_{\#}\rho$ instead of $(R\,\mathbf{I})_{\#}\rho$. Then our formal asymptotics yield

$$\lim_{t\to\infty} \inf_{R>0} \frac{\log d_2(u(t), R_{\#}\rho)}{t} \leqq \begin{cases} -(\frac{p}{2}+1)^2/(p+n) & \text{if } p \in \,]2,6], \\ -4(p-2)/(p+n) & \text{if } p \in [6, n+4], \\ -2 & \text{if } p \in [n+4, \infty], \end{cases} \tag{1.14}$$

and, quotienting over all nonsingular transformations $\mathbf{A}$, the improvement

$$\lim_{t\to\infty} \inf_{\det \mathbf{A}\neq 0} \frac{\log d_2(u(t), \mathbf{A}_{\#}\rho)}{t} \leqq -(3p+n-4)/(p+n) \quad \text{if} \quad p \geqq n+4. \tag{1.15}$$

This rate could be improved still further if we were willing to quotient out over larger (but still finite-dimensional) families M of configurations around the Barenblatt profile ρ, obtained by extending affine maps to a larger family of nonlinear maps $\mathbf{A} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ in (1.13). However, the presence of a continuous spectrum in our problem imposes a limit on these improvements:

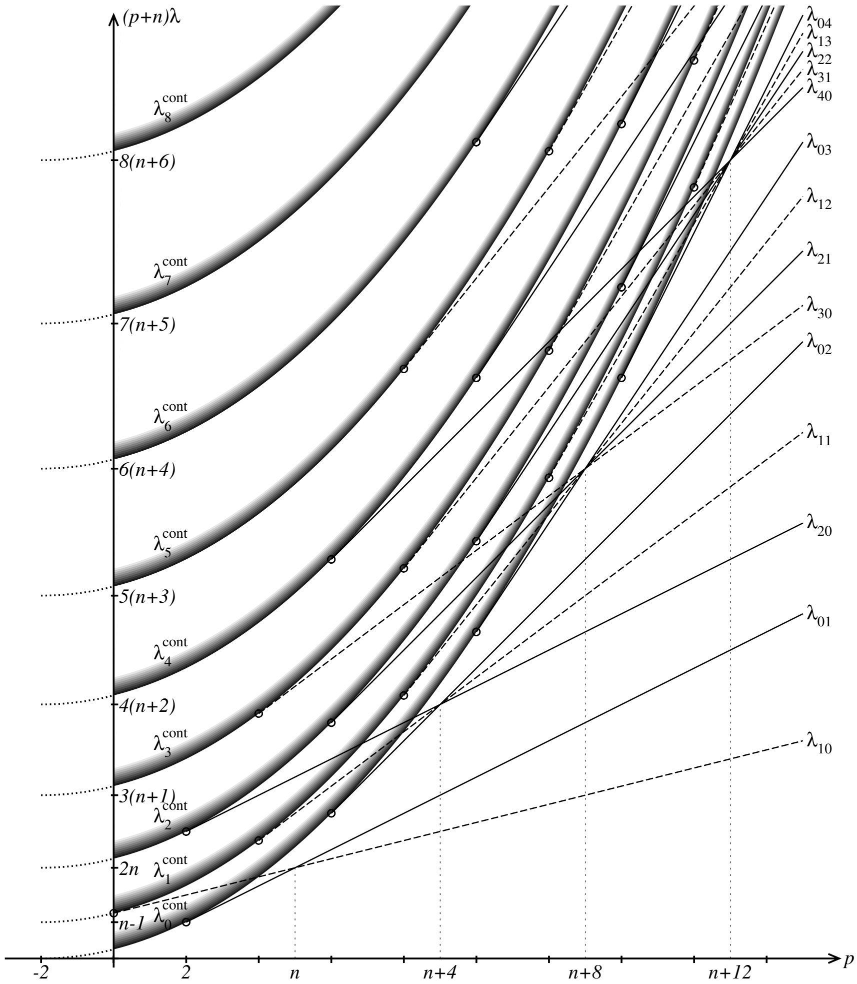

Fig. 3. The spectrum in dimension $n = 5$ as a function of number p of moments.

$$\lim_{t\to\infty} \inf_{\tilde{u}\in M} \frac{\log d_2(u(t), \tilde{u})}{t} \leqq -(\frac{p}{2}+1)^2/(p+n) \quad \text{for} \quad p \in]2, \infty[\qquad (1.16)$$

is the best possible estimate which can hold for generic initial data in (2.1), if M is finite dimensional. This limit on the rate of convergence grows like the number of moments p of ρ. This is quite different from the heat equation, where we get asymptotics to all orders by Fourier transform, or the central-limit theorem, where we have the Edgeworth expansion and analogous Berry-Esseen results [18]. It is precisely the divergence of the bound (1.16) as $p \to \infty$ – equivalent to the rapid decay of ρ – which permits asymptotics to all orders in these problems, and the porous medium regime $p < -n = -1$ [3, 44]. For the asymptotic expansion expected in the present context, see (3.6).

Let us now turn to the linearization procedure which yields these results.

1.3. Epilog

Before this manuscript was completed, we learned of a parallel investigation into the long-time asymptotics of fast diffusion by CARRILLO & VÁZQUEZ [11]. Their work complements ours nicely in many respects. Let $\tilde{v}(\tau, \boldsymbol{y})$ denote the spreading source-type solution (1.8), re-expressed in the original variables (1.2). When τ_0 can be chosen so that the variance of $v_0 - \tilde{v}(\tau_0)$ is finite, Carrillo and Vázquez establish the global bound

$$\|v(\tau) - \tilde{v}(\tau + \tau_0)\|_{L^1(\mathbf{R}^n)} \leqq \frac{C(v_0)}{\tau^{1/2}} \qquad 0 < p \leqq +\infty. \tag{1.17}$$

Note that this choice of τ_0 becomes crucial if $p \leqq 2$. Measured in Wasserstein distance between rescaled solutions, this corresponds to the rate of convergence (1.10) anticipated by CARRILLO *et al.* [14]. Our spectral gap calculation suggests that if v_0 is centered or $p \leqq n$, the exponent can be improved by a factor of two. Supporting this guess, Carrillo and Vázquez establish the sharp rate of convergence

$$\|v(\tau) - \tilde{v}(\tau + \tau_0)\|_{L^1(\mathbf{R}^n)} \leqq \frac{C(v_0)}{\tau} \qquad 0 < p \leqq +\infty \tag{1.18}$$

under the additional assumption of spherical symmetry $v_0(\boldsymbol{y}) = v_0(|\boldsymbol{y}|)$. In the radial case, this resolves our dilation-persistence conjecture. Thus their results apply to the nonlinear problem over the full range of parameters $p > 0$, but do not yield the sharp rate of convergence except for radial initial data v_0. Our results, although limited to the linearized problem and $p > 2$, give higher asymptotics as well as sharp decay rates. Both manuscripts elucidate the nature of the phase-transition which occurs at $p = n$ from a dilation-governed to a translation-governed regime.

2. Gradient flows and Hessian with respect to Wasserstein distance

2.1. Relevant facts from Otto's formal manifold approach

The starting points for our analysis are two of Otto's profound insights [31]:

(i) the space of Borel probability densities u with finite second moments

$$\mathcal{M}_2(\mathbf{R}^n) := \left\{ 0 \leqq u \in L^1(\mathbf{R}^n) \;\middle|\; \int_{\mathbf{R}^n} u(\boldsymbol{x})\, d\boldsymbol{x} = 1 \right. \\ \left. \text{and} \quad \int_{\mathbf{R}^n} \boldsymbol{x}^2 u(\boldsymbol{x})\, d\boldsymbol{x} < +\infty \right\} \tag{2.1}$$

has the formal structure of an infinite dimensional *Riemannian* manifold, on which the Wasserstein metric $d_2(u, v)$ gives the *geodesic distance* between u and v;

(ii) the nonlinear diffusion (1.1) amounts to nothing more than steepest descent of the Lyapunov functional $\|v\|^m_{L^m(\mathbf{R}^n)}/(m-1)$ on this manifold. Equivalently, the confined diffusion equation (1.3) amounts to steepest descent of the *entropy*

$$E(u) = -\frac{p+n}{2}\int_{\mathbf{R}^n} [u(\boldsymbol{x})^{-\frac{2}{p+n}} - 1]u(\boldsymbol{x})\,d\boldsymbol{x} + \frac{1}{2}\int_{\mathbf{R}^n} \boldsymbol{x}^2 u(\boldsymbol{x})\,d\boldsymbol{x} \qquad (2.2)$$

introduced earlier by NEWMAN [29] and RALSTON [37]. The shift -1 in (2.2) has been chosen so the first integral tends to Boltzmann's entropy $\|u\log u\|_{L^1(\mathbf{R}^n)}$ as $|p|\to\infty$ (i.e., $m\to 1$).

For $p>2$, $E(u)$ is finite-valued on $\mathcal{M}_2(\mathbf{R}^n)$. The only term that needs discussion is $\int_{\mathbf{R}^n} u(\boldsymbol{x})^m\,d\boldsymbol{x}$. Recalling $m = 1-\frac{2}{p+n} < 1$, it can be estimated by means of Hölder's inequality, in terms of $\int u(\boldsymbol{x})(1+\boldsymbol{x}^2)\,d\boldsymbol{x}$. This is definitely not the case for $p \leqq 2$: here, the Barenblatt profile ρ has $E(\rho) = -\infty$, with ρ^m and $\boldsymbol{x}^2\rho$ having the same divergence near infinity, but the negative coefficient of $\int \rho^m$ dominant. Nevertheless, directional derivatives of E in directions that correspond to compactly supported functions Ψ can be defined for any $m<1$, because in their definition, the integrals vanish outside the support of Ψ, and the uncontrollable tail behavior for $p\leqq 2$ does no harm. (Alternately, with hindsight our calculation for $p \leqq 2$ is justified more satisfactorily using the idea of LEDERMAN & MARKOWICH [24] as in CARRILLO & VÁZQUEZ [11, (3.4)], where an $L^1_{\mathrm{loc}}(\mathbf{R}^n)\setminus L^1(\mathbf{R}^n)$ counterterm was introduced into the integrand (2.2); see also [16]. Depending only on $\boldsymbol{x}$ but not u, this counterterm makes the integrand positive for all u, thus raising the minimum energy to $E_{\mathrm{new}}(\rho) := 0$ without changing the derivatives of $E(u)$.)

However, for $p>2$, the Barenblatt profile ρ is the unique minimizer of (even the unrenormalized) energy $E(u)$. This is most easily seen using the norm topology and linear structure which $\mathcal{M}_2(\mathbf{R}^n)$ inherits as a subset of the Banach space $L^1(\mathbf{R}^n, (1+\boldsymbol{x}^2)\,d\boldsymbol{x})$. With respect to this linear structure, E is strictly convex and ρ is a critical point, i.e., $\frac{d}{d\varepsilon}E(\rho+\varepsilon\varphi)|_{\varepsilon=0} = 0$ if $\int\varphi = 0$. The same holds for $p\leqq 2$, in the Euler equation sense, i.e., vanishing directional derivative in a dense set of directions in function space, namely the smooth functions $C_c^\infty(\mathbf{R}^n)$ of compact support. Since the dynamics (1.3) are the gradient flow of the entropy $E(\rho)$ with respect to the Wasserstein distance, we propose to compute the spectrum of the Hessian $\mathrm{Hess}_\rho E$ on the tangent space to $\mathcal{M}_2(\mathbf{R}^n)$ at the fixed point ρ. Since ρ is a minimum, this Hessian is a symmetric non-negative operator; any spectral gap $\mathrm{Hess}_\rho E \geqq \Lambda_0 > 0$ implies rapid convergence of nearby trajectories to ρ under the dynamics with exponential rate constant Λ_0 – as measured in the ambient distance d_2 on the manifold $\mathcal{M}_2(\mathbf{R}^n)$. Of course, the topology induced by the Wasserstein metric d_2 is a weak one: $d_2(u_k,u)\to 0$ if and only if $u_k\,d\boldsymbol{x} \rightharpoonup u\,d\boldsymbol{x}$ against $C_c(\mathbf{R}^n)$ test functions and $\int \boldsymbol{x}^2 u_k(x)\,d\boldsymbol{x} \to \int \boldsymbol{x}^2 u(\boldsymbol{x})\,d\boldsymbol{x}$; however, as mentioned above, effective techniques have emerged for converting rates of convergence from d_2 to $\|\cdot\|_{L^1}$ in situations akin to the present setting [31, 32, 13, 43], so we do not address this point further here.

We shall compute $\mathrm{Hess}_\rho E$ using Otto's Riemannian calculus [31, Section 4.4]. Although this is in principle equivalent to linearizing equation (1.3) as per CARRILLO et al. [14], it has the advantage that the local metric prescribed by Otto on the tangent space to $\mathcal{M}_2(\mathbf{R}^n)$ clearly identifies the Hilbert space on which $\mathrm{Hess}_\rho E$ should act – a crucial ingredient in determining the spectrum. To our knowledge this work represents the first spectral analysis attempted in the framework of Otto's calculus, i.e., in the framework of Wasserstein distance.

Recall that the *Wasserstein distance* between two probability densities $u, v \in \mathcal{M}_2(\mathbf{R}^n)$ is given by

$$d_2(u,v)^2 := \inf_{\gamma\in\Gamma(u,v)} \int_{\mathbf{R}^n\times\mathbf{R}^n} |\boldsymbol{x}-\boldsymbol{y}|^2\, d\gamma(\boldsymbol{x},\boldsymbol{y}). \tag{2.3}$$

Here the infimum is computed over the space $\Gamma(u,v)$ of non-negative measures γ on $\mathbf{R}^n \times \mathbf{R}^n$ having marginals u and v: i.e.,

$$\int_{\mathbf{R}^n\times\mathbf{R}^n} f(\boldsymbol{x})\, d\gamma(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbf{R}^n} f(\boldsymbol{x})\, u(\boldsymbol{x}) d\boldsymbol{x}\ , \quad \int_{\mathbf{R}^n\times\mathbf{R}^n} g(\boldsymbol{y})\, d\gamma(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbf{R}^n} g(\boldsymbol{y})\, v(\boldsymbol{y}) d\boldsymbol{y}$$

for all smooth test functions $f, g \in C_c^\infty(\mathbf{R}^n)$ of compact support. It is well known that d_2 gives a metric on $\mathcal{M}_2(\mathbf{R}^n)$ and the infimum is assumed in this case [21]. Moreover, definition (2.3) extends unambiguously to arbitrary non-negative distributions u and v on $\mathbf{R}^n$ (in the sense of Schwartz), whether or not they have densities, second moments, or finite mass – only then $d_2(u,v)$ may or may not be finite. Finiteness of $d_2(u(0),\rho)$ is the natural tail condition under which we conjectured (1.12). (However, for near-critical diffusion, $p \leqq 2$, the hypothesis introduced by Carrillo and Vázquez to derive (1.17) is more appropriate.)

A basic ingredient in Otto's calculus is the identification of the tangent space $\mathcal{T}_u\mathcal{M}$ at $u \in \mathcal{M}_2(\mathbf{R}^n)$ as the Sobolev space $\mathcal{T}_u\mathcal{M} = W_u^{1,2}(\mathbf{R}^n) \subset W_{\mathrm{loc}}^{1,2}(\mathbf{R}^n)$ of weakly differentiable functions [31, (9)]

$$W_u^{1,2}(\mathbf{R}^n) := \left\{ \Psi : \mathbf{R}^n \longrightarrow \mathbf{R} \;\middle|\; \int_{\mathbf{R}^n} |\nabla\Psi|^2 u(\boldsymbol{x})\, d\boldsymbol{x} < \infty \right\} \Big/ \{\|\cdot\| = 0\}; \tag{2.4}$$

the last symbol indicates that any two functions are identified with each other if their difference is constant a.e. The inner product on $W_u^{1,2}(\mathbf{R}^n)$ is given by

$$\|\Psi\|^2_{W_u^{1,2}(\mathbf{R}^n)} := \langle \Psi; \Psi\rangle_{W_u^{1,2}(\mathbf{R}^n)} := \int_{\mathbf{R}^n} |\nabla\Psi|^2 u(\boldsymbol{x})\, d\boldsymbol{x}, \tag{2.5}$$

which makes $W_u^{1,2}(\mathbf{R}^n)$ a Hilbert space, or pre-Hilbert, if we restrict ourselves to compactly supported Ψ. Indeed, at the Barenblatt profile $u = \rho$, Corollary 14 eventually asserts that $W_\rho^{1,2}(\mathbf{R}^n)$ can also be realized as the closure of the smooth functions $C_c^\infty(\mathbf{R}^n)$ of compact support with respect to the norm (2.5).

The *exponential map* from $W_u^{1,2}(\mathbf{R}^n)$ to $\overline{\mathcal{M}_2(\mathbf{R}^n)}$ gives a local coordinate chart on $\mathcal{M}_2(\mathbf{R}^n)$ equivalent to specifying the geodesics passing through u. Given a tangent vector $\Psi \in W_u^{1,2}(\mathbf{R}^n)$, the geodesic through u in direction Ψ is a path $u_s = \exp_u s\Psi = [id + s\nabla\Psi]_\# u$ in $\overline{\mathcal{M}_2(\mathbf{R}^n)}$ defined by gradually displacing the mass of u in the direction given by the initial vector field $\nabla\Psi$. More precisely, for each $s \in \mathbf{R}$, the mass is *pushed forward* through the map $F_s(\boldsymbol{x}) = \boldsymbol{x} + s\nabla\Psi(\boldsymbol{x})$ of $\mathbf{R}^n$, which is a diffeomorphism for small s and smooth Ψ; this means

$$\int_{\mathbf{R}^n} f(\boldsymbol{y})u_s(\boldsymbol{y})\,d\boldsymbol{y} := \int_{\mathbf{R}^n} f(\boldsymbol{x} + s\nabla\Psi(\boldsymbol{x}))u(\boldsymbol{x})\,d\boldsymbol{x} \tag{2.6}$$

for every smooth test function $f \in C_c^\infty(\mathbf{R}^n)$. Not having specified the topology or differentiable structure on $\mathcal{M}_2(\mathbf{R}^n)$, the Hessian $\mathrm{Hess}_\rho E$ will be *defined* as a quadratic form in terms of second derivatives of the entropy along minimizing geodesic segments [31, (83)–(85)]. As a second derivative in a functional analytic sense however, it is only formal:

$$\begin{aligned}\mathrm{Hess}_u E(\Psi, \Psi) := \left.\frac{d^2E(u_s)}{ds^2}\right|_{s=0} &= \|\Psi\|^2_{W_u^{1,2}(\mathbf{R}^n)} \\ &+ \int_{\mathbf{R}^n} \left\{|\mathrm{Hess}\Psi(\boldsymbol{x})|_2^2 - \frac{2(\Delta\Psi(\boldsymbol{x}))^2}{p+n}\right\} u^{1-\frac{2}{p+n}}(\boldsymbol{x})\,d\boldsymbol{x}. \end{aligned}\tag{2.7}$$

Here $|\mathrm{Hess}\Psi|_2$ is the Hilbert-Schmidt norm of the matrix of second partials $\partial_i\partial_j\Psi$; for any square matrix $\mathbf{A}$, $|\mathbf{A}|_2^2 := \sum_{ij} |\mathbf{A}_{ij}|^2 = \mathrm{trace}\,\mathbf{A}^T\mathbf{A}$. With compactly supported smooth Ψ, (2.7) holds for every $u \in \mathcal{M}_2(\mathbf{R}^n)$, and any $m < 1$.

Our task is to prove the existence of a *spectral gap*: namely the estimate $\mathrm{Hess}_\rho E(\Psi, \Psi) \geqq \Lambda_0 \|\Psi\|^2_{W_\rho^{1,2}(\mathbf{R}^n)}$, for some $\Lambda_0 > 0$ and all $\Psi \in W_\rho^{1,2}(\mathbf{R}^n)$. That the sharp constant Λ_0 is given by (3.7) is the first rigorous result that we claim. In fact, we shall give the complete spectral analysis of the displacement Hessian $\mathrm{Hess}_\rho(\Psi, \Psi) = \langle\Psi; \mathbf{H}\Psi\rangle_{W_\rho^{1,2}(\mathbf{R}^n)}$, viewed as an (unbounded) self-adjoint operator in $W_\rho^{1,2}(\mathbf{R}^n)$. The remarkable fact that $\mathbf{H}$ turns out to be a simple *differential* operator of second order is the key to our description of the spectrum. In the absence of special algebraic structure, based on a straightforward application of the Euler equations and Lagrange multipliers, we would have an eigenvalue problem of the type $H_4\Psi = \lambda H_2\Psi$, with differential operators H_4 and H_2 of orders 4 and 2 respectively, so $\mathbf{H}$ might be expected to be second-order pseudodifferential (as would actually happen for the similar problem at a non-critical point $u \neq \rho$ on $\mathcal{M}_2(\mathbf{R}^n)$).

2.2. The displacement Hessian as an operator

From the defining equation, it is not evident that the Hessian is non-negative, much less non-degenerate. Only for $p \geqq n$ does the Cauchy-Schwarz inequality

$$\Delta\Psi = \mathrm{trace}[\mathrm{Hess}\Psi] \leqq \sqrt{n}\ |\mathrm{Hess}\Psi|_2$$

control the sign of the integrand (2.7). In this case we may take $\Lambda_0 = 1$; indeed, $\mathrm{Hess}_\rho E(\Psi, \Psi) \geqq \|\Psi\|^2_{W_\rho^{1,2}(\mathbf{R}^n)}$ independently of our special choice of the Barenblatt profile ρ, and the nonlinear contraction rate (1.6) follows from uniform geodesic convexity of the entropy. This is the essence of Otto's argument; the geodesic convexity of $E(\rho)$ had already been established under the name *displacement convexity* by McCann [27]. For faster diffusions $p < n$, the integrand (2.7) can take negative values. As it involves second derivatives it seems quite surprising that it should be controlled by a norm $\|\Psi\|_{W^{1,2}_\rho(\mathbf{R}^n)}$ involving only first derivatives of Ψ. This fact is intimately linked to the very special form of the Barenblatt profile ρ, as we shall see in the next proposition and its corollary. There the correspondence between the operator $\mathbf{H} := -m\rho^{m-1}\Delta + \boldsymbol{x}\cdot\nabla$ on $W^{1,2}_\rho(\mathbf{R}^n)$ and the quadratic form $\mathrm{Hess}_\rho E$ is established. We begin by recalling a simple lemma.

Lemma 1 (Bochner identity)**.** *Any* $\Psi \in C^3(\mathbf{R}^n)$ *satisfies*

$$|\mathrm{Hess}\Psi|_2^2 = \tfrac{1}{2}\Delta(|\nabla\Psi|^2) - \nabla\Psi\cdot\nabla(\Delta\Psi). \tag{2.8}$$

Proof. We have

$$\begin{aligned}(\partial_i\partial_j\Psi)(\partial_i\partial_j\Psi) &= \partial_i\Big\{(\partial_j\Psi)(\partial_i\partial_j\Psi)\Big\} - (\partial_j\Psi)(\partial_j\Delta\Psi)\\ &= \partial_i\partial_i\Big\{\frac{1}{2}|\nabla\Psi|^2\Big\} - \nabla\Psi\cdot\nabla(\Delta\Psi). \quad \square\end{aligned}$$

Proposition 2 (Displacement Hessian operator)**.** *Let* $m > 0$*. For any distribution* $u \in L^m_{\mathrm{loc}}(\mathbf{R}^n)\cap L^1_{\mathrm{loc}}(\mathbf{R}^n)$ *and any smooth, compactly supported test function* Ψ *on* $\mathbf{R}^n$*, the formal Hessian, defined in* (2.7) *can be written as*

$$\begin{aligned}\mathrm{Hess}_u(\Psi,\Psi) &= \|\Psi\|^2_{W^{1,2}_u(\mathbf{R}^n)} + \int_{\mathbf{R}^n} u^m\Big\{|\mathrm{Hess}\Psi|_2^2 - (1-m)(\Delta\Psi)^2\Big\}\,d\boldsymbol{x}\\ &= \langle\Psi;\mathbf{H}\Psi\rangle_{W^{1,2}_u(\mathbf{R}^n)} + \frac{1}{2}\int_{\mathbf{R}^n}\Big\{\Delta u^m + \mathrm{div}(\boldsymbol{x}u)\Big\}|\nabla\Psi|^2\,d\boldsymbol{x} \end{aligned}\tag{2.9}$$

with the operator $\mathbf{H} : \Psi \mapsto (-mu^{m-1}\Delta + \boldsymbol{x}\cdot\nabla)\Psi$*. In particular, if u is the Barenblatt profile* ρ*, then* $\mathbf{H}\Psi = -m\rho^{m-2}\,\mathrm{div}(\rho\nabla\Psi)$*, and the last integrand vanishes in* (2.9).

Proof. Using Bochner's formula (2.8) and integrating by parts, we calculate, initially under the assumption $u \in C^2(\mathbf{R}^n)$:

$$\begin{aligned}&-\frac{1}{2}\int\Big(\Delta u^m + \mathrm{div}(\boldsymbol{x}u)\Big)|\nabla\Psi|^2 + \int u^m\Big\{|\mathrm{Hess}\Psi|_2^2 - (1-m)(\Delta\Psi)^2\Big\}\\ &= -\frac{1}{2}\int \mathrm{div}(\boldsymbol{x}u)|\nabla\Psi|^2 + \int u^m\Big\{-\nabla\Psi\cdot\nabla(\Delta\Psi) - (1-m)(\Delta\Psi)^2\Big\}\\ &= \frac{1}{2}\int u\boldsymbol{x}\cdot\nabla(|\nabla\Psi|^2) - \int u^m\nabla\Psi\cdot\nabla(\Delta\Psi)\\ &\quad -(1-m)\int u^m\Big(\mathrm{div}(\nabla\Psi\,\Delta\Psi) - \nabla\Psi\cdot\nabla(\Delta\Psi)\Big)\\ &= \int u\nabla\Psi\cdot(\mathrm{Hess}\Psi)\boldsymbol{x} - m\int u^m\nabla\Psi\cdot\nabla(\Delta\Psi)\\ &\quad +(1-m)\int(\nabla u^m)\cdot(\nabla\Psi)\Delta\Psi\\ &= \int u\nabla\Psi\cdot\nabla(\boldsymbol{x}\cdot\nabla\Psi - \Psi) - m\int u\nabla\Psi\cdot\nabla(u^{m-1}\Delta\Psi)\\ &= \langle\Psi;(\mathbf{H}-1)\Psi\rangle_{W^{1,2}_u(\mathbf{R}^n)}\,,\end{aligned}$$

hence (2.9), as desired, for smooth u. In the second-last line, we have made use of the identity $(1-m)\nabla u^m = -mu\nabla u^{m-1}$. Having estimated (2.9) for $u \in C^2$, we now approximate, for $u \in L^1_{\mathrm{loc}} \cap L^m_{\mathrm{loc}}$ and $m < 1$, the function $u^m \in L^{1/m}_{\mathrm{loc}} \cap L^1_{\mathrm{loc}} = L^{1/m}_{\mathrm{loc}}$ by C^2 functions w_k, using standard density arguments in L^p spaces; the $w_k^{1/m}$ are also in C^2 and approximate u. The distributional interpretation of (2.9) follows immediately; ($m \geqq 1$ would be even simpler, because it would not require passing to u^m to avoid the exotic spaces $L^{<1}$).

The following formulas for ρ follow right from the definition:

$$\partial_r(\rho(r)^m) = -r\rho(r) \ , \quad \partial_r\rho(r) = -\frac{r}{m}\rho(r)^{2-m}. \tag{2.10}$$

They imply immediately

$$\Delta\rho^m + \operatorname{div}(\boldsymbol{x}\rho) = r^{1-n}\partial_r(r^{n-1}\partial_r\rho^m) + n\rho + r\partial_r\rho = 0\ . \quad \square$$

A simple integration by parts shows positivity of this displacement Hessian:

Corollary 3 (Positivity and symmetry)**.** *For any smooth, compactly supported test functions Φ and Ψ on $\mathbf{R}^n$,*

$$\mathrm{Hess}_\rho(\Phi, \Psi) = \langle \Phi; \mathbf{H}\Psi\rangle_{W^{1,2}_\rho(\mathbf{R}^n)} = m\int_{\mathbf{R}^n} \rho^{m-2}\operatorname{div}[\rho\nabla\Phi]\operatorname{div}[\rho\nabla\Psi]\,d\boldsymbol{x}. \tag{2.11}$$

Proof. Express $\langle \Phi; \mathbf{H}\Psi\rangle_{W^{1,2}_\rho(\mathbf{R}^n)}$ by using $\mathbf{H}\Psi = -m\rho^{m-2}\operatorname{div}[\rho\nabla\Psi]$ in (2.5). $\square$

3. Overview of spectral results

In this section, we describe the spectral properties found below for the operator $\mathbf{H} = -m\rho^{m-1}\Delta + \boldsymbol{x}\cdot\nabla$ corresponding to the Hessian (with respect to the Wasserstein metric) of the energy $E(u)$ at the minimizing Barenblatt profile $u = \rho$. This Hessian represents a positive-definite self-adjoint unbounded operator on the Hilbert space $W^{1,2}_\rho(\mathbf{R}^n)$ with scalar product (2.5). The analysis is carried out by first noting that $\mathbf{H}$ commutes with the total angular momentum operator $-\Delta_{S^{n-1}}$, and then finding the spectrum of the radial problem one spherical harmonic at a time. We continue to assume $n \geqq 2$ tacitly, except where the contrary is stated; see Section 4.8 for the case $n = 1$.

For $\ell = 0, 1, 2, \ldots$, let $\mathbf{H}_\ell$ denote the restriction of $\mathbf{H}$ to the eigenspace of $-\Delta_{\mathbf{S}^{n-1}}$ corresponding to eigenvalue $\boldsymbol{L}^2 = \ell(\ell+n-2)$. Our conclusions are the following: for $1-\frac{2}{n} < m < 1$, the spectrum $\sigma(\mathbf{H}_\ell) = \sigma_{\mathrm{cont}}(\mathbf{H}_\ell) \cup \sigma_{\mathrm{p}}(\mathbf{H}_\ell)$ consists of a disjoint union of a (non-empty) continuous part, together with at most finitely many eigenvalues

$$\sigma(\mathbf{H}_\ell) = [\lambda_\ell^{\mathrm{cont}}, \infty[\cup \{\lambda_{\ell 0}, \lambda_{\ell 1}, \ldots, \lambda_{\ell K}\} \setminus \{0\}. \tag{3.1}$$

The main feature of this spectrum (see Fig. 2) is that all eigenvalues are given by linear functions of m which interpolate between $\lambda_{\ell k}|_{m=1} = \ell + 2k > 0$ and a point of tangency with the threshold of the continuous spectrum

$$\lambda_\ell^{\mathrm{cont}} = \frac{\boldsymbol{L}^2 + (\frac{p}{2}+1)^2}{p+n} \tag{3.2}$$
$$= \frac{1}{2(1-m)} + \frac{1-m}{2}\left(\frac{n}{2}+\ell-1\right)^2 - \frac{n}{2} + 1.$$

This tangency can only occur at $p = 2\ell - 2 + 4k$, after which the eigenvalue dissolves into the continuous spectrum and is lost for all smaller values of m. Thus the number $K + \min\{\ell, 1\}$ (if any) of eigenvalues is determined by the largest integer $K \geqq 0$ satisfying $\ell + 2K - 1 < p/2 = (1-m)^{-1} - n/2$. The eigenvalues themselves are given by

$$\lambda_{\ell k} := \ell+2k-2k(k+\ell+n/2-1)(1-m) \quad \text{for } k = 0, 1, \dots, K,$$
$$= \frac{\boldsymbol{L}^2 + (\frac{p}{2}+1)^2 - (\frac{p}{2}+1-\ell-2k)^2}{p+n}, \quad k \in \mathbf{Z} \cap [0, \tfrac{1+p/2-\ell}{2}[, \tag{3.3}$$

except that $\lambda_{00} = 0$ is not an eigenvalue. For any value of $p > 0$, they are ordered by

$$\ell = \lambda_{\ell 0} < \lambda_{\ell 1} < \lambda_{\ell 2} < \cdots < \lambda_{\ell K} < \lambda_\ell^{\mathrm{cont}} < \lambda_{\ell+1}^{\mathrm{cont}}. \tag{3.4}$$

If no eigenvalues are present ($p \leqq 2\ell - 2$ or $m \leqq 1 - \frac{2}{n+2\ell-2}$) we still have $\ell \leqq \lambda_\ell^{\mathrm{cont}} < \lambda_{\ell+1}^{\mathrm{cont}}$ (with equality where $\lambda_{\ell 0}$ disappears into the continuous spectrum). Notice that the continuum threshold diverges $\lambda_\ell^{\mathrm{cont}} \to +\infty$ as $m \to 1^-$, while the spectrum degenerates to the positive integers. This comes as no surprise, since our Hessian converges to the Ornstein-Uhlenbeck generator $\mathbf{H} = -\Delta + \boldsymbol{x} \cdot \nabla$, well known to be conjugate via similarity transformation to the harmonic oscillator Hamiltonian $-\Delta + \boldsymbol{x}^2/4$ $(= \rho_\infty^{1/2}\mathbf{H}\rho_\infty^{-1/2} + n/2)$. The Gaussian $\rho_\infty = (2\pi)^{-n/2}\exp(-r^2/2)$ is the limit of the Barenblatt profile as $p \to \infty$.

Except for recovering the limit $m = 1$, it is more convenient to visualize the spectrum of the operator $(p+n)\mathbf{H}$ as a function of $p \in]0, +\infty[$, where it corresponds to a sequence of half-lines and parabolas; see Fig. 3. Indeed the continuous thresholds (3.2) become a sequence of congruent parabolas assuming their minimum values $\boldsymbol{L}^2 = \ell(\ell+n-2)$ at $p = -2$. The eigenvalues, now linear in p, become a sequence of semi-infinite rays with positive integer slopes, increasing from a point of tangency with the parabola $y(p) = (p+n)\lambda_\ell^{\mathrm{cont}}$ to $p = +\infty$. Each such line corresponds to an eigenvalue family if and only if its slope is an integer $\ell + 2k$ $(k = 0, 1, 2, \dots)$ sharing the parity (even or oddness) of ℓ.

The corresponding eigenfunctions are almost as easy to describe. Each eigenvalue $\lambda_{\ell k}$ has the same degeneracy (4.4) for $\mathbf{H}_\ell$ as $\boldsymbol{L}^2 = \ell(\ell+n-2)$ has for $-\Delta_{\mathbf{S}^{n-1}}$. The corresponding eigenfunctions $\mathbf{H}\Psi_{\ell k\mu} = \lambda_{\ell k}\Psi_{\ell k\mu}$ are polynomials of degree $\ell + 2k$. Since ρ has fewer than p moments, it is clear why the restriction $\ell+2k < 1+p/2$ gives the square integrability (2.5) required for $\Psi_{\ell k\mu} \in W_\rho^{1,2}(\mathbf{R}^n)$. The non-integrability for larger values of k hints that continuous spectrum should be anticipated. Quite explicitly,

$$\Psi_{\ell k\mu}(\boldsymbol{x}) = \psi_{\ell k}(|\boldsymbol{x}|)\, Y_{\ell\mu}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \qquad \begin{matrix} \ell, k = 0, 1, 2, \ldots \\ 0 < \ell + 2k < 1 + p/2 \end{matrix}$$

where $Y_{\ell\mu}$ is a spherical harmonic $-\Delta_{\mathbf{S}^{n-1}} Y_{\ell\mu} = \ell(\ell + n - 2) Y_{\ell\mu}$ and

$$\psi_{\ell k}(r) = r^{\ell}\, {}_2F_1\left(\begin{matrix} k + \ell - 1 - p/2, \quad -k \\ \ell + n/2 \end{matrix}; \frac{-r^2}{C}\right) \tag{3.5}$$

is a *hypergeometric function* (5.3). Since $-k$ is an integer, the hypergeometric series ${}_2F_1(z)$ terminates at the $(k + 1)$-th term, forming a polynomial of degree k. The constant C comes from the normalization of the Barenblatt profile (1.4), (1.5).

Recalling from (2.6) that the Wasserstein geodesics $\exp_\rho s\Psi \in \mathcal{M}_2(\mathbf{R}^n)$ are given by displacing the mass of ρ inertially along the vector field $\nabla\Psi(\boldsymbol{x})$ gives an interpretation to these eigenfunctions. When $\ell + 2k \leqq 2$ the eigenfunction $\Psi_{\ell k\mu}$ is quadratic, so the map $F_s(\boldsymbol{x}) = \boldsymbol{x} + s\nabla\Psi(\boldsymbol{x})$ is affine. Thus $\nabla\Psi_{011} = -2p\boldsymbol{x}/(Cn)$ generates dilations; $\nabla\Psi_{10\mu} = \hat{\mathbf{e}}_\mu$ generates coordinate translations $\mu = 1, \ldots, n$; and $\nabla\Psi_{20\mu} = \mathbf{A}(\mu)\boldsymbol{x}$ with symmetric, trace-free matrices $\mathbf{A}(\mu)$, generate the remaining $\left(\binom{n}{2} + n - 1\right)$-dimensional symmetric space of affine images of ρ, in other words, the group of affine transformations of $\mathbf{R}^n$, modulo the group of rotations (symmetries of ρ); cf. [27, Example 1.7]. For $\ell + 2k > 2$, the transformation $F_s(\boldsymbol{x})$ of ρ is no longer affine, but – apart from this fact – not much more difficult to understand. Thus as $t \to \infty$ we may conjecture an asymptotic expansion for $u_t = \exp_\rho \Psi(t, \boldsymbol{x})$ given by polynomials in t depending only on the initial data:

$$\begin{aligned} &\left\| \Psi(t, \boldsymbol{x}) - \sum_{\boldsymbol{\iota}} \left(c_{\boldsymbol{\iota}0}(u_0)\Psi^{\boldsymbol{\iota}0}(x) + t c_{\boldsymbol{\iota}1}(u_0)\Psi^{\boldsymbol{\iota}1}(\boldsymbol{x}) + \cdots \right) \exp[-\boldsymbol{\iota}\cdot\lambda t] \right\| \\ &= O(e^{-t(\lambda_0^{\text{cont}} - \varepsilon)}) \end{aligned} \tag{3.6}$$

for any $\varepsilon > 0$. Here, the sum runs over all nonzero multi-indices $\boldsymbol{\iota} = (\iota_{\ell k\mu})$, such that $\boldsymbol{\iota}\cdot\boldsymbol{\lambda} := \sum \iota_{\ell k\mu}\lambda_{\ell k} < \lambda_0^{\text{cont}}$, and the polynomial coefficients $\Psi^{\boldsymbol{\iota}j}(\boldsymbol{x})$ depend only on m, n, and $\boldsymbol{\iota}$; if $\boldsymbol{\iota}$ has length 1 (i.e., represents an eigenvalue), then $\Psi^{\boldsymbol{\iota}0}(\boldsymbol{x})$ is the corresponding eigenfunction. The polynomials in t should reduce to constants in the absence of resonances. The question of resonances is discussed by ANGENENT [3], who corrected a formal expansion of ZEL'DOVICH & BARENBLATT [44] to establish rigorous asymptotics in the one-dimensional porous medium equation ($n = 1$, $m \geqq 1$). Since there is no continuous spectrum in that setting, he derives an asymptotic expansion to all orders in place of the finite sum (3.6). It is worth pointing out that the eigenvalues found by these authors coincide with the extension (4.41) of our spectral lines $\{\lambda_{0k}, \lambda_{1,k-1}\}_{k\geqq 1}$ to the region $p < -n$, i.e., $m \geqq 1$. Here the offset $\boldsymbol{L}^2 = 0$ between odd and even spherical harmonics vanishes, so there are no eigenvalue crossings in this regime (4.41).

On the other hand, for multidimensional fast diffusion ($n \geqq 2$, $p > 0$) our analysis shows $\mathbf{H} \geqq \Lambda_0 > 0$ is strictly positive for $p \geqq 0$, with a spectral gap given by $\Lambda_0 = \min\{\lambda_0^{\text{cont}}, \lambda_{01}, \lambda_{10}\}$, or explicitly

$$\Lambda_0 = \begin{cases} \lambda_0^{\text{cont}} = (\frac{p}{2}+1)^2/(p+n) & \text{if } p \in [0,2], \\ \lambda_{01} = 2p/(p+n) \quad \text{(dilation-governed)} & \text{if } p \in [2,n], \\ \lambda_{10} = 1 \quad \text{(translation-governed)} & \text{if } p \in [n,\infty]. \end{cases} \tag{3.7}$$

The eigenvalues λ_{01} and λ_{10} correspond to dilations and translations of ρ, and therefore come with multiplicities 1 and n respectively.

The next spectral level $\Lambda_1 = \min\{\lambda_0^{\text{cont}}, \max\{1, \lambda_{01}\}\}$ of the overall Hamiltonian $\mathbf{H}$ is not terribly interesting, since it also corresponds to dilation or translation, depending on which side of the level crossing we are at. Notice that the spectral multiplicity $\#(\Lambda_0) :=\dim \operatorname{Ker}(\mathbf{H} - \Lambda_0)$ and $\#(\Lambda_1)$ belong to $\{0, 1, n\}$, while their sum lies in $\#(\Lambda_1) + \#(\Lambda_0) \in \{0, 1, n+1\}$. The third spectral level $\Lambda_2 = \min\{\lambda_0^{\text{cont}}, \lambda_{02}, \lambda_{20}\}$, however, is more interesting, since it governs the rate of convergence of "shape" (1.14) to the submanifold consisting of translations and dilations of the self-similar profile ρ. Its explicit value is given by

$$\Lambda_2 = \begin{cases} \lambda_0^{\text{cont}} & \text{if } p \in]0,6], \\ \lambda_{02} = 4(p-2)/(p+n) & \text{if } p \in [6, n+4], \\ \lambda_{20} = 2 \quad \text{(affinely-governed)} & \text{if } p \in [n+4, \infty]. \end{cases} \tag{3.8}$$

Note that although λ_{11} might have contributed to Λ_2 according the ordering (3.4), its contribution is in fact precluded by the remarkable intersection which occurs at $p = n + 4j - 4$ for each integer j of the $j+1$ spectral lines

$$\lambda_{0,j} = \lambda_{1,j-1} = \lambda_{2,j-2} = \cdots = \lambda_{j,0} = j \ ;$$

(part of the strange numerology of the spectrum). Of course, at this multiple intersection point, λ_{11} still contributes positively to the multiplicity $\#(\Lambda_2)$. Finally, convergence (1.15) to affine images of ρ is controlled by

$$\Lambda_3 = \begin{cases} \lambda_0^{\text{cont}} & \text{if } p \in]0, 4 + 2\sqrt{n-1}] \\ \lambda_{11} = (3p+n-4)/(p+n) & \text{if } p \in [4 + 2\sqrt{n-1}, \infty] \end{cases} \tag{3.9}$$

in the latter range. Comparison with (3.7), (3.9) yields (1.7) and (1.12), (1.15); the presence of the continuous spectrum above λ_0^{cont} implies the limitation (1.16).

4. The Spectrum

This section derives the exact spectrum of the operator $\mathbf{H} := -m\rho^{m-1}\Delta + \boldsymbol{x}\cdot\nabla$ described above. The spectrum is found by solving the partial differential equation $(\mathbf{H} - \lambda\mathbf{I})\Psi = \Phi$ and then checking whether the resolvent operator $(\lambda\mathbf{I} - \mathbf{H})^{-1}$ defines a bounded transformation on $W_\rho^{1,2}(\mathbf{R}^n)$. This is accomplished by separation of variables into angular and radial parts. It is good fortune that the radial part of the problem reduces to a hypergeometric equation whose solutions are well known in special function theory, permitting a complete determination of $\sigma(\mathbf{H})$. The precise form (1.4) of the Barenblatt profile ρ accounts for this happy outcome: singularities of the radial equation can occur only at $r^2 = 0$ (the coordinate singularity), $r^2 = -C$ (singularity of ρ), and $r^2 = \infty$. The issue is whether all of these singularities are regular singularities in the sense of the Fuchsian theory (see, e.g., POOLE [36]). Any linear ordinary differential equation with only three *regular* singularities can be transformed into a hypergeometric differential equation.

The spectrum is real by the self-adjointness of $\mathbf{H}$ established in the next lemma:

Lemma 4 (Essentially self-adjoint). *The operator* $\mathbf{H}\Psi := -m\rho^{m-2}\operatorname{div}[\rho\nabla\Psi]$ *is essentially self-adjoint on* $C_c^\infty(\mathbf{R}^n) \subset W_\rho^{1,2}(\mathbf{R}^n)$*; i.e., its closure is self-adjoint, and forms the only self-adjoint extension of* $\mathbf{H}$. *The domain of self-adjointness is precisely*

$$D(\mathbf{H}) = \left\{ \Psi \in W^{3,2}_{\mathrm{loc}} \cap W^{1,2}_\rho(\mathbf{R}^n) \;\middle|\; \rho^{m-2}\operatorname{div}[\rho\nabla\Psi] \in W^{1,2}_\rho(\mathbf{R}^n) \right\}. \tag{4.1}$$

It should be stressed that self-adjointness refers to the scalar product in $W^{1,2}_\rho(\mathbf{R}^n)$, *not in* L^2.

Proof. From the theory of unbounded operators, recall: (i) any symmetric operator densely defined on a Hilbert space $\mathcal{H}$ is closable – meaning the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ is again the graph of a symmetric linear operator; (ii) any self-adjoint operator has a closed graph [39, Sections 13.9, 13.20]. Corollaries 3 and 14 show that $\mathbf{H}$ restricted to $C_c^\infty(\mathbf{R}^n)$ is symmetric and densely defined on the Hilbert space $W^{1,2}_\rho(\mathbf{R}^n)$, so let us denote the closure of this operator by $\bar{\mathbf{H}}$. Letting X be the space in (4.1), a routine approximation argument yields that the domain of $\bar{\mathbf{H}}$ contains X. We show the restriction $\tilde{\mathbf{H}} := \bar{\mathbf{H}}|_X$ is a self-adjoint operator by proving $\tilde{\mathbf{H}}^* \subset \tilde{\mathbf{H}}$. The obvious chain of reverse inclusions $\tilde{\mathbf{H}} \subset \bar{\mathbf{H}} \subset \bar{\mathbf{H}}^* \subset \tilde{\mathbf{H}}^*$ then completes the proof that $\bar{\mathbf{H}} = \tilde{\mathbf{H}}$ is self-adjoint; also, any other self-adjoint extension of $\mathbf{H}$ would be sandwiched between $\bar{\mathbf{H}}$ and $\bar{\mathbf{H}}^*$ in this chain, and hence coincide with $\bar{\mathbf{H}}$.

The desired inclusion requires us to show that if $\Phi \in W^{1,2}_\rho(\mathbf{R}^n)$ is in the domain of $\tilde{\mathbf{H}}^*$ – i.e., if there exists a $\Xi \in W^{1,2}_\rho(\mathbf{R}^n)$ such that $\langle \Phi; \tilde{\mathbf{H}}\Psi\rangle_{W^{1,2}_\rho(\mathbf{R}^n)} = \langle \Xi; \Psi\rangle_{W^{1,2}_\rho(\mathbf{R}^n)}$ for all $\Psi \in X$ – then $\Phi \in X$. Restricting to test functions $\Psi \in C_c^\infty(\mathbf{R}^n)$, the equation $\langle \Phi; \tilde{\mathbf{H}}\Psi\rangle_{W^{1,2}_\rho(\mathbf{R}^n)} = \langle \Xi; \Psi\rangle_{W^{1,2}_\rho(\mathbf{R}^n)}$ means

$$\Delta_\rho(-m\rho^{m-2}\Delta_\rho\Phi - \Xi) = 0 \tag{4.2}$$

in the sense of distributions, where $\Delta_\rho \Xi := \operatorname{div}(\rho\nabla\Xi)$. Fix a smooth, bounded domain $\Omega \subset\subset \mathbf{R}^n$, and consider the restriction of the operator Δ_ρ to the Sobolev space $\mathring{W}^{1,2}(\Omega) := \overline{C_c^\infty(\Omega)}$ of weakly differentiable functions with zero boundary trace. Soft functional analysis asserts that

$$\Delta_\rho : \mathring{W}^{1,2}(\Omega) \to \mathring{W}^{1,2}(\Omega)^* =: W^{-1,2}(\Omega)$$

is an isomorphism, and moreover, all distributions χ which are ρ-harmonic (i.e., $\Delta_\rho\chi = 0$) are actually smooth functions; see RUDIN [39, Section 8.12], with his L our $\rho^{-1}\Delta_\rho = \Delta + \frac{\nabla\rho}{\rho}\cdot\nabla$. These two facts permit us to conclude from (4.2) that $m\rho^{m-2}\operatorname{div}(\rho\nabla\Phi) \in W^{1,2}_{\mathrm{loc}}$, because $\Xi \in W^{1,2}_{\mathrm{loc}}$, and then from elliptic regularity that $\Phi \in W^{3,2}_{\mathrm{loc}}$. We now claim

Lemma 5. *If* $\Delta_\rho \chi = 0$ *with* $\Delta_\rho = \rho\Delta + (\nabla\rho)\cdot\nabla$ *and* $\chi = \Xi + m\rho^{m-2}\Delta_\rho\Phi \in C^\infty$ *and* $\Phi, \Xi \in W^{1,2}_\rho(\mathbf{R}^n)$*, then* χ *is constant.*

This lemma immediately implies that

$$m\rho^{m-2}\Delta_\rho\Phi = \chi - \Xi \in W^{1,2}_\rho(\mathbf{R}^n),$$

hence $\Phi \in X$, which concludes Lemma 4, apart from the proof of Lemma 5, which is postponed to Section 4.7 below. $\square$

A description of the null space of $\rho^{-1}\Delta_\rho = \Delta + \frac{\nabla\rho}{\rho}\cdot\nabla$ in weighted Sobolev spaces, based merely on qualitative properties like smoothness and asymptotic behavior of $\frac{\nabla\rho}{\rho}$ near infinity is rather subtle, if not impossible: see the discussion in NIRENBERG & WALKER [30]. Our example just barely fails the assumptions of their Theorem 4.1, which would ascertain a finite-dimensional null space, if it were applicable. Their Section 5 indicates the sharpness of their assumptions, even though their counterexample, based on an example by PLIŚ [35], does not explicitly rule out stronger results for second order. Our argument to prove Lemma 5 says essentially that 0 is not an eigenvalue, and is based on explicit solutions of the eigenvalue problem in terms of the hypergeometric functions which follow. The proof is therefore deferred to Section 4.7. Essentially, the radial symmetry of the potential permits us to treat the problem as one-dimensional.

We will henceforth use the unaccented $\mathbf{H}$ for the operator on its domain of self-adjointness (4.1).

4.1. Separation of variables in spherical coordinates

Transforming $\boldsymbol{x} \in \mathbf{R}^n$ into spherical coordinates $(r, \boldsymbol{\omega}) \in [0, \infty[\times \mathbf{S}^{n-1}$ given by $(r, \boldsymbol{\omega}) = (|\boldsymbol{x}|, \boldsymbol{x}/|\boldsymbol{x}|)$, we recall that the Barenblatt profile $\rho(\boldsymbol{x}) = \rho(r)$ is a function of the radius only. The Laplacian is given by the familiar expression

$$\Delta_{\mathbf{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\Delta_{\mathbf{S}^{n-1}}}{r^2}, \tag{4.3}$$

where $\Delta_{\mathbf{S}^{n-1}}$ is the *Laplace-Beltrami* or *angular momentum operator* on the unit sphere. From the formula $\mathbf{H} = -m\rho^{m-1}\Delta + \boldsymbol{x}\cdot\nabla$ it is now clear that the operators $\mathbf{H}$ and $\Delta_{\mathbf{S}^{n-1}}$ commute, hence can be simultaneously diagonalized.

Let us therefore recall the spectrum of the Laplace-Beltrami operator $\Delta_{\mathbf{S}^{n-1}}$ (see BERGER, GAUDUCHON & MAZET [7, pp. 159–163]). Its eigenvalues are $\ell(\ell + n - 2) =: \boldsymbol{L}^2$, and their respective multiplicities M_ℓ are

$$M_\ell = \frac{(n+\ell-3)!\,(n+2\ell-2)}{\ell!\,(n-2)!}. \tag{4.4}$$

(This is understood as $M_\ell = 1$, if $\ell = 0$, or if $\ell = n = 1$). Our choice of notation $\boldsymbol{L}^2$ is motivated by the fact that these eigenvalues are the quantum analog for the magnitude squared of the angular momentum vector. So we have

$$-\Delta_{\mathbf{S}^{n-1}} Y_{\ell\mu} = \boldsymbol{L}^2\, Y_{\ell\mu}, \qquad \begin{array}{l} \ell = 0, 1, 2, \ldots \\ \mu = 1, 2, \ldots, M_\ell. \end{array} \tag{4.5}$$

The eigenfunctions $Y_{\ell\mu}$ are the spherical harmonics $Y_{\ell\mu} : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, which form a complete orthonormal basis (4.8) for $L^2(\mathbf{S}^{n-1}, d\boldsymbol{\omega})$; they are restrictions of the homogeneous harmonic polynomials of degree ℓ to the unit sphere. (Our enumeration of spherical harmonics by μ is different from the one used for $n = 3$ in quantum mechanics, but as explicit formulas for the $Y_{\ell\mu}$ do not play a role here, no confusion should arise.)

The next proposition gives the decomposition of our Hilbert space $W^{1,2}_\rho(\mathbf{R}^n)$ into angular momentum eigenspaces, and a formula for the restriction $\mathbf{H}_\ell$ of $\mathbf{H}$:

Proposition 6 (Restriction to angular momentum eigenspaces)**.** *Defining* $W^{1,2}_\ell$ *by* (4.12), (4.14) *yields a Hilbert space isomorphism* $W^{1,2}_\rho(\mathbf{R}^n) = \bigoplus_{\ell=0}^{\infty} \bigoplus_{\mu=1}^{M_\ell} W^{1,2}_\ell$ *given by*

$$\Psi(r\boldsymbol{\omega}) = \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{M_\ell} f_{\ell\mu}(r) Y_{\ell\mu}(\boldsymbol{\omega}) \tag{4.6}$$

for $f_{\ell\mu} \in W^{1,2}_\ell$*. Furthermore,* (4.15) *and* (4.16) *define a non-negative self-adjoint operator* $\mathbf{H}_\ell$ *such that* $\mathbf{H}\Psi = \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{M_\ell} (\mathbf{H}_\ell f_{\ell\mu}) Y_{\ell\mu}$.

Proof. First fix $\Psi \in W^{1,2}_\rho(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n \setminus \{\mathbf{0}\})$, and define the corresponding Fourier components $f_{\ell\mu} \in C^\infty(]0, \infty[)$ by

$$f_{\ell\mu}(r) := \int_{\mathbf{S}^{n-1}} \Psi(r\boldsymbol{\omega}) Y_{\ell\mu}(\boldsymbol{\omega})\, d\boldsymbol{\omega}\,, \quad f'_{\ell\mu}(r) = \int_{\mathbf{S}^{n-1}} (\boldsymbol{\omega} \cdot \nabla\Psi)(r\boldsymbol{\omega}) Y_{\ell\mu}(\boldsymbol{\omega})\, d\boldsymbol{\omega}\,. \tag{4.7}$$

The spectral decomposition for $\Delta_{\mathbf{S}^{n-1}}$ on $L^2(\mathbf{S}^{n-1}, d\boldsymbol{\omega})$ using an orthonormal basis of spherical harmonics

$$\int_{\mathbf{S}^{n-1}} Y_{\ell\mu}(\boldsymbol{\omega}) Y_{\tilde{\ell}\tilde{\mu}}(\boldsymbol{\omega})\, d\boldsymbol{\omega} = \delta_{\ell\tilde{\ell}} \delta_{\mu\tilde{\mu}}, \tag{4.8}$$

yields (4.6), and similarly

$$\frac{\partial \Psi}{\partial r}(r\boldsymbol{\omega}) = \sum_{\ell} \sum_{\mu} f'_{\ell\mu}(r) Y_{\ell\mu}(\boldsymbol{\omega})\,, \tag{4.9}$$

$$\nabla_{\mathbf{S}^{n-1}} \Psi(r\boldsymbol{\omega}) = \sum_{\ell} \sum_{\mu} f_{\ell\mu}(r) \nabla_{\mathbf{S}^{n-1}} Y_{\ell\mu}(\boldsymbol{\omega})\,. \tag{4.10}$$

In each case the convergence takes place in $L^2(\mathbf{S}^{n-1}, d\boldsymbol{\omega})$, and convergence of (4.10) was deduced from (4.6) using finiteness of

$$\int_{\mathbf{S}^{n-1}} |\nabla_{\mathbf{S}^{n-1}} \Psi(r\boldsymbol{\omega})|^2\, d\boldsymbol{\omega} = -\int_{\mathbf{S}^{n-1}} \Psi \Delta_{\mathbf{S}^{n-1}} \Psi\, d\boldsymbol{\omega} = \sum_{\ell} \sum_{\mu} \boldsymbol{L}^2\, |f_{\ell\mu}(r)|^2 \tag{4.11}$$

and the following useful variant of the orthonormality relations:

$$\int_{\mathbf{S}^{n-1}} \nabla_{\mathbf{S}^{n-1}} Y_{\ell\mu}(\boldsymbol{\omega}) \cdot \nabla_{\mathbf{S}^{n-1}} Y_{\tilde{\ell}\tilde{\mu}}(\boldsymbol{\omega})\, d\boldsymbol{\omega} = \boldsymbol{L}^2\, \delta_{\ell\tilde{\ell}} \delta_{\mu\tilde{\mu}}\, .$$

Combining $\nabla_{\mathbf{R}^n} = \partial/\partial r + \nabla_{\mathbf{S}^{n-1}}/r$ with (4.9) and (4.11) yields

$$\int_{\mathbf{S}^{n-1}} |\nabla_{\mathbf{R}^n} \Psi(r\boldsymbol{\omega})|^2\, d\boldsymbol{\omega} = \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{M_\ell} \left(f'_{\ell\mu}(r)^2 + \boldsymbol{L}^2\, \frac{f_{\ell\mu}(r)^2}{r^2} \right).$$

Integrating against ρr^{n-1} defines

$$\begin{aligned} \|f\|^2_{W^{1,2}_\ell} &:= \int_0^\infty \Big(f'(r)^2 + \frac{\boldsymbol{L}^2}{r^2} f(r)^2 \Big) \rho(r) r^{n-1}\, dr \\ &= \|f Y_{\ell\mu}\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} \end{aligned} \tag{4.12}$$

and gives the desired isometry

$$\|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} = \sum_\ell \sum_\mu \|f_{\ell\mu}\|^2_{W^{1,2}_\ell}\, ; \tag{4.13}$$

the exchange of limits implicit in the last formula is legitimized by finiteness of the left-hand side. We are therefore naturally led to the Sobolev spaces

$$W^{1,2}_\ell := \Big\{ f :]0,\infty[\longrightarrow \mathbf{R} \;\Big|\; \|f\|_{W^{1,2}_\ell} < \infty \Big\} \Big/ \{\|\cdot\| = 0\}\, , \tag{4.14}$$

where $/\{\|\cdot\| = 0\}$ applies to $\ell = 0$ (i.e., $\boldsymbol{L}^2 = 0$) only, dividing out constants.

Now $C^\infty(\mathbf{R}^n) \cap W^{1,2}_\rho(\mathbf{R}^n)$ is dense in $W^{1,2}_\rho(\mathbf{R}^n)$, by a standard mollification argument exploiting the tail behavior of ρ, given in the proof of Corollary 14. Thus we have shown that $W^{1,2}_\rho(\mathbf{R}^n)$ embeds isometrically into $\bigoplus_{\ell=1}^{\infty} \bigoplus_{\mu=1}^{M_\ell} W^{1,2}_\ell$. To show this isometry is onto, use a new sequence $\{f_{\ell\mu}\} \in \bigoplus_{\ell=1}^{\infty} \bigoplus_{\mu=1}^{M_\ell} W^{1,2}_\ell$ with only finitely many non-zero entries $f_{\ell\mu} \in C^\infty(]0,\infty[)$ to *define* $\Psi \in C^\infty(\mathbf{R}^n \setminus \{\mathbf{0}\})$ via the *finite sum* (4.6); then (4.9)–(4.13) follow immediately and imply $\Psi \in W^{1,2}_\rho(\mathbf{R}^n)$ and hence (4.7). Such sequences form a dense subset of the latter space, thus completing the proof.

A straightforward calculation for $\mathbf{H} := -m\rho^{m-1}\Delta + \boldsymbol{x} \cdot \nabla$, using (4.3)–(4.5), shows:

$$\text{If } \Psi(r\boldsymbol{\omega}) := f(r)\, Y_{\ell\mu}(\boldsymbol{\omega})\ , \ \text{then } (\mathbf{H}\Psi)(r\boldsymbol{\omega}) = (\mathbf{H}_\ell\, f)(r)\, Y_{\ell\mu}(\boldsymbol{\omega})$$

with

$$(\mathbf{H}_\ell\, f)(r) := -m\rho^{m-1} \left(f''(r) + \frac{n-1}{r} f'(r) - \frac{\boldsymbol{L}^2}{r^2} f(r) \right) + r f'(r). \tag{4.15}$$

The operator $\mathbf{H}_\ell$ is defined on the projection of the domain of $\mathbf{H}$ (4.1) onto the eigenspace with eigenvalue $\boldsymbol{L}^2$. More precisely, $f \in D(\mathbf{H}_\ell) \iff f\,Y_{\ell\mu} \in D(\mathbf{H})$, independent of the choice of μ; namely

$$D(\mathbf{H}_\ell) = \left\{ f \in W^{3,2}_{\mathrm{loc}}([0,\infty[, r^{n-1}\,dr) \cap W^{1,2}_\ell \,\middle|\, \mathbf{H}_\ell f \in W^{1,2}_\ell \right\}. \tag{4.16}$$

□

Remark 7. A similar but simpler proof shows (4.6) also yields the Hilbert space isomorphism

$$L^2\left(\mathbf{R}^n, \frac{\rho(\boldsymbol{x})\,d\boldsymbol{x}}{|\boldsymbol{x}|^2 + \delta_{n,2}}\right) = \bigoplus_{\ell=0}^{\infty} \bigoplus_{\mu=1}^{M_\ell} L^2\left(]0,\infty[, \frac{\rho(r) r^{n-1} dr}{r^2 + \delta_{n,2}}\right). \quad n \geqq 2. \tag{4.17}$$

The proposition makes clear that finding the spectrum of $\mathbf{H}$ is equivalent to finding the spectrum of $\mathbf{H}_\ell$ for each $\ell = 0, 1, 2, \ldots$, which requires analyzing the radial ordinary differential equation. This will be accomplished in several steps.

4.2. Radial eigenvalue problem solved in hypergeometric functions

To find the spectrum of the self-adjoint operator $\mathbf{H}_\ell$ on its domain $D(\mathbf{H}_\ell) \subset W^{1,2}_\ell(]0,\infty[)$, we need to know for which $\lambda \in \mathbf{R}$ the resolvent $(\lambda\mathbf{I} - \mathbf{H}_\ell)^{-1}$ is bounded. This means understanding the solutions of the ordinary differential equation $(\mathbf{H}_\ell - \lambda) f = g \in W^{1,2}_\ell$. In this section we solve the eigenvalue problem $\mathbf{H}_\ell \psi = \lambda\psi$, whose eigenfunctions turn out to be hypergeometric functions in $W^{1,2}_\ell$. This study gives also a heuristic basis for our understanding of the remaining spectrum.

Explicitly, the equation $\mathbf{H}_\ell \psi = \lambda\psi$ takes the form

$$\psi''(r) + \left(\frac{n-1}{r} - \frac{(p+n)r}{r^2 + C}\right)\psi'(r) + \left(\frac{\lambda(p+n)}{r^2 + C} - \frac{\boldsymbol{L}^2}{r^2}\right)\psi(r) = 0, \tag{4.18}$$

according to (1.4) and (4.15). The key to our analysis is the following proposition. It relies on results and notation from the theory of special functions summarized in Appendix 5. To sketch the subsequent logic briefly (neglecting some technicalities for $n = 2$ that will be taken care of below), there are two linearly independent solutions to (4.18) among the hypergeometric functions. Only one of these is analytic at the origin: it has the form (3.5) with $k = k(\lambda)$ the lesser root of

$$\boldsymbol{L}^2 + \left(\frac{p}{2} + 1\right)^2 - \left(\frac{p}{2} + 1 - \ell - 2k\right)^2 = (p+n)\lambda;$$

the other solution has a singularity at $r = 0$ which prevents it from belonging to $W^{1,2}_\ell$. Whether or not the remaining solution lies in $W^{1,2}_\ell$ depends on its growth at infinity; it grows too quickly unless k is a non-negative integer – a necessary and sufficient condition for the hypergeometric series (5.3) to terminate, forming a polynomial $\psi = \psi_{\ell k}$ of degree $2k + \ell$. Since the Barenblatt profile has up to p moments, this polynomial belongs to $W^{1,2}_\ell$ if and only if $2k + \ell < p/2 + 1$. More precisely:

Proposition 8 (Hypergeometric radial solutions)**.** *Let ${}_2F_1$ be the Gauss hypergeometric function defined by* (5.3), *and*

$$T := \boldsymbol{L}^2 + (p/2+1)^2 - \lambda(p+n)\ .$$

If n is odd, the linear second-order equation (4.18) *admits two linearly independent solutions, analytic on $r \in \,]0,\infty[$:*

$$\psi_1(r) := \quad r^{\ell} \quad {}_2F_1\left({\frac{1}{2}(\ell - \frac{p}{2} - 1 \pm \sqrt{T}) \atop \frac{n}{2} + \ell} ; \frac{-r^2}{C} \right), \tag{4.19}$$

$$\psi_2(r) := r^{2-n-\ell}\ {}_2F_1\left({\frac{1}{2}(1 - n - \ell - \frac{p}{2} \pm \sqrt{T}) \atop 2 - \frac{n}{2} - \ell} ; \frac{-r^2}{C} \right). \tag{4.20}$$

If n is even, ψ_1 continues to be a solution, but ψ_2 is ill defined, except for $n = 2$, $\ell = 0$, in which case $\psi_2 = \psi_1$. In either of these two cases, a linearly independent solution $\hat{\psi}_2$ exists whose asymptotic behavior is $\hat{\psi}_2(r) \sim \psi_1(r)\log r + \gamma r^{2-n-\ell}(1 + O(r^2))$ as $r \to 0+$, with a nonvanishing constant γ. The $O(r^2)$ correction is analytic near 0.

Unless T is a perfect square, an alternate solution basis is given by

$$\tilde{\psi}_1(r) := r^{1+p/2+\sqrt{T}}\ {}_2F_1\left({-[\frac{p}{4} + \frac{n}{4} + \frac{\sqrt{T}}{2} \pm (\frac{\ell}{2} + \frac{n}{4} - \frac{1}{2})] \atop 1 - \sqrt{T}} ; -\frac{C}{r^2} \right), \tag{4.21}$$

$$\tilde{\psi}_2(r) := r^{1+p/2-\sqrt{T}}\ {}_2F_1\left({-[\frac{p}{4} + \frac{n}{4} - \frac{\sqrt{T}}{2} \pm (\frac{\ell}{2} + \frac{n}{4} - \frac{1}{2})] \atop 1 + \sqrt{T}} ; -\frac{C}{r^2} \right). \tag{4.22}$$

If $T = 0$, then $\tilde{\psi}_1 = \tilde{\psi}_2$; if $T > 0$ is a perfect square, then $\tilde{\psi}_1$ is ill-defined. In both cases, a solution $\check{\psi}_1$, linearly independent of $\tilde{\psi}_2$, replaces $\tilde{\psi}_1$, and satisfies $\check{\psi}_1(r) \sim \tilde{\psi}_2(r)\log r + \gamma r^{1+p/2+\sqrt{T}}(1 + O(r^{-2}))$ as $r \to +\infty$, with a nonvanishing constant γ. The $O(r^{-2})$ correction is analytic at $r = \infty$.

Unless T is a perfect square, ψ_1 can be expressed in the form

$$\psi_1(r) = \frac{c_+ \tilde{\psi}_1(r)}{\Gamma\left(\frac{1}{2}(\ell - \frac{p}{2} - 1 + \sqrt{T})\right)} + \frac{c_- \tilde{\psi}_2(r)}{\Gamma\left(\frac{1}{2}(\ell - \frac{p}{2} - 1 - \sqrt{T})\right)}, \tag{4.23}$$

$$c_\pm = C^{(\ell - \frac{p}{2} - 1 \mp \sqrt{T})/2} \frac{\Gamma\left(\frac{n}{2} + \ell\right)\Gamma(\pm\sqrt{T})}{\Gamma\left(\frac{1}{2}(n + \ell + \frac{p}{2} + 1 \pm \sqrt{T})\right)} \in \mathbf{R}.$$

Proof. The differential equation (4.18) is of Fuchsian type: its coefficients are rational functions and the only singularities are *regular*, and they occur at the four points $0, \pm i\sqrt{C}, \infty$. By introducing a new variable $\mathrm{R} = -r^2/C$, the singular points are brought to the standard positions $0, 1, \infty$. Any second-order equation with only three regular singularities can be reduced to the Gaussian hypergeometric type. The relevant theory can be found, e.g., in POOLE [36]. A power-series ansatz $\psi(r) = r^\alpha \sum_{i=0}^{\infty} a_i r^{2i}$, $a_0 \neq 0$, in even powers of r leads to the characteristic equation for α, namely: $\alpha(\alpha + n - 2) = \ell(\ell + n - 2)$. So either $\alpha_1 = 2 - n - \ell$ or

$\alpha_2 = \ell$. In contrast, for the standard hypergeometric equation, one of the characteristic exponents at 0 vanishes. This is why (e.g.) the substitution $\psi(r) = (-\mathrm{R})^{\frac{\ell}{2}} f(\mathrm{R})$ leads to the hypergeometric equation (5.4),

$$\mathrm{R}(1-\mathrm{R})f''(\mathrm{R}) + (c-(a+b+1)\mathrm{R})f'(\mathrm{R}) - abf(\mathrm{R}) = 0 \tag{4.24}$$

with

$$a, b = \frac{\ell - p/2 - 1 \pm \sqrt{\ell(\ell+n-2) + (\frac{p}{2}+1)^2 - \lambda(p+n)}}{2}, \tag{4.25}$$

$$c = \frac{n}{2} + \ell\,. \tag{4.26}$$

If n is odd, then c is not a positive integer, and, comparing (4.24) with (5.4), (5.5), we verify the claimed solutions (4.19), (4.20). Notice that if we had chosen the other characteristic exponent for our transformation, the definition of T would be unchanged and the two solutions would merely have been interchanged.

For even n, the basis of solutions can be found in [1, (15.5.16–17) or (15.5.18–19)]. For $\hat{\psi}_2$, we have merely quoted the behavior near 0 from that source, because later, both ψ_2 and $\hat{\psi}_2$ will be discarded for the spectral problem.

The set of alternate solutions (4.21), (4.22) follows from [1, (15.5.7–8)], or from applying the connection formula (5.7) to ψ_1 and ψ_2. For ψ_1 and our choice of a, b, c, this specializes to (4.23). The solution $\check{\psi}_1$ that can replace $\tilde{\psi}_1$ when T is a perfect square can be obtained like $\hat{\psi}_2$ was obtained, using the self-transformation of the hypergeometric equation mentioned in the appendix. This proves the proposition. □

We shall see in a moment that the "nuisance cases" requiring $\hat{\psi}_2$ instead of ψ_2, or $\check{\psi}_1$ instead of $\tilde{\psi}_1$, do not affect the spectral properties: For a solution of (4.18) to have the integrability properties (4.12) required for $W_\ell^{1,2}$ locally near 0, it must be a multiple of ψ_1; and for a solution to have the corresponding integrability properties near ∞, it must be a multiple of $\tilde{\psi}_2$, and with this information, the spectral condition can be deduced from (4.23).

4.3. The point spectrum

We now begin to confirm assertions made in Section 3 above, by determining the point spectrum and eigenfunctions of $\mathbf{H}_\ell$. The calculation will give enough insight to guess the continuous spectrum as well – a guess which will be verified subsequently.

Corollary 9 (Radial eigenfunctions and eigenvalues)**.** *The eigenvalue problem* $\mathbf{H}_\ell\,\psi = \lambda\psi$ *has a solution in* $D(\mathbf{H}_\ell) \subset W_\ell^{1,2}$ *if and only if* $\lambda = \lambda_{\ell k}$ *is given by* (3.3)*, with* $\ell \geqq 0$, $(\ell, k) \neq (0,0)$*. The corresponding eigenfunction* $\psi = \psi_{\ell k}$ *is unique and given by* (3.5)*; the hypergeometric series reduces to a Jacobi polynomial in the eigenvalue case.*

Proof. Suppose $\psi \in D(\mathbf{H}_\ell)$ is an eigenfunction, $\mathbf{H}_\ell \psi = \lambda\psi$ in $W^{1,2}_\ell$; then by the standard bootstrapping, the eigenvalue equation holds classically, and in this case we may express $\psi(r) = c_1\psi_1(r) + c_2\psi_2(r)$ (or $\hat{\psi}_2$) as a linear combination of the two solutions from Proposition 8. For $n > 1$ odd, $\psi_2 \notin W^{1,2}_\ell$, because near zero, $|\psi_2'|^2\rho r^{n-1} \sim \text{const}\, r^{1-n-2\ell}$ is not integrable. For n even, the same argument applies to $\hat{\psi}_2$ instead of ψ_2. In either case, any solution to (4.18) lying in $W^{1,2}_\ell$ can only be a multiple of ψ_1; thus $c_2 = 0$ and the subsequent reasoning does not need to distinguish the parity of n any more.

We now assume, for the time being, that T is not a perfect square. The connection formula (4.23) gives the behavior of ψ_1 near ∞. Its growth determines whether ψ_1 is actually an eigenfunction. For $T > 0$, the dominant contribution comes from the first term in (4.23), i.e., from $\tilde{\psi}_1$, unless the coefficient of this term happens to vanish. But $\tilde{\psi}_1 \notin W^{1,2}_\ell$ since $|\tilde{\psi}_1'|^2 \sim \text{const} r^{p+2\sqrt{T}}$ and ρr^{n-1} fails to have p-th moments. Essentially the same reasoning applies even if $T > 0$ is a perfect square. In this case, the right-hand side of (4.23) must be replaced by its limit, as T tends to the desired value. This limit exists, because the left-hand side is analytic in the parameters. Single terms in the power series defining ${}_2F_1$ in $\tilde{\psi}_1$ diverge as the Pochhammer symbol $(1-\sqrt{T})_k$ in the denominator starts including a factor 0, and also the coefficient c_- of $\tilde{\psi}_2$ diverges. These divergences cancel termwise, by combining like powers. The leading terms of $c_+\tilde{\psi}_1(r)/\Gamma(\frac{1}{2}(\ell - \frac{p}{2} - 1 + \sqrt{T}))$ (their number being $\sqrt{T}$) do not contribute divergences and are not paired with like terms from $\tilde{\psi}_2$. The first of them determines the asymptotic behavior as $r \to \infty$. Therefore, whether T is a perfect square or not, the vanishing of $1/\Gamma(\frac{1}{2}(\ell - \frac{p}{2} - 1 + \sqrt{T}))$ due to a pole of the Γ function, is a necessary condition for an eigenvalue contributed by $T > 0$. Note that c_+ does not vanish, since $\Gamma(z)$ has no zeros and its poles occur precisely at the non-positive integers; cf. (5.1).

Thus we need $a = \frac{1}{2}(\ell - \frac{p}{2} - 1 + \sqrt{T}) = -k \in \mathbf{Z}$ for some integer $k \geqq 0$ to have a $T > 0$ eigenvalue. But this number is exactly the a from (4.25). These cases lead indeed to eigenvalues: The series (5.3) for ${}_2F_1$ in (4.19) terminates at the k-th term, so $\psi_1(r)$ is a polynomial of degree $\ell + 2k$. As $r \to \infty$, we have $\text{const}|\psi_1'|^2 \sim r^{p-2\sqrt{T}} \sim \text{const}|\psi_1|^2/r^2$; again ρr^{n-1} has up to p moments so both integrals converge in (4.12). (Near zero, ψ_1 is analytic and $\rho r^{n-1}|\psi_1|^2/r^2 \sim \text{const} r^{2\ell+n-3}$ is also integrable if $\ell \geqq 1$, as required). Thus ψ_1 lies in the Hilbert space $W^{1,2}_\ell$, and indeed in $D(\mathbf{H}_\ell)$. We see $\lambda = \lambda_{\ell k}$ by comparing (3.3) with the explicit form of T given in the proposition. Comparing (3.5) with (4.19) we also read off $\psi/c_1 = \psi_1 = \psi_{\ell k}$. Apart from its normalization and domain, ψ coincides with the Jacobi polynomial $r^\ell P_k^{(2\ell+n-2,-2-p-n)/2}(1 + 2r^2/C)$ of [1, (15.4.6)]. Note that λ_{00} is not an eigenvalue, because the corresponding ψ_{00} is a constant function, i.e., vanishes in the quotient space $W^{1,2}_{\ell=0}$ according to (4.14).

Finally for $T < 0$, both terms in (4.23) contribute equally to the growth of ψ_1, with an oscillatory coefficient whose amplitude grows like $r^{1+p/2}$, thus we get logarithmic divergence of the integral $\int^\infty |\psi_1'|^2\rho r^{n-1}\,dr$. So no eigenfunctions arise for $T < 0$, although we will see later that this case contributes the continuous spectrum. The same reasoning applies to the case $T = 0$, again by a limit argument, and the corollary is complete. □

We recapitulate the last part of the proof for future reference: when $T \leqq 0$, the asymptotic behavior (4.21)–(4.23) of ψ_1 makes the integrals of $\rho\psi_1'^2 r^{n-1}$ and $\rho\psi_1^2 r^{n-3}$ defining $\|\psi_1\|_{W_\ell^{1,2}}$ just barely divergent (logarithmically) at $+\infty$. The corresponding values of λ should therefore belong to the continuous spectrum. Indeed, condition $T \leqq 0$ is equivalent to $\lambda \geqq \lambda_\ell^{\rm cont}$, with $\lambda_\ell^{\rm cont}$ given by (3.2).

4.4. Continuous spectrum

Next we have to prove that, in the case $T \leqq 0$, we have indeed a continuous spectrum, i.e., that there exists a sequence of approximate eigenfunctions $f_k \in W_\ell^{1,2}$ such that $\|(\mathbf{H}_\ell - \lambda) f_k\|_1^2 / \|f_k\|_1^2 \to 0$, but no genuine eigenfunction.

Proposition 10 (Continuous spectrum)**.** *The continuous spectrum of the operator* $\mathbf{H}_\ell$ *on* $D(\mathbf{H}_\ell) \subset W_\ell^{1,2}$ *from* (4.15) *includes the interval* $[\lambda_\ell^{\rm cont}, +\infty[$ *of* (3.2).

Proof. Assume $\lambda \geqq \lambda_\ell^{\rm cont}$. We claim that λ belongs to the spectrum of $\mathbf{H}_\ell$ on $D(\mathbf{H}_\ell) \subset W_\ell^{1,2}$ but is not an eigenvalue. The latter is clear from Corollary 9 and the ordering $\lambda_{\ell k} < \lambda_\ell^{\rm cont}$ of (3.2), (3.3). To see that the inverse of $\mathbf{H}_\ell - \lambda\mathbf{I}$ fails to be bounded, we construct a sequence of approximate eigenfunctions $f_k \in D(\mathbf{H}_\ell) \subset W_\ell^{1,2}$ such that $\|(\mathbf{H}_\ell - \lambda) f_k\|_{W_\ell^{1,2}} / \|f_k\|_{W_\ell^{1,2}} \to 0$. This construction begins with the solution ψ_1 to $\mathbf{H}_\ell\, \psi_1 = \lambda\psi_1$ which was analytic at the origin in Proposition 8. Since $\lambda \geqq \lambda_\ell^{\rm cont}$ implies we are in the case $T \leqq 0$ of that proposition, we know $\|\psi_1\|^2_{W_\ell^{1,2}}$ diverges (logarithmically as $r \to +\infty$). The approximate eigenfunctions are fashioned from ψ_1 by means of a cutoff function η:

$$\eta \in C_0^\infty[0, 2[\,, \quad \eta \equiv 1 \text{ on } [0, 1]\,, \quad \eta_k(r) := \eta(r/k)\,, \quad f_k := \psi_1\eta_k. \tag{4.27}$$

Clearly $f_k \in C_c^\infty([0, \infty[) \subset D(\mathbf{H}_\ell)$. Moreover, the i-th derivative $\eta^{(i)} = d^i\eta/dr^i$ satisfies the usual decay estimates:

$$\begin{aligned} \left\|\eta_k^{(i)}\right\|_{L^\infty} &= O(k^{-i}) \ \text{ as } k \to +\infty, \\ \sup_k \eta_k^{(i)}(r) &= O(r^{-i}) \ \text{ as } r \to +\infty. \end{aligned}$$

We may neglect the case $T = 0$ now, because the spectrum is closed. Therefore, recalling $m\rho^{m-1} = (r^2 + C)/(p + n)$ from (1.4), and the asymptotics $\psi_1 = O(r^{1+p/2})$, $\psi_1' = O(r^{p/2})$ and $\psi_1'' = O(r^{p/2-1})$ from (4.21)–(4.23), we compute

$$\begin{aligned} \mathbf{H}_\ell\, f_k &= (\mathbf{H}_\ell\, \psi_1)\eta_k + \rho^{m-1} O\left(|\psi_1'\eta_k'| + |\psi_1\eta_k''| + |\psi_1\eta_k'/r|\right) + O\left(|\psi_1\eta_k'|\right) r \\ &= (\mathbf{H}_\ell\, \psi_1)\eta_k + O(r^{1+p/2}), \\ (\mathbf{H}_\ell\, f_k)' &= (\mathbf{H}_\ell\, \psi_1)'\eta_k + O\left(|\psi_1'\eta_k'| + |\psi_1\eta_k''| + |\psi_1\eta_k'/r|\right) r \\ &\quad + \rho^{m-1} O(|\psi_1''\eta_k'| + |\psi_1'\eta_k''| + |\psi_1\eta_k'''| \\ &\quad + |\psi_1\eta_k'/r^2| + |\psi_1'\eta_k'/r| + |\psi_1\eta_k''/r|) \\ &= (\mathbf{H}_\ell\, \psi_1)'\eta_k + O(r^{p/2}) \end{aligned}$$

as $r \to +\infty$. Using $(\mathbf{H}_\ell - \lambda)\psi_1 = 0$ pointwise in (4.12) yields

$$\begin{aligned}
&\|(\mathbf{H}_\ell - \lambda) f_k\|^2_{W^{1,2}_\ell} \\
&= \int_0^\infty \left\{ [(\mathbf{H}_\ell f_k - \lambda\psi_1\eta_k)']^2 + \frac{\boldsymbol{L}^2}{r^2}(\mathbf{H}_\ell f_k - \lambda\psi_1\eta_k)^2 \right\} \rho r^{n-1}\, dr \\
&= \int_k^{2k} \left\{ [O(r^{p/2}) - \lambda\psi_1\eta_k']^2 + \frac{O(r^{1+p/2})^2}{r^2} \right\} O\left(\frac{r^{n-1}}{r^{p+n}}\right) dr \\
&= \int_k^{2k} O\left(r^{-1}\right) dr = O(1)
\end{aligned}$$

as $k \to +\infty$. On the other hand, Lebesgue's monotone-convergence theorem gives

$$\langle f_k; f_k \rangle_{W^{1,2}_\ell} \geqq \int_0^k \rho \left(f_k'^2 + \frac{\boldsymbol{L}^2}{r^2} f_k^2 \right) r^{n-1}\, dr \to \|\psi_1\|^2_{W^{1,2}_\ell} = +\infty$$

from (4.27). This shows the inverse of $\mathbf{H}_\ell - \lambda\mathbf{I}$ on $W^{1,2}_\ell$ cannot be bounded, concluding the proof that $\lambda \in \sigma(\mathbf{H}_\ell)$. $\square$

We still need to show that there is no further continuous spectrum; this is accomplished in Section 4.6 by showing that for $T > 0$ not in the eigenvalue case, λ belongs to the resolvent set. Before beginning that task, we must devote a section to developing the auxiliary inequalities that will be required.

4.5. Weighted Poincaré inequalities and Sobolev spaces

The quotient (4.14) in the definition of $W^{1,2}_\ell$ applies to the case $\boldsymbol{L}^2 = 0$ only, where constant functions are to be removed. Even in this case, we will need to control $\inf_c \int |f(r) - c)|^2 \rho(r) r^{n-1}\, dr/(r^2 + 1_{n\leqq 2})$ in terms of $\|f\|^2_{W^{1,2}_\ell}$ to deduce the density of $C^\infty_c(\mathbf{R}^n)$ in $W^{1,2}_\rho(\mathbf{R}^n)$ and establish the resolvent set of $\mathbf{H}_\ell$. This control is obtained as a consequence of the following Hardy / Poincaré type inequality, with weight function $\sin^2 x$ vanishing quadratically at both ends of its domain $[0, \pi]$.

Lemma 11 (Hardy/Poincaré inequality with doubly degenerate weight)**.** *The embedding* $W^{1,2}_{\sin^2 x}[0, \pi] \subset L^2[0, \pi]/\{\text{const}\}$ *is continuous: i.e., any function* $g : [0, \pi] \longrightarrow \mathbf{R}$ *satisfies*

$$\inf_c \int_0^\pi |g(x) - c|^2\, dx \leqq \pi^2 \int_0^\pi \sin^2 x\, |g'(x)|^2\, dx. \tag{4.28}$$

Proof. We first show the estimate for trigonometric cosine polynomials $g(x) = \sum_{k=1}^{2N} a_k \cos kx$. It costs no generality to discard the constant term a_0 and choose $c = a_0 = 0$ in (4.28). Let the space of these polynomials be called $\mathbf{F}_{2N} = \mathbf{F}^{\mathrm{e}}_{2N} \oplus \mathbf{F}^{\mathrm{o}}_{2N}$, where the superscripts refer to even and odd Fourier indices k respectively. In the odd subspace, $g(\frac{\pi}{2} + x) = -g(\frac{\pi}{2} - x)$, so $g(\frac{\pi}{2}) = 0$ and

$$\int_0^{\pi/2} g^2\, dx = \left[x g^2 \right]_0^{\pi/2} - \int_0^{\pi/2} x\, 2gg'\, dx \leqq 2 \left(\left(\int_0^{\pi/2} g^2\, dx \right) \int_0^{\pi/2} x^2 g'^2\, dx \right)^{1/2}.$$

This one-liner yields a variant of the classical Hardy inequality:

$$\int_0^{\pi/2} g(x)^2\,dx \leqq 4\int_0^{\pi/2} x^2 g'(x)^2\,dx.$$

Now $2x \leqq \pi\sin(x)$ on the interval $]0,\pi/2[$, and symmetry around $\pi/2$ gives the desired result (4.28) for odd trigonometric polynomials $g \in \mathbf{F}^{\mathrm{o}}_{2N}$.

To complete the lemma, we now show that the inequality for odd polynomials implies the inequality for all polynomials $g \in \mathbf{F}^{\mathrm{e}}_{2N}$. Using relations like

$$\int_0^{\pi}\sin(jx)\sin(kx)\cos(2x)\,dx = \frac{\pi}{4}[\delta_{j,k+2}+\delta_{j,k-2}-\delta_{j,2-k}-\delta_{j,-2-k}], \qquad j,k\in\mathbf{Z},$$

with Kronecker's symbol $\delta_{j,k}\in\{0,1\}$ and $g'(x) = -\sum_{k=1}^{2N} ka_k \sin kx$, we evaluate

$$\begin{aligned}
&\frac{4}{\pi}\int_0^{\pi} g'(x)^2\sin^2 x\,dx\\
&\quad= \frac{2}{\pi}\int_0^{\pi} g'(x)^2(1-\cos(2x))\,dx\\
&\quad= \sum_{k=1}^{\infty}\frac{ka_k}{2}[2ka_k-(k+2)a_{k+2}-(k-2)a_{k-2}+(2-k)a_{2-k}].
\end{aligned}$$

Thus

$$\inf\left\{\int_0^{\pi} g'(x)^2\sin^2 x\,dx \;\middle|\; \int_0^{\pi} g(x)^2\,dx = 1\,,\ g\in\mathbf{F}^{\mathrm{e}}_{2N}\text{ or }\mathbf{F}^{\mathrm{o}}_{2N}\right\}$$

is the lowest eigenvalue of a symmetric tridiagonal matrix $\mathbb{A}^{\mathrm{e}}$ or $\mathbb{A}^{\mathrm{o}}$ respectively:

$$\mathbb{A}^{\mathrm{o}} = \frac{1}{2}\begin{bmatrix} \frac{3}{2} & -\frac{1\cdot 3}{2} & & & \\ -\frac{1\cdot 3}{2} & 3^2 & -\frac{3\cdot 5}{2} & & \\ & -\frac{3\cdot 5}{2} & 5^2 & -\frac{5\cdot 7}{2} & \\ & & -\frac{5\cdot 7}{2} & 7^2 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}, \quad \mathbb{A}^{\mathrm{e}} = \frac{1}{2}\begin{bmatrix} 2^2 & -\frac{2\cdot 4}{2} & & & \\ -\frac{2\cdot 4}{2} & 4^2 & -\frac{4\cdot 6}{2} & & \\ & -\frac{4\cdot 6}{2} & 6^2 & -\frac{6\cdot 8}{2} & \\ & & -\frac{6\cdot 8}{2} & 8^2 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

(no misprint: the $(k,k) = (1,1)$ term in $\mathbb{A}^{\mathrm{o}}$ is special). Now $\mathbb{B} := \mathbb{A}^{\mathrm{e}} - \mathbb{A}^{\mathrm{o}} = \{b_{jk}\}_{j,k=1}^{\infty}$ gives a positive-definite form: indeed $b^T\mathbb{B}\,b = \sum_{j=1}^{N-1}\frac{4j+1}{2}(b_{j+1}-b_j)^2 + \frac{4N+1}{2}b_N^2$. Therefore, the lowest eigenvalue comes from the odd subspace $\mathbf{F}^{\mathrm{o}}_{2N}$, and we have proved the lemma for cosine polynomials.

It suffices to prove the lemma for $g\in C^{\infty}[0,\pi]$, by virtue of this space being dense in the weighted Sobolev space $W^{1,2}_{\sin^2 x}[0,\pi]$ (see Kufner [23, Theorem 7.2]). Such g still extends to a piecewise C^1, 2π-periodic, even function, so that its Fourier partial sums g_N satisfy $g_N' \to g'$ and $g_N \to g$ in (unweighted) L^2. The inequality (4.28) survives this limit. This ends the proof of the lemma. □

The indicator function $1_{n\leqq 2}$ in the denominator below ensures summability of the weight in the proposition for all dimensions $n \geqq 1$.

Proposition 12 (Weighted Poincaré inequality)**.** *Fix $p > 0$, $n \geqq 1$, and let ρ be the corresponding Barenblatt profile* (1.4)*. Each function $f \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ satisfies*

$$\inf_c \int_0^\infty \frac{|f(r)-c|^2}{r^2 + 1_{n\leqq 2}} \rho(r) r^{n-1}\, dr \leqq C(p,n) \int_0^\infty |f'(r)|^2 \rho(r) r^{n-1}\, dr \tag{4.29}$$

with the indicator function $1_{n\leqq 2} \in \{0,1\}$ vanishing unless $n \leqq 2$. Thus there is a continuous embedding

$$W^{1,2}_{\ell=0} \hookrightarrow L^2\left(\mathbf{R}^+, \frac{\rho(r) r^{n-1}\, dr}{r^2 + 1_{n\leqq 2}}\right) / \{\mathrm{const}\}.$$

Proof. Fix $f \in \mathcal{D}'(]0,\infty[)$. The same proof works whether or not f is a smooth function, provided all objects are interpreted distributionally using $C_c^\infty(]0,\infty[)$ test functions.

Note that for any finite measure $d\mu$ – unless the infimum is infinite – we have

$$\inf_c \int |f(r)-c|^2\, d\mu(r) = \int |f(r)|^2\, d\mu(r) - \left(\int f(r)\, d\mu(r)\right)^2 \Big/ \int d\mu(r)\ ;$$

then the infimum is attained at the average value $c = \int f(r)\, d\mu(r) / \int d\mu(r)$, and it costs no generality to assume that $\int f\, d\mu = 0$ and set $c = 0$. If the infimum is infinite, it is also attained trivially at $c = 0$. Assuming $n \geqq 3$ for the moment, employ the transformation $x(r) = \int_0^r \rho(s) s^{n-3}\, ds$, and call its inverse $r = \sigma(x)$. Note that $x_\infty := \lim_{r\to\infty} x(r)$ is finite. For $g(x) := f(\sigma(x))$ our claim (4.29) becomes

$$\begin{aligned} \int_0^{x_\infty} g(x)^2\, dx &= \int_0^\infty f(r)^2 \rho(r) r^{n-3}\, dr \\ &\leqq C(p,n) \int_0^\infty f'(r)^2 \rho(r) r^{n-1}\, dr \\ &= C(p,n) \int_0^{x_\infty} g'(x)^2 \frac{\sigma(x)^2}{\sigma'(x)^2}\, dx. \end{aligned} \tag{4.30}$$

From the asymptotic behavior of $x = \sigma^{-1}(r)$ and its derivative as $r \to 0$ and $+\infty$, we see (by continuity of the weights on $x \in [x, x_\infty]$) that $C(p,n)$ large enough implies

$$\frac{\pi^2}{C(p,n)} \frac{\sin^2(x\pi/x_\infty)}{(\pi/x_\infty)^2} \leqq \frac{\sigma^2(x)}{\sigma'(x)^2} \sim \begin{cases} c_1 x^2 & \text{as } x \to 0^+, \\ c_2(x_\infty - x)^2 & \text{as } x \to x_\infty^-. \end{cases} \tag{4.31}$$

Here $c_1 = (n-2)^2$ and $c_2 = (p+2)^2$. At this point, a trivial length scaling reduces the assertion (4.30) of the proposition to Lemma 11.

For $n \leqq 2$, we must use a modified coordinate transformation, to avoid divergence at 0. Namely, we take $x(r) = \int_0^r \rho(s) s^{n-1}/(s^2+1)\, ds$. The claim then transforms to

$$\int_0^{x_\infty} g(x)^2\, dx \leqq C(p,n) \int_0^{x_\infty} g'(x)^2 \frac{1+\sigma(x)^2}{\sigma'(x)^2}\, dx.$$

and for $n = 1, 2$, we have $\frac{1+\sigma^2}{\sigma'^2} = (1+r^2)|x'(r)|^2 \sim n\rho(0)x^{n-1}$ as $x \to 0$. Except for this modification, (4.31) carries over and the proposition follows. $\square$

Corollary 13 (Sobolev space embedding)**.** *For $n \geqq 2$, $p > 0$ and ρ as above, the embeddings*

$$W_\rho^{1,2}(\mathbf{R}^n) \hookrightarrow L^2\left(\mathbf{R}^n, \frac{\rho(\boldsymbol{x})d\boldsymbol{x}}{|\boldsymbol{x}|^2 + 1_{n\leqq 2}}\right)/\{\text{const}\} \hookrightarrow L^2(\mathbf{R}^n, \rho^{2-m}\, d\boldsymbol{x})/\{\text{const}\}$$

are continuous:

$$\inf_c \int_{\mathbf{R}^n} |\Psi(\boldsymbol{x}) - c|^2 \frac{\rho(\boldsymbol{x})}{|\boldsymbol{x}|^2 + 1_{n\leqq 2}}\, d\boldsymbol{x} \leqq \max\{C(p,n), 1\} \int_{\mathbf{R}^n} |\nabla\Psi(\boldsymbol{x})|^2 \rho(\boldsymbol{x})\, d\boldsymbol{x}. \tag{4.32}$$

Proof. Proposition 6 and Remark 7 show that (4.32) follows from the analogous embeddings $W_\ell^{1,2} \hookrightarrow L^2\left(\mathbf{R}^+, \frac{\rho(r)r^{n-1}\,dr}{r^2+1_{n\leqq 2}}\right)$ provided the embedding constant is uniform in $\ell \in \mathbf{N}$. For $\ell = 0$, this embedding is given by Proposition 12. For $\ell \neq 0$, the embeddings (without the $\inf_c$ or $1_{n\leqq 2}$) follow trivially from (4.12):

$$\int_0^\infty \frac{|f(r)|^2}{r^2} \rho(r) r^{n-1}\, dr \leqq \frac{1}{\boldsymbol{L}^2} \|f\|^2_{W_\ell^{1,2}} \leqq \|f\|^2_{W_\ell^{1,2}}. \tag{4.33}$$

This establishes (4.32). Continuity of the second embedding $L^2\left(\mathbf{R}^n, \frac{\rho(\boldsymbol{x})d\boldsymbol{x}}{|\boldsymbol{x}|^2+\delta_{n,2}}\right) \hookrightarrow L^2(\mathbf{R}^n, \rho^{2-m}\, d\boldsymbol{x})$ follows immediately from $\rho(\boldsymbol{x})^{m-1}/\text{const} = |\boldsymbol{x}|^2 + C$. $\square$

Note the relation between this Sobolev embedding corollary and the Poincaré inequality discussed by CARRILLO *et al.* for $\Psi = v\rho^{m-2}$ [14, (2.20)]. The result can also be used to deduce the density of compactly supported functions in $W_\rho^{1,2}(\mathbf{R}^n)$. In fact, since the Sobolev space (2.4) is defined modulo constants, functions with compact support in $\mathbf{R}^n \setminus \{\mathbf{0}\}$ should be enough. However we need only the weaker assertion:

Corollary 14 (Density of smooth functions with compact support)**.** *The functions $C_c^\infty(\mathbf{R}^n)$ form a dense subset of the Sobolev space $W_\rho^{1,2}(\mathbf{R}^n)$.*

Proof. Let us first verify the density of smooth functions $C^\infty(\mathbf{R}^n)$. Given $\Psi \in W_{\text{loc}}^{1,2}(\mathbf{R}^n)$, with $\int |\nabla\Psi|^2 \rho\, d\boldsymbol{x} < \infty$, define $\Psi_\varepsilon := \Psi * \sigma_\varepsilon \in C^\infty(\mathbf{R}^n)$ by convolution with a smooth mollifier $\sigma_\varepsilon(\boldsymbol{x}) = \varepsilon^{-n}\sigma(\boldsymbol{x}/\varepsilon)$ supported in the unit ball: $0 \leqq \sigma \leqq \chi_{B_1^n(\mathbf{0})}$. Jensen's inequality yields the pointwise relation $|\nabla\Psi_\varepsilon|^2 \leqq |\nabla\Psi|^2 * \sigma_\varepsilon$, which, integrated with Fubini's theorem, leads to

$$\int_{|\boldsymbol{x}|>R} |\nabla\Psi_\varepsilon(\boldsymbol{x})|^2 \rho(\boldsymbol{x})\, d\boldsymbol{x} \leqq \int_{|\boldsymbol{y}|>R-\varepsilon} |\nabla\Psi(\boldsymbol{y})|^2 \rho_\varepsilon(\boldsymbol{y})\, d\boldsymbol{y}. \tag{4.34}$$

The estimate $\rho_\varepsilon(r\boldsymbol{\omega}) \leqq \rho(\boldsymbol{\omega}[r-\varepsilon]_+)$ and tail behavior of the Barenblatt profile (1.4) show that $\rho_\varepsilon(\boldsymbol{x})/\rho(\boldsymbol{x}) \leqq 1 + O(\varepsilon)$ as $\varepsilon \to 0$; here the error term is uniformly bounded when $\boldsymbol{x} \to \infty$. Thus taking R large enough makes (4.34) uniformly small for all $\varepsilon \in [0, 1]$. A standard argument (see, e.g., ADAMS [2]) gives $\Psi_\varepsilon \longrightarrow \Psi$ strongly in $W^{1,2}(B^n_R(\mathbf{0}))$, and, since the truncation error (4.34) is small, in the desired space $W^{1,2}_\rho(\mathbf{R}^n)$.

The corollary will be completed by showing that any smooth $\Psi \in C^\infty \cap W^{1,2}_\rho(\mathbf{R}^n)$ can be approximated by one with compact support. Introduce a cutoff function $\eta \in C^\infty_c(\mathbf{R}^n)$ such that $\chi_{B^n_1(\mathbf{0})} \leqq \eta \leqq \chi_{B^n_2(\mathbf{0})}$, and define $\Psi_R(\boldsymbol{x}) := \Psi(\boldsymbol{x})\eta_R(\boldsymbol{x})$ using $\eta_R(\boldsymbol{x}) := \eta(\boldsymbol{x}/R)$. Then $\nabla\Psi_R - \nabla\Psi = (\eta_R - 1)\nabla\Psi + R^{-1}\Psi\nabla\eta(\boldsymbol{x}/R)$ implies

$$\frac{1}{2}\int\limits_{\mathbf{R}^n} |\nabla\Psi_R - \nabla\Psi|^2 \rho\, d\boldsymbol{x} \leqq \int\limits_{|\boldsymbol{x}|>R} |\nabla\Psi|^2 \rho\, d\boldsymbol{x} + \|\nabla\eta\|_{L^\infty(\mathbf{R}^n)} \int\limits_{R<|\boldsymbol{x}|<2R} \frac{4\Psi^2}{|\boldsymbol{x}|^2}\rho\, d\boldsymbol{x}. \tag{4.35}$$

Now $\Psi \in L^2\left(\mathbf{R}^n, \rho(\boldsymbol{x})\, d\boldsymbol{x}/(|\boldsymbol{x}|^2 + 1_{n\leqq 2})\right)$ by Corollary 13, so taking R large enough makes both terms in (4.35) small. Thus $\|\Psi_R - \Psi\|_{W^{1,2}_\rho(\mathbf{R}^n)} \to 0$ as $R \to \infty$, concluding the corollary. □

Remark 15. In the abstract, we announced $W^{1,2}_\rho(\mathbf{R}^n)$ as closure of $C^\infty_c(\mathbf{R}^n)$ with respect to the norm $\|\cdot\|_{W^{1,2}_\rho(\mathbf{R}^n)}$. Precisely speaking, this involves, next to the above corollary, also the statement that for any Cauchy sequence Ψ_j, in $C^\infty_c(\mathbf{R}^n)$ there is a *sequence of constants* c_j and a function $\Psi \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$, such that a subsequence $\Psi_j - c_j$ converges to Ψ pointwise a.e., and in L^2_{loc}, and such that $\left\|\Psi_j - \Psi\right\|_{W^{1,2}_\rho(\mathbf{R}^n)} \to 0$. The proof uses a similar cutoff argument; the constants c_j can be taken as the average of Ψ_j over, say, the unit ball, and the Poincaré inequality over an arbitrary ball $|\boldsymbol{x}| < R$ together with a diagonal sequence argument must be used to construct the function Ψ first.

4.6. The resolvent set

To complete our description of the spectrum of $\mathbf{H}_\ell$, we need to prove that each $\lambda < \lambda^{\mathrm{cont}}_\ell$ which is not an eigenvalue $\lambda \neq \lambda_{\ell k}$ is in fact in the resolvent set. This is the content of the next theorem. Its proof goes by first solving the inhomogeneous differential equation $(\mathbf{H}_\ell - \lambda) f = g \in W^{1,2}_\ell$, and then checking that the solution gives a bounded operator taking $g \mapsto (\mathbf{H}_\ell - \lambda)^{-1} g = f \in W^{1,2}_\ell$.

Theorem 16 (Spectrum)**.** *Let* $\mathbf{H}_\ell$ *be the self-adjoint operator* (4.15)*, on its domain* (4.16). *Its spectrum* $\sigma(\mathbf{H}_\ell)$ *is given by* (3.1)–(3.3).

Proof. Fix $0 \leqq \ell \in \mathbf{Z}$ and assume $\lambda < \lambda^{\mathrm{cont}}_\ell$ is not an eigenvalue. Variation of parameters yields a solution f for $(\mathbf{H}_\ell - \lambda) f = g \in W^{1,2}_\ell$; in terms of the functions ψ_1 and $\tilde\psi_2$ introduced at (4.19) and (4.22), this solution is given by

$$\begin{aligned} f(r) &= \tilde\psi_2(r)\int_0^r \bar g(s)\psi_1(s)\rho(s)\, s^{n-1} ds + \psi_1(r)\int_r^\infty \bar g(s)\tilde\psi_2(s)\rho(s)\, s^{n-1} ds, \\ f'(r) &= \tilde\psi_2'(r)\int_0^r \bar g(s)\psi_1(s)\rho(s)\, s^{n-1} ds + \psi_1'(r)\int_r^\infty \bar g(s)\tilde\psi_2(s)\rho(s)\, s^{n-1} ds, \end{aligned} \tag{4.36}$$

where $\bar g(s) = g(s)\rho(s)^{1-m}/\gamma$ and $\gamma := ms^{n-1}\rho(s)[\tilde\psi_2(s)\psi_1'(s) - \tilde\psi_2'(s)\psi_1(s)]$ is a nonvanishing constant whose precise value is not relevant here.

If this solution is well defined and indeed in $W^{1,2}_\ell$ (and we will show it is), then it is clearly the unique solution in this space, since λ is assumed not to be an eigenvalue. Note that in the case $\ell = 0$, the space $W^{1,2}_\ell$ is a quotient space modulo constants; equation (4.36) is compatible with this, even though it is not manifest in the formula. For a constant function g, (4.36) will be seen to yield the constant solution f.

We estimate f, assuming $p > 0$. This will in particular guarantee the convergence of the integrals in (4.36).

Indeed, Proposition 8 and the analyticity of ${}_2F_1(z)$ near ${}_2F_1(0) = 1$ imply

$$\begin{aligned} \psi_1(r) &= O(r^\ell) \quad \text{as } r \to 0, & |\psi_1(r)| &\leqq O(r^{1+p/2+\sqrt{T}}) \text{ as } r \to +\infty;\\ |\tilde\psi_2(r)| &\leqq O(r^{2-n-\ell}) \text{ as } r \to 0, & \tilde\psi_2(r) &= O(r^{1+p/2-\sqrt{T}}) \text{ as } r \to +\infty; \end{aligned} \tag{4.37}$$

with $T > 0$ since $\lambda < \lambda^{\text{cont}}_\ell$.

Let us first deal with the case $\ell \geqq 1$; the modifications for $\ell = 0$ will be discussed afterwards. With the norm (4.12) in mind, we distribute an extra factor s/s when using the Hölder inequality in the spaces $L^q(\mathbf{R}^+, d\mu), d\mu(s) := \rho(s)s^{n-1}\,ds$, with $1/q + 1/q' = 1$. Ultimately $q = q' = 2$, but first we need the extra flexibility. We do assume q, q' to be sufficiently close to 2, depending on $T > 0$. From (4.37) and $\bar g(s)/\text{const} = g(s)/(s^2 + C)$, we estimate vanishing and growth rates for the variable coefficients in (4.36):

$$\begin{aligned} \left|\int_0^r \bar g\psi_1\rho s^{n-1}\,ds\right| &\leqq \left(\int_0^r |\bar g s|^q \rho s^{n-1}\,ds\right)^{1/q} \left(\int_0^r |\psi_1/s|^{q'}\rho s^{n-1}\,ds\right)^{1/q'}\\ &\leqq \left\|\frac{g}{s+\frac{C}{s}}\right\|_{L^q(d\mu)} \times \begin{cases} O(r^{\ell-1+n/q'}) & \text{as } r \to 0\\ O(r^{p/2-p/q'+\sqrt{T}}) & \text{as } r \to \infty \end{cases},\\ \left|\int_r^\infty \bar g\tilde\psi_2\rho s^{n-1}\,ds\right| &\leqq \left(\int_r^\infty |\bar g s|^q \rho s^{n-1}\,ds\right)^{1/q} \left(\int_r^\infty \left|\tilde\psi_2/s\right|^{q'}\rho s^{n-1}\,ds\right)^{1/q'}\\ &\leqq \left\|\frac{g}{s+\frac{C}{s}}\right\|_{L^q(d\mu)} \times \begin{cases} O(r^{1-\ell-n/q}) & \text{as } r \to 0\\ O(r^{p/2-p/q'-\sqrt{T}}) & \text{as } r \to \infty \end{cases}. \end{aligned} \tag{4.38}$$

In these estimates, we have assumed that q, q' are sufficiently close to 2 such that $p/2 - p/q' + \sqrt{T} > 0$ and $p/2 - p/q' - \sqrt{T} < 0$. Otherwise the first term would be only $O(1)$ as $r \to \infty$, and the second term would diverge at ∞. Similarly, we have made use of $\ell \geqq 1$ to ensure $1 - \ell - n/q < 0$ and $\ell - 1 + n/q' > 0$. (Note, for later use, that these inequalities are also justified for $\ell = 0$ if $n \geqq 3$.)

The estimates (4.38), with $q = q' = 2$, show already that f, f' are well defined (and smooth) on $]0, \infty[$ by (4.36), because (4.12) controls $\|g/s\|_{L^2(d\mu)}$ by $\|g\|_{W^{1,2}_\ell}$. We also conclude immediately that

$$|f(r)/r| \leqq \text{const}\|g/s\|_{L^q(d\mu)} \times \begin{cases} r^{-n/q} & \text{as } r \to 0 \\ r^{p/q} & \text{as } r \to \infty \end{cases}.$$

For any q, in particular $q = 2$, these estimates just barely fail to control $\|f/r\|_{L^q(d\mu)}$. However, they show that our solution operator $g/r \mapsto f/r$ is of weak type (q, q) for every q (≈ 2 such that the estimates hold). The Marcinkiewicz interpolation theorem then bounds $\|f/r\|_{L^2(d\mu)}$ in terms of $\|g/r\|_{L^2(d\mu)}$. For the definition of "weak type" and the Marcinkiewicz interpolation theorem, the reader may consult, e.g., MALÝ & ZIEMER [26, pp. 51–53]. The very same argument holds for f' instead of f/r. We have shown that, for $\ell \neq 0$, and λ as specified, the resolvent is a continuous operator from $W_\ell^{1,2}$ into itself.

For $n \geqq 3$ and $\ell = 0$, the estimates (4.38) can also be used with Marcinkiewicz to estimate $\|f'\|_{L^2(d\mu)}$ in terms of $\|g/r\|_{L^2(d\mu)}$. The latter is finite by (4.29), since $c/r \in L^2(d\mu)$. Let us now argue the case for $n = 2$, $\ell = 0$, where a few modifications must be made. To begin with, the asymptotic behavior (4.37) of $\tilde{\psi}_2$ near zero is altered, according to Proposition 8:

$$\begin{array}{ll} \psi_1(r) = O(1), & \tilde{\psi}_2(r) = O(\log r) \\ \psi_1'(r) = O(r), & \tilde{\psi}_2'(r) = O(r^{-1}) \end{array} \qquad \text{as } r \to 0, \quad (n = 2). \tag{4.39}$$

Even though the asymptotics of ψ_1 have not changed, the first estimate in (4.38) is no longer good enough for varying q around 2. However, if, instead of s/s, we distribute $\sqrt{s^2+1}/\sqrt{s^2+1}$, we gain one power of r near zero:

$$\left|\int_0^r \bar{g}\psi_1 \rho s\, ds\right| \leqq \left\|\bar{g}\sqrt{s^2+1}\right\|_{L^q(d\mu)} O(r^{2/q'}) \leqq \text{const}\left\|\frac{g}{\sqrt{s^2+1}}\right\|_{L^q(d\mu)} r^{2/q'}$$

as $r \to 0$. Here $\|g/\sqrt{s^2+1}\|_{L^2(d\mu)}$ is finite according to (4.29), since $c/\sqrt{s^2+1}$ belongs to $L^2(d\mu)$. The estimate as $r \to \infty$ is not affected, nor need it change.

The asymptotics (4.39) for $\tilde{\psi}_2$ are worse however, so redoing the second growth estimate in (4.38) (distributing s/s, but estimating $\bar{g}(s)$ in terms of $g/\sqrt{s^2+1}$) yields:

$$\left|\int_r^\infty \bar{g}\tilde{\psi}_2 \rho s\, ds\right| \leqq \left\|\frac{g}{\sqrt{s^2+1}}\right\|_{L^q(d\mu)} O(1 + r^{1-2/q}|\log r|) \quad \text{as} \quad r \to 0. \tag{4.40}$$

In any case, for $q \approx 2$ the bound (4.40) grows slower than $1/r$ and hence, being multiplied by the cofactor $\psi_1'(r) = O(r)$, does not contribute to the growth of $f'(r)$ near zero. Indeed, estimating (4.36) using (4.39), (4.40) and the unchanged asymptotics at $r = \infty$ yields

$$|f'(r)| \leqq \text{const}\left\|\frac{g}{\sqrt{s^2+1}}\right\|_{L^q(d\mu)} \times \begin{cases} r^{-1+2/q} & \text{as } r \to 0 \\ r^{p/q} & \text{as } r \to \infty \end{cases}.$$

Now $|f'|^q$ is integrable at the origin, and Marcinkiewicz bounds $\|f'\|_{L^2(d\mu)} = \|f\|_{W^{1,2}_{\ell=0}}$ in terms of $\|g/\sqrt{s^2+1}\|_{L^2(d\mu)}$.

For all dimensions $n \geqq 2$, and $\ell = 0$, we have shown $f \in W^{1,2}_{\ell=0}$. If g happens to be constant, then f must also be a constant, since the equation $(\mathbf{H}_\ell - \lambda)\psi = g$ admits a constant solution, and has a unique solution $\psi \in W^{1,2}_{\ell=0}$. Thus formula (4.36) respects the quotient-space structure, and we may immediately improve our estimate of $\|f'\|_{L^2(d\mu)} = \|f\|_{W^{1,2}_{\ell=0}}$ via $\|g/\sqrt{s^2 + 1_{n\leqq 2}}\|_{L^2(d\mu)}$, to an estimate in terms of $\inf_c \|(g-c)/\sqrt{s^2 + 1_{n\leqq 2}}\|_{L^2(d\mu)}$. This latter is controlled by $\|g\|_{W^{1,2}_\ell}$ according to Proposition 12.

We have therefore seen that $\left\|(\mathbf{H}_\ell - \lambda)^{-1} g\right\|_{W^{1,2}_\ell} = \|f\|_{W^{1,2}_\ell} \leqq \text{const}\, \|g\|_{W^{1,2}_\ell}$ with the constant depending on λ and ℓ but not g. This estimate holds for those λ satisfying $T > 0$ which are not among the eigenvalues $\lambda_{\ell k}$ specified in (3.3).

This shows that λ is in the resolvent set. Since the eigenvalues (3.3) were established in Corollary 9, while $\lambda \geqq \lambda^{\text{cont}}_\ell$ lies in the continuous spectrum according to Proposition 10, this establishes the theorem and completes our rigorous analysis of the spectrum. □

4.7. Proof of Lemma 5

For the proof of Lemma 5 we let $\eta := \chi_{\ell\mu}$, $\varphi := \Phi_{\ell\mu}$, $\xi := \Xi_{\ell\mu}$, as defined in (4.6), (4.7). In doing so, note that (4.6) holds for each r and is not dependent on the asymptotic behavior implied by the radial spaces used there. Note that $\eta \in C^\infty[0,\infty[$, and that $\eta(0) = 0$ if $\ell \neq 0$. In deviation from the rest of the paper, here we will use $\mathbf{H}_\ell$ as the differential expression (4.15), not with the specific domain (4.16). From $-m\rho^{m-2}\Delta_\rho \chi = 0$, we get $\mathbf{H}_\ell\, \eta = 0$, and from $\chi = \Xi + m\rho^{m-2}\Delta_\rho \Phi$, we get $\eta = \xi - \mathbf{H}_\ell\, \varphi$. We also get $\varphi, \xi \in W^{1,2}_\ell$ and $\varphi \in W^{3,2}_{\text{loc}}$.

We note that $\mathbf{H}_\ell\, \eta = 0$ and $\eta \in C^\infty[0,\infty[$ imply that η is a multiple of ψ_1; use equation (4.19) with $\lambda := 0$ for this. In the case $\ell = 0$, this means that η is a constant. We are left with showing that, for $\ell \neq 0$, it follows $\eta \equiv 0$. We will do this by concluding from the asymptotic behavior of η (or ψ_1) as $r \to \infty$ to an asymptotic behavior of φ, based on $\mathbf{H}_\ell\, \varphi = \xi - \eta = \xi - a\psi_1$ and (4.36) above. It will turn out that, unless $a = 0$, the asymptotic behavior of φ is incompatible with $\varphi \in W^{1,2}_\ell$. So $a = 0$ and hence $\eta \equiv 0$.

Let us analyze the asymptotic behavior as $r \to \infty$ of a solution φ to $\mathbf{H}_\ell\, \varphi = \xi - a\psi_1$, $\varphi(0) = 0$ in the case $\ell \neq 0$, $\lambda = 0$. We must show $\varphi \in W^{1,2}_\ell \implies a = 0$. We have already seen that $\lambda = 0$ is in the resolvent set, and the estimates give us a $\tilde\varphi \in W^{1,2}_\ell$ such that $\mathbf{H}_\ell\, \tilde\varphi = \xi$. That leaves us with solving $\mathbf{H}_\ell\, \varphi_0 = a\psi_1$ for $\tilde\varphi - \varphi =: \varphi_0 \in W^{1,2}_\ell$. All solutions to this equation are given by a modified version of (4.36), namely

$$\begin{aligned}\varphi_0(r) = \frac{a}{\gamma}\biggl(&\tilde\psi_2(r)\int_0^r \rho^{2-m}(s)\psi_1^2(s)\, s^{n-1}ds \\ &-\psi_1(r)\int_0^r \rho^{2-m}(s)\psi_1(s)\tilde\psi_2(s)\, s^{n-1}ds\biggr) + c_1\psi_1 + c_2\psi_2.\end{aligned}$$

We need $c_2 = 0$ to have φ_0 square integrable (with respect to $r^{n-1}\rho\, dr$) at 0. From (4.19), (4.21)–(4.23), (4.37), we get the respective asymptotics of the three terms as $r \to \infty$:

$$
\begin{aligned}
\tilde{\psi}_2(r) \textstyle\int_0^r \rho^{2-m}(s)\psi_1^2(s)\, s^{n-1} ds &\sim r^{1+\frac{p}{2}+\sqrt{T}}, \\
\psi_1(r) &\sim r^{1+\frac{p}{2}+\sqrt{T}}, \\
\psi_1(r) \textstyle\int_0^r \rho^{2-m}(s)\psi_1(s)\tilde{\psi}_2(s)\, s^{n-1} ds &\sim r^{1+\frac{p}{2}+\sqrt{T}} \log r.
\end{aligned}
$$

The last term is the dominant one; unless $a = 0$, this behavior is incompatible with $\varphi_0 \in W_\ell^{1,2}$, because it makes $\int^\infty \varphi_0^2(r) r^{-2} \rho(r)\, r^{n-1}\, dr \sim \int^\infty r^{2\sqrt{T}-1} (\log r)^2\, dr$ divergent. This concludes the proof of Lemma 5. $\square$

4.8. Modifications for one dimension ($n = 1$)

We have assumed $n \geqq 2$ so far. However, the formulas obtained apply to the simpler case $n = 1$ as well, with a few obvious changes.

We do not have a spherical Laplacian on $\mathbf{S}^0$, but the operator $\mathbf{H}$ commutes with the parity operator $\mathbb{P}$ defined by $(\mathbb{P} f)(x) = f(-x)$, and we can identify the restriction of $\mathbf{H}$ to even functions with $\ell = 0$, and the restriction to odd functions with $\ell = 1$. Higher values of ℓ do not occur, so $\boldsymbol{L}^2 := 0$.

The operators $\mathbf{H}$, $\mathbf{H}_{\ell=0}$ and $\mathbf{H}_{\ell=1}$ all have the same differential expression, but operate in the full, the even, and the odd space respectively. (We may write x instead of r.) Formulas (4.19) and (4.20) in Proposition 8 remain valid, however we have $\psi_1|_{\ell=0} = \psi_2|_{\ell=1}$ and $\psi_2|_{\ell=0} = \psi_1|_{\ell=1}$. With either ℓ, we must discard ψ_2 for parity reasons. The formulas for the eigenvalues $\lambda_{\ell k}$ and for the threshold $\lambda_\ell^{\text{cont}}$ defined by $T = 0$ remain intact, as well as the reasoning in Section 4.3 and 4.4. However, it becomes convenient for $n = 1$ to merge the point spectra (3.3) for $\ell = 0$ and $\ell = 1$ by setting $2k = \kappa$ for $\ell = 0$, and $2k + 1 = \kappa$ for $\ell = 1$. The continuous thresholds (3.2) coincide anyway: $\lambda_{\ell=0}^{\text{cont}} = \lambda_{\ell=1}^{\text{cont}}$, so we have a complete absense of eigenvalue crossings. Indeed, for $n = 1$:

$$
\begin{aligned}
\lambda_\kappa &= \kappa - \frac{\kappa(\kappa-1)}{2}(1-m) = \kappa - \frac{\kappa(\kappa-1)}{p+1}, \quad \kappa \in \mathbf{Z} \cap [1, 1 + p/2[, \\
\lambda^{\text{cont}} &= \frac{1}{2(1-m)} + \frac{1-m}{8} + \frac{1}{2} = \frac{(\frac{p}{2}+1)^2}{p+1}.
\end{aligned}
\tag{4.41}
$$

Extrapolating these eigenvalues to the regime $m \geqq 1$ yields the spectrum discovered by ZEL'DOVICH & BARENBLATT [44] and ANGENENT [3] for the porous medium equation in one dimension. The continued absence of level crossings among the extrapolated eigenvalues explains why phase transitions were never observed in these one dimensional studies. Indeed, the dynamics remain translation-governed in the entire supercritical regime $m \geqq -1$, because the phase transition at $p = n = 1$ is suppressed: dilations cease to be a d_2 small perturbation at $p = 2$, where the eigenvalue $\lambda_{\kappa=2} = \lambda_{01}$ dissolves into the continuous spectrum.

The resolvent estimate in Section 4.6 requires a re-counting of exponents as in the case $n = 2$, simplified by the fact that both $\psi_1(r)$ and $\tilde{\psi}_2(r)$ are analytic at $r = 0$; this yields an estimate of $\|f'\|_{L^2(d\mu)}$ in terms of $\|g/\sqrt{r^2+1}\|_{L^2(d\mu)}$. Proposition 12 extends to $n = 1$. However, for $\ell = 1$, we cannot get rid of the constants, but still need an analogous estimate, namely

$$\int_0^\infty \frac{|f(r)|^2}{r^2+1}\rho(r)\,dr \leqq C(p)\int_0^\infty |f'(r)|^2\rho(r)\,dr \tag{4.42}$$

provided $f(0) = 0$. This is proved like the Hardy inequality, by integration by parts: Let $h(r) := -\frac{r^2+1}{\rho(r)}\int_r^\infty \frac{\rho(s)}{s^2+1}\,ds$. Then

$$\begin{aligned}\int_0^\infty & f(r)^2\frac{\rho(r)}{r^2+1}\,dr \\ &= -\int_0^\infty 2f(r)f'(r)h(r)\frac{\rho(r)}{r^2+1}\,dr \\ &\leqq 2\left(\int_0^\infty f(r)^2\frac{\rho(r)}{r^2+1}\,dr\right)^{1/2}\left(\int_0^\infty f'(r)^2\frac{h(r)^2}{r^2+1}\rho(r)\,dr\right)^{1/2}.\end{aligned}$$

Since $\frac{h(r)^2}{r^2+1}$ is bounded above on $[0,\infty[$, the desired inequality (4.42) follows.

4.9. Form domain and sharp spectral-gap inequality

According to Corollary 3, $\mathbf{H}$ is a positive, symmetric operator on $C_c^\infty(\mathbf{R}^n)$. Its *form domain* $Q(\mathbf{H})$ is defined as the closure of $C_c^\infty(\mathbf{R}^n)$ with respect to the norm $\mathrm{Hess}_\rho(\Psi,\Psi) + \|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)}$. Thanks to Lemma 4, there is a unique self-adjoint extension of $\mathbf{H}$, which must therefore coincide with the *Friedrich's extension* described, e.g., in RIESZ & SZ-NAGY [38, VIII Section 124]. In particular, it follows that the form domain $Q(\mathbf{H})$ contains the domain $D(\mathbf{H})$ of the extended operator, and that this extension does not change the lower bound of the quadratic form, i.e., the *spectral gap* $\inf\{\mathrm{Hess}_\rho(\Psi,\Psi) \mid \|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} = 1\}$. We have calculated the spectrum of $\mathbf{H}$ to lie in $[\Lambda_0,\infty)$, the sharp threshold $\Lambda_0 > 0$ being given by (3.7). By the spectral theorem, this spectral gap implies the Poincaré type inequality:

$$\begin{aligned}\|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} &:= \int_{\mathbf{R}^n} |\nabla\Psi|^2\rho\,d\boldsymbol{x} \\ &\leqq \frac{m}{\Lambda_0}\int_{\mathbf{R}^n}(\mathrm{div}[\rho\nabla\Psi]/\rho)^2\,\rho^m\,d\boldsymbol{x} = \frac{1}{\Lambda_0}\mathrm{Hess}_\rho(\Psi,\Psi)\end{aligned} \tag{4.43}$$

for all $p > 0$ and $\Psi \in W^{1,2}_\rho(\mathbf{R}^n)$. Apart from constants, this equality is saturated only by linear functions of $\boldsymbol{x}$ in the translation-governed regime $p > n$, and only by multiples of the quadratic function $\Psi(\boldsymbol{x}) = |\boldsymbol{x}|^2$ in the dilation-governed regime $p \in]2, n[$. (Recall Fig. 1.) It is not saturated but remains sharp in the near-critical regime $p \in]0, 2[$, except that when $n = 1$ it is saturated by linear functions of $\boldsymbol{x}$ for all $p > 0$. Together with Theorem 16, this completes the proof of all results announced in our earlier note [16].

It is worthwhile to have a direct characterization of the form domain. Our final lemma will show $Q(\mathbf{H})$ to consist of those $W^{2,2}_{\mathrm{loc}}(\mathbf{R}^n)$ functions lying in the Sobolev space

$$W^{2,2}_\rho(\mathbf{R}^n) := \left\{ \Psi : \mathbf{R}^n \longrightarrow \mathbf{R} \;\middle|\; \|\Psi\|_{W^{2,2}_\rho(\mathbf{R}^n)} < \infty \right\} \Big/ \{\|\cdot\| = 0\}, \tag{4.44}$$

$$\|\Psi\|^2_{W^{2,2}_\rho(\mathbf{R}^n)} := \int_{\mathbf{R}^n} \left\{ (1+|\boldsymbol{x}|^2)|\mathrm{Hess}\Psi|_2^2 + |\nabla\Psi|^2 \right\} \rho \, d\boldsymbol{x}. \tag{4.45}$$

Lemma 17 (Form domain)**.** *The norms* $\mathrm{Hess}_\rho(\Psi, \Psi) + \|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)}$, $\mathrm{Hess}_\rho(\Psi, \Psi)$, *and* $\|\Psi\|^2_{W^{2,2}_\rho(\mathbf{R}^n)}$ *all induce the same topology on* $C^\infty_c(\mathbf{R}^n)$.

Proof. The first two norms are equivalent because of the spectral gap (4.43):

$$\mathrm{Hess}_\rho(\Psi, \Psi) \leqq \mathrm{Hess}_\rho(\Psi, \Psi) + \|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} \leqq (1 + 1/\Lambda_0)\mathrm{Hess}_\rho(\Psi, \Psi).$$

To see the last two are equivalent, recall (2.9):

$$\begin{aligned} \mathrm{Hess}_\rho(\Psi, \Psi) &\leqq \|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} + \int_{\mathbf{R}^n} \rho^{m-1} |\mathrm{Hess}\Psi|_2^2 \, \rho \, d\boldsymbol{x} \\ &= \mathrm{Hess}_\rho(\Psi, \Psi) + (1-m) \, \|\Delta\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})} . \end{aligned} \tag{4.46}$$

The right-hand side of (4.46) gives a norm easily seen to be equivalent to (4.45), in view of (1.4): $\mathrm{const}\rho^{m-1} = |\boldsymbol{x}|^2 + C$. To conclude the lemma, it remains only to bound $\|\Delta\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})}$ above by $\mathrm{Hess}_\rho(\Psi, \Psi)$. Introducing the differential operator

$$\mathbf{A}\Psi \;:=\; \mathrm{div}[\rho\nabla\Psi]/\rho \;=\; \Delta\Psi - \frac{\rho^{1-m}}{m} \boldsymbol{x} \cdot \nabla\Psi, \tag{4.47}$$

(2.10) and (2.11) imply

$$m \, \|\mathbf{A}\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})} = \mathrm{Hess}_\rho(\Psi, \Psi). \tag{4.48}$$

Also, (4.47) gives

$$\begin{aligned} \|\Delta\Psi - \mathbf{A}\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})} &\leqq \frac{1}{m^2} \int_{\mathbf{R}^n} \rho^m \rho^{1-2m} |\boldsymbol{x}|^2 |\nabla\Psi|^2 \rho \, d\boldsymbol{x} \\ &\leqq \frac{\mathrm{const}}{m^2} \int_{\mathbf{R}^n} |\nabla\Psi|^2 \rho \, d\boldsymbol{x}, \end{aligned} \tag{4.49}$$

since $|\boldsymbol{x}|^2 \rho^{1-m}(\boldsymbol{x})$ is bounded. Combining (4.48), (4.49) with (4.43) yields

$$\begin{aligned} \frac{1}{2} \, \|\Delta\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})} &\leqq \|\mathbf{A}\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})} + \|\Delta\Psi - \mathbf{A}\Psi\|^2_{L^2(\mathbf{R}^n, \rho^m d\boldsymbol{x})} \\ &\leqq \frac{1}{m} \left(1 + \frac{\mathrm{const}}{m\Lambda_0} \right) \mathrm{Hess}_\rho(\Psi, \Psi), \end{aligned}$$

to give the desired bound on (4.46) and complete the lemma. □

Remark 18 (**L^2 embedding**)**.** Corollary 13 shows that the norms $\|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)}$ and $\|\Psi\|^2_{W^{1,2}_\rho(\mathbf{R}^n)} + \|\Psi\|^2_{L^2(\mathbf{R}^n,\rho d\boldsymbol{x}/(|\boldsymbol{x}|^2+1))}$ induce the same topology on the codimension-one subspace $\{1\}^\perp$ orthogonal to the constant functions in the space $L^2\left(\mathbf{R}^n, \rho\, d\boldsymbol{x}/(|\boldsymbol{x}|^2+1)\right)$. Thus $W^{1,2}_\rho(\mathbf{R}^n)$ and $W^{2,2}_\rho(\mathbf{R}^n)$ are continuously embedded in $\{1\}^\perp$. One power of $|\boldsymbol{x}|$ enters the weight at $+\infty$ for each derivative of Ψ in these norms and (4.45).

5. Special functions: notation and background lore

This appendix collects the few facts about special functions which prove relevant to our investigation. For the Gamma function $\Gamma(z)$ and hypergeometric functions ${}_2F_1(a, b, c; z)$ we refer to Abramowitz & Stegun [1] or Stalker [40]. For a discussion of spherical harmonics $Y_{\ell\mu}(\boldsymbol{\omega})$ see Berger, Gauduchon & Mazet [7].

Let $\Gamma : \mathbf{C} \longrightarrow \mathbf{C} \cup \{\infty\}$ be defined by the Newman-Weierstrass product [40, (1.3.6)]

$$\Gamma(z) := \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right)^{-1} e^{z/k} \tag{5.1}$$

with $\gamma = \lim\limits_{k\to\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log k)$; it gives the meromorphic continuation of

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{5.2}$$

from $z \in]0, \infty[$ to the complex plane $\mathbf{C}$ [40, (1.1.18)]. The product representation shows that $\Gamma(z)$ does not vanish, its poles are all simple, and their locations coincide with the non-positive integers. The integral representation shows $\Gamma(1) = 1$ and the functional equation $\Gamma(z + 1) = z\Gamma(z)$ is satisfied.

For fixed $a, b, c \in \mathbf{C}$ the Pochhammer symbol $(a)_k := (a)(a + 1) \cdots (a + k - 1) = \Gamma(a + k)/\Gamma(a)$ may be used to define the hypergeometric series

$${}_2F_1\left({a, b \atop c}; z\right) := 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k = F(z) \tag{5.3}$$

(provided c is not an integer $\leqq 0$) within its radius of convergence $|z| < 1$. As is well known, the hypergeometric function arising by analytic continuation from the series (5.3) solves the following differential equation of Fuchsian type [40, (1.6.19)] or [1, (15.5.1)]:

$$z(1 - z)\frac{d^2F}{dz^2} + (c - (a + b + 1)z)\frac{dF}{dz} - abF = 0, \tag{5.4}$$

away from $z = 0, 1, \infty$. A second linearly independent solution is

$$z^{1-c}{}_2F_1\left({a - c + 1, b - c + 1 \atop 2 - c}; z\right), \tag{5.5}$$

unless c is a positive integer.

In the nongeneric cases, when c *is* a positive integer, certain linear combinations of solutions for nonsingular c have a limit as c approaches the singular value, and this limit will actually be a derivative with respect to c. The actual formulas, involving logarithmic terms, can be found in [1] and need not be repeated here.

The analytic continuation of hypergeometric series can be effected by means of certain integral formulas. One of them (valid for real parameters a and $c > b > 0$) is [40, (1.6.22)]

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty t^{b-1}(1+t)^{a-c}(1+t-tz)^{-a} dt \tag{5.6} $$

(for real parameters a and $c > b > 0$). In particular, it gives the analytic continuation to $-\infty < z < 1$. This and similar integral representations can be used to show certain connection formulas, due to Thomé 1879 (see STALKER's footnote [40, p. 48]). The following is germane to (4.21)–(4.23) (see [40, (1.6.39)] or [1, (15.3.7)]):

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}; \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}; \frac{1}{z}\right). \end{aligned} \tag{5.7} $$

The singular points 0, 1 and ∞ of the hypergeometric equation can be treated on the same footing, and the permutation group on these three points, represented as a group of Möbius transformations in $\mathbf{C} \cup \{\infty\}$, induces corresponding self-transformations of equation (5.4). For instance, the substitution $F(z) = z^{-a}G(z^{-1})$, $z^{-1} = w$, transforms (5.4) into the same equation for $G(w)$, but with different parameters a, b, c, namely a, $1-c+a$, $1-b+a$ respectively. This sheds light on (5.7), and permits us to reduce the discussion of nongeneric cases for (5.4) at ∞ (namely when $a-b$ is an integer) to the nongeneric cases at 0 (namely when c is an integer).

Acknowledgements. It is a pleasure to acknowledge fruitful discussions concerning this work with ERIC CARLEN, DARIO CORDERO-ERAUSQUIN, WILFRID GANGBO, ANASTASIA RUZMAIKINA, and MICHAEL SIGAL. The question of long-time asymptotics for very fast diffusion was raised in a lecture by PETER MARKOWICH at the 2000 Summer School "Mass Transportation Methods in Kinetic Theory and Hydrodynamics," organized in Azores by Maria Carvalho and the European Kinetic Theory TMR. We are grateful to JOSÉ CARRILLO and JUAN LUIS VÁZQUEZ for sharing their work with us in preprint form.

This research was supported by United States National Science Foundation grants DMS 0074037 and DMS 0354729, a Natural Sciences and Engineering Research Council of Canada grant 217006-99 RGPIN, and an Ontario Premier's Research Award of Excellence. The authors are grateful to the Universities of Toronto [JD], Wisconsin [RJM] and Texas [RJM] for their generous hospitality during various stages of this work.

References

1. ABRAMOWITZ, M., STEGUN, I.A.: *Handbook of Mathematical Functions*. Dover Publications Inc., New York, 1992

2. ADAMS, R.A.: *Sobolev Spaces*. Academic Press, New York, 1975
3. ANGENENT, S.B.: Large time asymptotics for the porous medium equation. In: *Nonlinear diffusion equations and their equilibrium states I*, volume 12 of *Math. Sci. Res. Inst. Publ.*, Springer, New York, 1988, pp. 21–34
4. BAKRY, D., EMERY, M.: Diffusions hypercontractives. In: *Sém. Proba. XIX*, number 1123 in Lecture Notes in Math., Springer, New York, 1985, pp. 177–206
5. BARENBLATT, G.I.: On some unsteady motions of a liquid or gas in a porous medium. *Akad. Nauk. SSSR. Prikl. Mat. Mekh.* **16**, 67–78 (1952)
6. BÉNILAN, P., CRANDALL, M.G.: The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$. *Indiana Univ. Math. J.* **30**, 161–177 (1981)
7. BERGER, M., GAUDUCHON, P., MAZET, E.: *Le spectre d'une variété riemannienne*, volume 194 of *Lecture Notes in Mathematics*. Springer, New York, 1971
8. BOUSSINESQ, J.: Recherches théoriques sur l'écoulement des nappes d'eau infiltrés dans le sol et sur le débit de sources. *Comptes Rendus Acad. Sci. / J. Math. Pures Appl.* **10**, 5–78 (1903/04)
9. CARLSON, J.M., CHAYES, J.T., GRANNAN, E.R., SWINDLE, G.H.: Self-organized criticality and singular diffusion. *Phys. Rev. Lett.* **65**, 2547–2550 (1980)
10. CARRILLO, J.A., TOSCANI, G.: Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity. *Indiana Univ. Math. J.* **49**, 113–141 (2000)
11. CARRILLO, J.A., VÁZQUEZ, J.L.: Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations* **28**, 1023–1056 (2003)
12. CARRILLO, J.A., MCCANN, R.J., VILLANI, C.: *Contractions in the* 2*-Wasserstein length space and thermalization of granular media*. Submitted 2004
13. CARRILLO, J.A., JÜNGEL, A., MARKOWICH, P.A., TOSCANI, G., UNTERREITER, A.: Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.* **133**, 1–82 (2001)
14. CARRILLO, J.A., LEDERMAN, C., MARKOWICH, P.A., TOSCANI, G.: Poincaré inequalities for linearizations of very fast diffusion equations. *Nonlinearity* **15**, 565–580 (2002)
15. CHAYES, J.T., OSHER, S.J., RALSTON, J.V.: On singular diffusion equations with applications to self-organized criticality. *Comm. Pure Appl. Math.* **46**, 1363–1377 (1993)
16. DENZLER, J., MCCANN, R.J.: Phase transitions and symmetry breaking in singular diffusion. *Proc. Natl. Acad. Sci. USA* **100**, 6922–6925 (2003)
17. DOLBEAULT, J., DEL PINO, M.: Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.* **81**, 847–875 (2002)
18. FELLER, W.: *An Introduction to Probability Theory and Its Applications II*. 2nd ed. John Wiley & Sons, New York, 1972
19. FRIEDMAN, A., KAMIN, S.: The asymptotic behaviour of a gas in an n-dimensional porous medium. *Trans. Am. Math. Soc.* **262**, 551–563 (1980)
20. GALAKTIONOV, V.A., PELETIER, L.A., VAZQUEZ, J.L.: Asymptotics of the fast-diffusion equation with critical exponent. *SIAM J. Math. Anal.* **31**, 1157–1174 (2000)
21. GIVENS, C.R., SHORTT, R.M.: A class of Wasserstein metrics for probability distributions. *Michigan Math. J.* **31**, 231–240 (1984)
22. HERRERO, M.A., PIERRE, M.: The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$. *Trans. AMS* **291**, 145–158 (1985)
23. KUFNER, A.: *Weighted Sobolev Spaces*. 2nd ed. John Wiley & Sons, New York, 1985
24. LEDERMAN, C., MARKOWICH, P.A.: On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass. *Comm. Partial Differential Equations* **28**, 301–332 (2003)
25. LYTCHAK, A., STURM, K.-Th., VON RENESSE, M.-K.: Work in progress. Private communication 2004
26. MALÝ, J., ZIEMER, W.P.: *Fine Regularity of Solutions of Elliptic Partial Differential Equations*. Number 51 in Mathematical Surveys and Monographs. AMS, Providence, 1997
27. MCCANN, R.J.: A convexity principle for interacting gases. *Adv. Math.* **128**, 153–179 (1997)

28. MUSKAT, M.: *The Flow of Homogeneous Fluids Through Porous Media*. McGraw-Hill, New York, 1937
29. NEWMAN, W.: A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity. I. *J. Math. Phys.* **25**, 3120–3123 (1984)
30. NIRENBERG, L., WALKER, H.F.: The null spaces of elliptic partial differential operators in R^n. *J. Math. Anal. Appl.* **42**, 271–301 (1975)
31. OTTO, F.: The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations* **26**, 101–174 (2001)
32. OTTO, F., VILLANI, C.: Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173**, 361–400 (2000)
33. PATTLE, R.E.: Diffusion from an instantaneous point source with concentration dependent coefficient. *Quart. J. Mech. Appl. Math.* **12**, 407–409 (1959)
34. DEL PINO, M., SAEZ, M.: On the extinction profile for solutions of $u_t = \Delta u^{(N-2)/(N+2)}$. *Indiana Univ. Math. J.* **50**, 612–628 (2001)
35. PLIŚ, A.: A smooth linear elliptic differential equation without any solutions in a sphere. *Comm. Pure Appl. Math.* **14**, 599–617 (1961)
36. POOLE, E.G.C.: *Introduction to the Theory of Linear Differential Equations*. Dover, New York, 1960
37. RALSTON, J.: A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity. II. *J. Math. Phys.* **25**, 3124–3127 (1984)
38. RIESZ, F., SZ.-NAGY, B.: *Functional Analysis*. Dover, New York, 1990
39. RUDIN, W.: *Functional Analysis*. 2nd ed. McGraw-Hill, New York, 1991
40. STALKER, J.: *Complex Analysis: Fundamentals of the Classical Theory of Functions*. Birkhäuser, Boston, 1998
41. VÁZQUEZ, J.L.: Asymptotic behaviour for the porous medium equation in the whole space. Dedicated to Philippe Bénilan. *J. Evolution Equations* **3**, 67–118 (2003)
42. VAZQUEZ, J.L., ESTEBAN, J.R., RODRIGUEZ, A.: The fast diffusion equation with logarithmic nonlinearity and the evolution of conformal metrics in the plane. *Advances in Differential Equations* **1**, 21–50 (1996)
43. VILLANI, C.: *Topics in Optimal Transportation*. Graduate Studies in Mathematics #58, American Mathematical Society, Providence, 2003
44. ZEL'DOVICH, Ya.B., BARENBLATT, G.I.: The asymptotic properties of self-modelling solutions of the nonstationary gas filtration equations. *Sov. Phys. Doklady* **3**, 44–47 (1958)
45. ZEL'DOVICH, Ya.B., KOMPANEETS, A.S.: Theory of heat transfer with temperature dependent thermal conductivity. In: *Collection in Honour of the 70th Birthday of Academician* A.F. Ioffe, Izdvo. Akad. Nauk. SSSR, Moscow, 1950, pp. 61–71
46. ZEL'DOVICH, Ya.B., RAIZER, Yu.P.: *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena II*. Academic Press, New York, 1966

Department of Mathematics
University of Tennessee
Knoxville, TN 37996-1300
USA
e-mail: denzler@math.utk.edu

and

Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3
Canada
e-mail: mccann@math.toronto.edu

(*Accepted June 23, 2004*)
Published online September 22, 2004 – © *Springer-Verlag* (*2004*)