Optimal Maps in Monge's Mass Transport Problem.

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October 10, 1995

Abstract: Choose a cost function $c(\mathbf{x}) \geq 0$ which is either strictly convex on $\mathbf{R}^d$, or a strictly concave function of the distance $|\mathbf{x}|$. Given two non-negative functions $f, g \in L^1(\mathbf{R}^d)$ with the same total mass, we assert the existence and uniqueness of a map which is measure-preserving between f and g, and minimizes the mass transport cost measured against $c(\mathbf{x} - \mathbf{y})$. An analytical proof based on the Euler-Lagrange equation of a dual problem is outlined. It assumes f, g to be compactly supported, and disjointly supported in the concave case.

Solutions optimales au problème de transport de masse de Monge.

Résumé. Considérons une fonction coût $c(\mathbf{x}) \geq 0$ supposée soit strictement convexe sur $\mathbf{R}^d$, ou soit une fonction strictement concave de la distance $|\mathbf{x}|$. Etant donnée deux fonctions $f, g \in L^1(\mathbf{R}^d)$, non négatives, d'égales masses totales, nous prouvons l'existence et l'unicité d'une application préservant les mesures, relativement à f et g, qui minimise le coût de transport par rapport à $c(\mathbf{x} - \mathbf{y})$. Nous donnons une preuve analytique basée sur l'équation d'Euler-Lagrange d'un problème dual en faisant l'hypothèse que f, g sont à supports compacts. Dans le cas concave, nous faisons l'hypothèse supplémentaire que les supports sont disjoints.

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Version française abrégée. Nous présentons la solution d'un problème variationnelle qui généralise un problème formulé par Monge [1]. Notre formulation est la suivante. Soit $c(\mathbf{x})$ définie sur $\mathbf{R}^d$ une fonction que nous appelons fonction coût et soient $f, g \geq 0$ dans $L^1(\mathbf{R}^d)$ deux fonctions d'égales masses totales $\int f = \int g$. Trouver l'application $\mathbf{t} : \mathbf{R}^d \longrightarrow \mathbf{R}^d$ qui minimise la fonctionnelle

$$\mathcal{C}(\mathbf{s}) := \int_{\mathbf{R}^d} c(\mathbf{x} - \mathbf{s}(\mathbf{x})) f(\mathbf{x}) d\mathbf{x} \tag{1}$$

sur l'ensemble $S(f, g)$ des applications de $\mathbf{R}^d$ dans $\mathbf{R}^d$, *préservant les mesures* relativement à f et g : c'est à dire que $S(f, g)$ est l'ensemble des applications $\mathbf{s}$ de $\mathbf{R}^d$ dans $\mathbf{R}^d$ satisfaisant la formule de changement de variables (2) pour tout h continue sur $\mathbf{R}^d$.

Une classe typique de fonctions coûts motivant ce travail est celle est donnée par $c(\mathbf{x}) = |\mathbf{x}|^p$ avec $p > 0$. Monge choisit la distance euclidienne ($p = 1$) comme fonction coût. Deux siècles s'écoulèrent avant que Sudakov prouva que $\mathcal{C}(\mathbf{s})$ admette un minimiseur sur $S(f, g)$ [2] (voir aussi [3]). D'autre part, il était bien clair depuis le début que dans le cas $p = 1$, le minimiseur de $\mathcal{C}(\mathbf{s})$ ne pouvait être unique. Brenier apporte un nouveau souffle au sujet en montrant que si la fonction coût est donnée par $c(\mathbf{x}) = |\mathbf{x}|^2$, il existe non seulement un unique minimiseur $\mathbf{t}$, mais de plus $\mathbf{t}$ a la propriété d'être le gradiant d'une fonction convexe [4]. Les puissances de la distance $p > 1$ trouvent des applications en probabilité et statistiques [5], où le problème correspond à trouver la meilleure correlation entre deux variables aléatoires suivant la norme L^p. Néanmoins, du point de vue des sciences economique, le cas le plus intéressant est le cas $p < 1$ où le coût est une fonction concave de la distance: dans ce cas $|\mathbf{x} - \mathbf{y}|^p$ est une métrique sur $\mathbf{R}^d$.

Dans ce travail, nous ne nous intéresseront qu'aux coûts $c(\mathbf{x}) \geq 0$ qui sont strictement convexes ou qui s'écrivent comme une fonction concave de la distance $|\mathbf{x}|$. Les fonctions f et g sont supposées être à support compacts où le *support de* f dénote le plus petit ensemble fermé $\operatorname{spt} f \subset \mathbf{R}^d$ sur lequel f est différent de zéro. Dans le cas où le coût est concave, nous supposons de plus que $\operatorname{spt} f$ et $\operatorname{spt} g$ sont disjoints. Grâce à un argument purement analytique, nous prouvons que $\mathcal{C}(\mathbf{s})$ atteint sont minimum sur $S(f, g)$ et qu'il existe une ensemble de mesure nulle relativement à f tel que restriction du minimiseur $\mathbf{t}$ sur le complémentaire de cet ensemble soit unique et injectif. Il s'ensuit aisément que le problème de Monge-Kantorovich dont l'existence d'un minimiseur était connu, n'admet pas plus qu'un minimiseur; ce minimiseur s'écrit explicitement en fonction de $\mathbf{t} \in S(f, g)$. (Nous référons le lecteur au livre de Rachev [5] pour une formulation du problème de Monge-Kantorovich et son historique.)

Résumons notre approche analytique en quelques mots; elle est basée sur une observation faite indépendamment par Caffarelli et Varadhan [6] et Gangbo [7] lorsque la fonction coût est $c(\mathbf{x}) = |\mathbf{x}|^2$. Fixons deux voisinages ouverts $U, V \subset \mathbf{R}^d$ de $\operatorname{spt} f \subset U$ et de $\operatorname{spt} g \subset V$ et faisont l'hypothèse supplémentaire que $\overline{U} \cap \overline{V} = \emptyset$ dans le cas concave. Etudions un problème dual à (1-2). Definissons par (3) la fonctionnelle $J(u, v)$ sur les paires $(u, v) \in Lip_c$ de (4). Il est bien connu que $J(u, v)$ atteint son maximum pour un certain $(\psi, \phi) \in Lip_c$ [8]. L'équation d'Euler-Lagrange satisfaite par (ψ, ϕ) produit une application $\mathbf{t}(\mathbf{x}) = \mathbf{x} - (\nabla c)^{-1}(\nabla \psi(\mathbf{x}))$ appartenant à $S(f, g)$. Nous vérifions aisément que cette application minimise $\mathcal{C}(\mathbf{s})$.

Nous donnons les propriétés géométriques de ces minimiseurs dans un travail en cours où nous présentons une preuve détaillée de nos résultats, basée sur une approche géométrique et constructive, similaire à [9]; dans cet travail en cours, nous relaxons les hypothèses sur f et g. Cette relaxation impose une reformulation du théorème dans le cas concave et tient compte du fait que toute masse $b(\mathbf{x}) := \min\{f(\mathbf{x}), g(\mathbf{x})\}$ commune a f et g n'a pas besoin d'être déplacée.

Introduction

A variational problem originating with Monge [1] is resolved for a large class of costs. The formulation we choose is as follows. Fix a cost function $c(\mathbf{x})$ on $\mathbf{R}^d$ and let $f, g \geq 0$ be $L^1(\mathbf{R}^d)$ functions with the same total mass $\int f = \int g$. Find the map $\mathbf{t} : \mathbf{R}^d \longrightarrow \mathbf{R}^d$ which minimizes

$$\mathcal{C}(\mathbf{s}) := \int_{\mathbf{R}^d} c(\mathbf{x} - \mathbf{s}(\mathbf{x})) f(\mathbf{x}) d\mathbf{x} \tag{1}$$

among the *measure-preserving* maps $S(f, g)$ between f and g: that is, the mass of g on each Borel $M \subset \mathbf{R}^d$ should coincide with the mass of f on the Borel set $\mathbf{s}^{-1}(M)$. Equivalently, the set $S(f, g)$ may be characterized as consisting of those Borel mappings $\mathbf{s}$ which satisfy the change of variables formula

$$\int_{\mathbf{R}^d} h(\mathbf{y})\, g(\mathbf{y}) d\mathbf{y} = \int_{\mathbf{R}^d} h(\mathbf{s}(\mathbf{x}))\, f(\mathbf{x}) d\mathbf{x}, \tag{2}$$

whenever $h(\mathbf{y})$ is continuous on $\mathbf{R}^d$. When f and g are continuous themselves, this represents a measure theoretic relaxation of the problem of minimizing $\mathcal{C}(\mathbf{s})$ among the C^1 maps from $\{\mathbf{x} \mid f(\mathbf{x}) > 0\}$ to $\mathbf{R}^d$ which satisfy $g(\mathbf{s}(\mathbf{x})) \det [D\mathbf{s}(\mathbf{x})] = f(\mathbf{x})$.

The class of costs which motivated the present developments are those of the form $c(\mathbf{x}) = |\mathbf{x}|^p$ with $p > 0$. Monge chose the Euclidean distance ($p = 1$) to be his cost function, but even for this special case, two centuries elapsed before Sudakov showed that $\mathcal{C}(\mathbf{s})$ attains a minimum on $S(f, g)$ [2] (see also [3]). On the other hand, it has long been appreciated that for $p = 1$ the minimizer would not be unique. Brenier breathed new life into the subject by showing that for the cost $c(\mathbf{x}) = |\mathbf{x}|^2$, not only does a unique minimizer $\mathbf{t}$ exist, but the map $\mathbf{t}$ is characterized as the gradient of a convex function [4]. Other powers $p > 1$ of the distance find applications in probability and statistics [5], while from an economic point of view, it is the concave powers $p < 1$ of the distance which form the most interesting class of costs: for them, $|\mathbf{x} - \mathbf{y}|^p$ is a metric on $\mathbf{R}^d$.

The discussion here will be restricted to cost functions $c(\mathbf{x}) \geq 0$ which are either strictly convex on $\mathbf{R}^d$ or strictly concave functions of the distance $|\mathbf{x}|$. The functions f and g are assumed to have bounded support, where *support* refers to the smallest closed set $\operatorname{spt} f \subset \mathbf{R}^d$ which carries the full mass of $f \in L^1(\mathbf{R}^d)$. For the case of concave costs, $\operatorname{spt} f$ is assumed disjoint from $\operatorname{spt} g$. Using a purely analytical argument, we prove that $\mathcal{C}(\mathbf{s})$ assumes its minimum on $S(f, g)$. Apart from a set with zero mass for f, the optimal map $\mathbf{t}$ will be uniquely determined and one-to-one. It is an immediate corollary that the solution to the associated Monge-Kantorovich problem — long known to exist — will be unique; this solution is derived from the mapping $\mathbf{t} \in S(f,g)$. (A formulation of the Monge-Kantorovich problem and its history may be found in Rachev's book [5].)

Our analytical approach may be outlined in a few words: it developed from an observation made independently by Caffarelli and Varadhan [6] and Gangbo [7] for the cost $c(\mathbf{x}) = |\mathbf{x}|^2$. Fixing bounded open neighbourhoods $U, V \subset \mathbf{R}^d$ of $\operatorname{spt} f \subset U$ and $\operatorname{spt} g \subset V$, with $\overline{U} \cap \overline{V} = \emptyset$ if the cost is concave, we turn our attention to a problem dual to (1-2). Define the functional

$$J(u,v) := \int u(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} + \int v(\mathbf{y}) g(\mathbf{y}) d\mathbf{y} \tag{3}$$

on pairs (u,v) of continuous functions in

$$Lip_c := \Big\{(u,v) \mid u,v \in C(\mathbf{R}^d),\ u(\mathbf{x}) + v(\mathbf{y}) \le c(\mathbf{x}-\mathbf{y}) \text{ on } U \times V\Big\}. \tag{4}$$

Since f and g have compact support, $J(u,v)$ is known to assume its maximum at some $(\psi, \phi) \in Lip_c$ [8]. From the Euler-Lagrange equation satisfied by (ψ,ϕ) we read off a map $\mathbf{t}(\mathbf{x}) = \mathbf{x} - (\nabla c)^{-1}(\nabla\psi(\mathbf{x}))$ in $S(f,g)$ which turns out to minimize $\mathcal{C}(\mathbf{s})$.

The geometry of these solutions is characterized in a forthcoming paper, where we also give a detailed proof of these results using the constructive geometrical approach of [9]; there, assumptions on f and g will be relaxed. This relaxation requires a reformulation of the theorem for concave costs, taking into account that any mass $b(\mathbf{x}) := \min\{f(\mathbf{x}), g(\mathbf{x})\}$ which is common to f and g must stay in its place. We point out that using discrete methods, Cuesta-Albertos and Tuero-Díaz have solved the analogous problem in which the target mass is not distributed throughout $\mathbf{R}^d$ according to $g \in L^1(\mathbf{R}^d)$, but concentrates on a finite set of points [10]; their result was extended to countably many points by Abdellaoui and Heinich [11].

Note added in revision: After the submission of this manuscript, the authors learned from Caffarelli [12] of his independent discovery of the same results for convex costs.

Existence and Uniqueness of Optimal Maps

Theorem 1 (Strictly Convex Costs) *Let $c: \mathbf{R}^d \longrightarrow \mathbf{R}$ be strictly convex, and $f, g \ge 0$ be $L^1(\mathbf{R}^d)$ functions of bounded support with the same total mass. The transport cost $\mathcal{C}(\mathbf{s})$ is minimized on $S(f,g)$ by some mapping $\mathbf{t}$ which is unique — $\mathcal{C}(\mathbf{s}) \le \mathcal{C}(\mathbf{t})$ for $\mathbf{s} \in S(f,g)$ implies $\mathbf{s}(\mathbf{x}) = \mathbf{t}(\mathbf{x})$ a.e. with respect to f — and one-to-one: there is a map $\mathbf{t}^* \in S(g,f)$ such that $\mathbf{t}^*(\mathbf{t}(\mathbf{x})) = \mathbf{x}$ a.e. with respect to f, while $\mathbf{t}(\mathbf{t}^*(\mathbf{y})) = \mathbf{y}$ a.e. with respect to g.*

Outline of proof. For simplicity, we assume the cost c to be $C^1(\mathbf{R}^d)$ and denote its gradient map by ∇c. The same proof adapts to non-smooth costs with the help of a little convex analysis [13]: in particular, one must recognize that convex functions are locally Lipschitz, and that the Legendre-Fenchel transform of c has a continuous gradient ∇c^* which can be used in place of $(\nabla c)^{-1}$ to prove Claim ♯2.

Let $U \supset \operatorname{spt} f$ and $V \supset \operatorname{spt} g$ be bounded open neighbourhoods — to be definite take U, V to be the same large ball $B(\mathbf{0}, R) \subset \mathbf{R}^d$ — and consider the maximization of $J(u,v)$ on Lip_c. Except as noted, *a.e.* refers to Lebesgue measure on U or V.

Claim ♯1: *$J(u, v)$ admits a maximizer (ψ, ϕ) on Lip_c.*
Proof: The proof is well-known [8, 5]. It is also known that one may assume $\psi = \phi^c$ and $\phi = \psi^c$, where

$$\phi^c(\mathbf{x}) := \inf_{\mathbf{y}\in\overline{V}} c(\mathbf{x}-\mathbf{y}) - \phi(\mathbf{y}), \qquad \psi^c(\mathbf{y}) := \inf_{\mathbf{x}\in\overline{U}} c(\mathbf{x}-\mathbf{y}) - \psi(\mathbf{x}). \tag{5}$$

This is because $\phi \in C(\mathbf{R}^d)$ yields $(\phi^c, \phi) \in Lip_c$, while $(\psi, \phi) \in Lip_c$ implies $\phi^c \geq \psi$; thus (ψ, ϕ) may be replaced by (ϕ^c, ϕ^{cc}) if necessary without decreasing $J(\psi, \phi)$.

Claim ♯2: *Since $\psi = \phi^c$, the equation $\psi(\mathbf{x}) + \phi(\mathbf{t}(\mathbf{x})) = c(\mathbf{x} - \mathbf{t}(\mathbf{x}))$ can be solved almost everywhere on U by a mapping $\mathbf{t} : U \longrightarrow \overline{V}$. This map is uniquely determined (a.e.) and Borel; it is given by $\mathbf{t}(\mathbf{x}) = \mathbf{x} - (\nabla c)^{-1}(\nabla\psi(\mathbf{x}))$.*
Proof: Let $\mathbf{x} \in \mathbf{R}^d$. From (5) and continuity of ϕ and c, one sees that for some $\mathbf{y} \in \overline{V}$,

$$\psi(\mathbf{x}) + \phi(\mathbf{y}) - c(\mathbf{x} - \mathbf{y}) = 0. \tag{6}$$

For any other $\mathbf{x}' \in \mathbf{R}^d$ one has

$$\psi(\mathbf{x}') + \phi(\mathbf{y}) - c(\mathbf{x}' - \mathbf{y}) \leq 0 \tag{7}$$

from (4). The roles of $\mathbf{x}$ and $\mathbf{x}'$ can also be interchanged, so subtracting (6) from (7) yields ψ Lipschitz on U; its Lipschitz constant $\|\psi\|_{U;Lip}$ is no greater that of c on $B(\mathbf{0}, 2R)$. By Rademacher's theorem, ψ is differentiable almost everywhere; its gradient $\nabla\psi$ is Borel measurable. Suppose that ψ is differentiable at $\mathbf{x} \in U$ and choose $\mathbf{y} \in \overline{V}$ so that (6) holds. Since $\mathbf{x}' = \mathbf{x}$ implies equality in (7), one sees that $\nabla\psi(\mathbf{x}) = \nabla c(\mathbf{x}-\mathbf{y})$. On the other hand, the strict convexity of $c \in C^1(\mathbf{R}^d)$ implies that ∇c is one-to-one; it is a homeomorphism of $\mathbf{R}^d$ onto $\nabla c(\mathbf{R}^d)$ by the open mapping theorem (invariance of domain). Its inverse $(\nabla c)^{-1}$ will be continuous, and one concludes that (6) determines $\mathbf{y} = \mathbf{x} - (\nabla c)^{-1}(\nabla\psi(\mathbf{x})) = \mathbf{t}(\mathbf{x})$ as a function of $\mathbf{x}$. The map $\mathbf{t}$ is Borel, so the claim is established.

Claim ♯3: *The map $\mathbf{t}$ is measure-preserving between f and g: $\mathbf{t} \in S(f, g)$.*
Proof: Fix any $h \in C(\mathbf{R}^d)$ and $|\epsilon| < 1$. For $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$ define $v_\epsilon(\mathbf{y}) := \phi(\mathbf{y}) + \epsilon h(\mathbf{y})$ and

$$u_\epsilon(\mathbf{x}) := \inf_{\mathbf{y}\in\overline{V}} c(\mathbf{x} - \mathbf{y}) - \phi(\mathbf{y}) - \epsilon h(\mathbf{y}).$$

In view of Claim ♯2, the infimum (5) defining $\phi^c(\mathbf{x})$ is uniquely attained for almost every $\mathbf{x} \in U$; at these points $u_\epsilon(\mathbf{x}) = \psi(\mathbf{x}) - \epsilon h(\mathbf{t}(\mathbf{x})) + o(\epsilon)$. Since $(u_o, v_o) = (\psi, \phi)$ minimizes $J(u, v)$ on Lip_c, the associated Euler-Lagrange equation is

$$\begin{aligned} 0 = \lim_{\epsilon\to 0} \frac{J(u_\epsilon, v_\epsilon) - J(u_o, v_o)}{\epsilon} &= \lim_{\epsilon\to 0} \int_U \frac{u_\epsilon(\mathbf{x}) - u_o(\mathbf{x})}{\epsilon} f(\mathbf{x})d\mathbf{x} + \int_V h(\mathbf{y})\, g(\mathbf{y})d\mathbf{y} \\ &= \int_U -h(\mathbf{t}(\mathbf{x}))\, f(\mathbf{x})d\mathbf{x} + \int_V h(\mathbf{y})\, g(\mathbf{y})d\mathbf{y}. \end{aligned}$$

The arbitrariness of h yields $\mathbf{t} \in S(f, g)$ via (2).

Claim ♯4: $\mathbf{t}$ *minimizes $\mathcal{C}(\mathbf{s})$ on $S(f, g)$ and duality holds: $\mathcal{C}(\mathbf{t}) = \sup_{(u,v)\in Lip_c} J(u, v)$.*
Proof: For all $(u, v) \in Lip_c$ and $\mathbf{s} \in S(f, g)$ one has $J(u, v) \leq \mathcal{C}(\mathbf{s})$:

$$\begin{aligned} \int uf + \int vg &= \int \Big(u(\mathbf{x}) + v(\mathbf{s}(\mathbf{x}))\Big) f(\mathbf{x})d\mathbf{x} \\ &\leq \int c(\mathbf{x} - \mathbf{s}(\mathbf{x}))\, f(\mathbf{x})d\mathbf{x}. \end{aligned} \tag{8}$$

Claims ♯2 and ♯3 show that equality holds in (8) when $(u, v) = (\psi, \phi)$ and $\mathbf{s} = \mathbf{t}$. Thus $J(\psi, \phi) = \mathcal{C}(\mathbf{t})$, which also shows $\mathcal{C}(\mathbf{t})$ to be a minimum.

Claim ♯5: *Up to a set where f vanishes, the map minimizing $\mathcal{C}(\mathbf{s})$ in $S(f, g)$ is unique.*
Proof: Let $\mathcal{C}(\mathbf{t}) = J(\psi, \phi)$ as in Claim ♯4, and suppose that $\mathbf{s} \in S(f, g)$ also has minimal cost: $\mathcal{C}(\mathbf{s}) = \mathcal{C}(\mathbf{t})$. Equality holds in (8) when $(u, v) = (\psi, \phi)$, so one must have equality in $\psi(\mathbf{x}) + \phi(\mathbf{s}(\mathbf{x})) \leq c(\mathbf{x} - \mathbf{s}(\mathbf{x}))$ on a set of full measure for f. By Claim ♯2, $\mathbf{t}(\mathbf{x}) = \mathbf{s}(\mathbf{x})$ except on a set where f may be taken to vanish.

Claim ♯6: *The map $\mathbf{t}^* : V \longrightarrow \overline{U}$ given by $\mathbf{t}^*(\mathbf{y}) = \mathbf{y} + (\nabla c)^{-1}(-\nabla\phi(\mathbf{y}))$ is in $S(g, f)$. Moreover, $\mathbf{t}^*(\mathbf{t}(\mathbf{x})) = \mathbf{x}$ a.e. with respect to f while $\mathbf{t}(\mathbf{t}^*(\mathbf{y})) = \mathbf{y}$ a.e. with respect to g.*
Proof: Claim ♯2 followed from $\psi = \phi^c$. But the argument applies equally well to $\phi = \psi^c$. Since the definitions of ψ^c and ϕ^c are symmetrical under interchange of U with V and $c(x)$ with $c(-x)$, the map $\mathbf{t}^*$ is the unique solution to $\psi(\mathbf{t}^*(\mathbf{y})) + \phi(\mathbf{y}) = c(\mathbf{t}^*(\mathbf{y}) - \mathbf{y})$ on a subset Y having full Lebesgue measure in V. Claim ♯3 showed that $\mathbf{t} \in S(f, g)$, whence $\mathbf{t}^{-1}(Y)$ carries the full mass of f. On $\mathbf{t}^{-1}(Y)$, the unique solution to (6) is given by $\mathbf{y} := \mathbf{t}(\mathbf{x})$. Since $\mathbf{t}(\mathbf{x}) \in Y$, one concludes that $\mathbf{t}^*(\mathbf{t}(\mathbf{x})) = \mathbf{x}$.

By the symmetry in $f \leftrightarrow g$ and $\mathbf{t} \leftrightarrow \mathbf{t}^*$, Claim ♯3 yields $\mathbf{t}^* \in S(g, f)$ while the preceding paragraph yields $\mathbf{t}(\mathbf{t}^*(\mathbf{y})) = \mathbf{y}$ a.e. with respect to g. ∎

Theorem 2 (Strictly Concave Costs) *Let $h(\lambda) > 0$ be strictly concave on $\lambda > 0$. Define $c(\mathbf{x}) := h(|\mathbf{x}|)$ on $\mathbf{R}^d \setminus \{\mathbf{0}\}$, and let f and g be as in Theorem 1. The conclusions of that theorem continue to hold provided* spt f *and* spt g *are disjoint.*

Sketch of proof. Assume $h \in C^1(0, \infty)$ for simplicity, and let $U \supset \operatorname{spt} f$ and $V \supset \operatorname{spt} g$ be open neighbourhoods as in the convex case. However, take $\overline{U}$ to be disjoint from $\overline{V}$. The assertions and proofs of Claim ♯1 and Claims ♯3-6 do not differ from the convex case, so we discuss only the proof of Claim ♯2. Here there are two central issues: first to deduce that the optimal potential $\psi = \phi^c$ of Claim ♯1 is Lipschitz on U (hence differentiable a.e.); then to show that at points of differentiability, $\mathbf{x} \in U$ and $\nabla\psi(\mathbf{x})$ determine $\mathbf{y}$ in (6) uniquely. Separation of $\overline{U}$ from $\overline{V}$ is used in the first step, where it prevents the singularity in $c(\mathbf{x} - \mathbf{y})$ at $\mathbf{x} = \mathbf{y}$ from spoiling regularity of ψ. Continuous differentiability of the cost — though not essential — facilitates the second step.

Proof of Claim ♯2: Let $r > 0$ denote the minimum distance between U and V. Since $h(\lambda)$ is strictly concave and remains positive, it must also be strictly increasing. Moreover,

$$h(\xi) - h(\lambda) \leq h'(r)\,|\xi - \lambda| \tag{9}$$

for $\xi, \lambda \geq r$. Given $\mathbf{x} \in U$, some $\mathbf{y} \in \overline{V}$ exists satisfying (6) just as in Theorem 1. Since (7) holds for any other $\mathbf{x}' \in U$, their difference yields

$$\psi(\mathbf{x}') - \psi(\mathbf{x}) \leq c(\mathbf{x}' - \mathbf{y}) - c(\mathbf{x} - \mathbf{y}) \leq h'(r)|\mathbf{x}' - \mathbf{x}|.$$

Here (9) has been been exploited to provide one side of a Lipschitz bound for ψ. Since no $\mathbf{y}$ appears in the final estimate, $\mathbf{x}$ and $\mathbf{x}'$ may be exchanged to yield ψ Lipschitz on U. It follows that ψ is differentiable almost everywhere on U with Borel gradient $\nabla\psi$.

Suppose ψ is differentiable at $\mathbf{x} \in U$ and choose $\mathbf{y} \in \overline{V}$ for (6) to hold. Then $\mathbf{x}' = \mathbf{x}$ implies equality in (7), so noting that $\mathbf{x}$ and $\mathbf{y}$ lie in disjoint sets one sees that

$$\nabla\psi(\mathbf{x}) = \nabla c(\mathbf{x} - \mathbf{y}) = h'(|\mathbf{x} - \mathbf{y}|)\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|} \neq \mathbf{0}. \tag{10}$$

Since $h'(\lambda)$ decreases strictly, $|\nabla\psi(\mathbf{x})| = h'(|\mathbf{x} - \mathbf{y}|)$ determines the magnitude of $\mathbf{x} - \mathbf{y}$ uniquely. The direction of $\mathbf{x} - \mathbf{y}$ must coincide with $\nabla\psi(\mathbf{x})$, so one can invert (10) to find $\mathbf{y} = \mathbf{x} - (\nabla c)^{-1}(\nabla\psi(\mathbf{x}))$. In this way, (6) forces $\mathbf{y} = \mathbf{t}(\mathbf{x})$ when ψ is differentiable at $\mathbf{x} \in U$. Continuity of $(\nabla c)^{-1}$ shows $\mathbf{t}(\mathbf{x})$ to be Borel, thereby concluding the proof. ∎

It is a pleasure to thank L. Craig Evans and Jill Pipher for bringing us together, while RJM wishes to acknowledge the support provided by an NSERC postdoctoral fellowship (Canada).

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