FORMULAS OF VERLINDE TYPE FOR NON-SIMPLY CONNECTED GROUPS

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ABSTRACT. We derive Verlinde's formula from the fixed point formula for loop groups proved in the companion paper [3], and extend it to compact, connected groups that are not necessarily simply-connected.

1. INTRODUCTION

In this paper we give applications of the fixed point formula proved in the companion paper [3]. Our original motivation was to understand a formula of E. Verlinde [32] for the geometric quantization of the moduli space of flat connections on a Riemann surface. In particular A. Szenes suggested to us that the Verlinde formula should follow from an equivariant index theorem, much as the Weyl or Steinberg formulas can be interpreted as fixed point formulas for flag varieties. It turns out that our approach also applies to moduli spaces for compact, connected groups that are not necessarily simply connected. In the present paper we will consider the case of at most one marking. The main result is Theorem 5.1 below. The case $G = SO(3)$ is due to Pantev [24], and $G = \mathrm{PSU}(p)$ with p prime to Beauville [8]. A formula for any number of markings was conjectured very recently by Fuchs and Schweigert [16]. The proof of this more general result, which involves so-called orbit Lie algebras, will appear in a later work.

Verlinde's formula appears in the literature in various guises. Our version computes the index of the pre-quantum line bundle over the moduli space of flat connections. Verlinde's original conjecture computes the dimension of the space of conformal blocks, for a wide range of two-dimensional conformal field theories. The cases we consider here arise from the Wess-Zumino-Witten model; see Felder, Gawedzki, and Kupiainen [14] for arbitrary simple groups. In the algebraic geometry literature, many authors refer to Verlinde's formula as computing the dimension of the space of holomorphic sections (generalized theta-functions). In many cases, the higher cohomology of the pre-quantum line bundle is known to vanish (see recent work by Teleman [28]) so that our version and the algebraic geometry version are the same.

Mathematically rigorous approaches to Verlinde's formula in algebraic geometry are due to Tsuchiya-Ueno-Yamada [31], Faltings [13], and Teleman [29, 30], to name a few. The comparison between conformal blocks and holomorphic sections can be found in

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Kumar-Narasimhan-Ramanathan [20], Beauville-Laszlo [7], and Pauly [25]. A nice survey can be found in Sorger [26].

A proof of the Verlinde formula via the Riemann-Roch theorem was outlined by Szenes in [27], and carried out for SU(2) and SU(3). The proof was extended by Jeffrey-Kirwan [18] to SU(n), and Bismut-Labourie [9] to arbitrary compact, connected simply-connected groups, for sufficiently high level. The idea of deriving the Verlinde formula from localization also appears in the physics papers by Gerasimov [17] and Blau and Thompson [10].

Notation.

This paper is designed to be read parallel to its companion *A fixed point formula for loop group actions* [FP]. Throughout this paper, G will denote a compact, connected, simply connected Lie group, and $G = G_1 \times \cdots \times G_s$ its decomposition into simple factors. Given a tuple $k = (k_1, \ldots, k_s)$ of positive numbers, $B = B_k$ will denote the invariant bilinear form which restricts to k_j times the basic inner product on the jth factor ([FP], Section 2.4).

We use the following Sobolev norms. Fix a number $f > 1$. For any manifold X (possibly with boundary) and $p \leq \dim X$, we denote by $\underline{\Omega}^p(X, \mathfrak{g})$ the set of $\mathfrak{g}$-valued p-forms of Sobolev class $f - p + \dim X/2$. Under this assumption, forms in $\underline{\Omega}^0(X, \mathfrak{g})$ are C^1 and those in $\underline{\Omega}^1(X, \mathfrak{g})$ are C^0. The space $\underline{\Omega}^0(X, \mathfrak{g})$ is the Lie algebra of the group $\mathcal{G}(X) = \mathrm{Map}(X, G)$ of Sobolev class $f + \dim X/2$.

Some other notations introduced in this paper are

$\mathcal{M}(\Sigma)$	moduli space of flat G-connections on surface Σ; 2.1
$M(\Sigma)$	holonomy manifold of $\mathcal{M}(\Sigma)$; 2.3
$L(\Sigma)$	pre-quantum line bundle over $\mathcal{M}(\Sigma)$; 2.2
h, r	genus, number of boundary components of Σ; 2.1
$\circledast$	fusion product; Appendix B
Γ, G'	central subgroup of G, quotient $G' = G/\Gamma$; 3
F_γ	fixed point component corresponding to $\gamma \in \Gamma^{2h}$; 3.3

2. The simply-connected case

2.1. The moduli space of flat connections. We begin with a brief review of the gauge theory construction of moduli spaces of flat connections. More details can be found in [4], [22], and [12]. Let $\Sigma = \Sigma_h^r$ denote a compact, connected, oriented surface of genus h with r boundary components. Let $\mathcal{A}(\Sigma) = \underline{\Omega}^1(\Sigma, \mathfrak{g})$ be the affine space of connections on the trivial G-bundle over Σ, equipped with the action of $\mathcal{G}(\Sigma)$ by gauge transformations

$$g \cdot A = \mathrm{Ad}_g(A) - g^*\overline{\theta}, \tag{1}$$

where $\overline{\theta}$ is the right-invariant Maurer-Cartan form. Let $\mathcal{G}_\partial(\Sigma) \subset \mathcal{G}(\Sigma)$ be the kernel of the restriction map $\mathcal{G}(\Sigma) \to \mathcal{G}(\partial\Sigma)$. Since G is simply connected, the restriction map is surjective, and therefore $\mathcal{G}(\Sigma)/\mathcal{G}_\partial(\Sigma) \cong \mathcal{G}(\partial\Sigma)$. We define

$$\mathcal{M}(\Sigma) := \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}_\partial(\Sigma), \tag{2}$$

the moduli space of flat G-connections under based gauge equivalence. If $\partial\Sigma \neq \emptyset$, it is a smooth $\mathcal{G}(\partial\Sigma)$-equivariant Banach manifold. Pull-back of connections to the boundary induces a map,

$$\widehat{\Phi}:\ \mathcal{M}(\Sigma) \to \underline{\Omega}^1(\partial\Sigma, \mathfrak{g}).$$

The map $\widehat{\Phi}$ is smooth and proper, and is equivariant for the gauge action of $\mathcal{G}(\partial\Sigma)$. Let $B = B_k$ be an invariant inner product on $\mathfrak{g}$. The symplectic form on $\mathcal{A}(\Sigma)$ is given by the integration pairing of 1-forms $(a_1, a_2) \mapsto \int_\Sigma B(a_1 \wedge a_2)$. As observed by Atiyah-Bott [4], the action of $\mathcal{G}_\partial(\Sigma)$ is Hamiltonian, with moment map the curvature. Hence (2) is a symplectic quotient and $\mathcal{M}(\Sigma)$ inherits a symplectic 2-form $\widehat{\omega}$. Moreover, the residual action of $\mathcal{G}(\partial\Sigma)$ on $\mathcal{M}(\Sigma)$ is Hamiltonian with moment map $\widehat{\Phi}$, using the pairing of $\underline{\Omega}^1(\partial\Sigma, \mathfrak{g})$ and $\underline{\Omega}^0(\partial\Sigma, \mathfrak{g})$ given by the inner product and integration over $\partial\Sigma$. A choice of parametrization of the boundary $\partial\Sigma = (S^1)^r$ induces isomorphisms

$$\mathcal{G}(\partial\Sigma) \cong LG^r, \quad \underline{\Omega}^1(\partial\Sigma, \mathfrak{g}) \cong \underline{\Omega}^1(S^1, \mathfrak{g}^r).$$

Thus $(\mathcal{M}(\Sigma), \widehat{\omega}, \widehat{\Phi})$ is an example of a Hamiltonian LG^r-manifold with proper moment map. For any $\mu = (\mu_1, \ldots, \mu_r) \in L(\mathfrak{g}^r)^*$, the symplectic quotient $\mathcal{M}(\Sigma)_\mu$ is the moduli space of flat connections for which the holonomy around the jth boundary component is contained in the conjugacy class of $\mathrm{Hol}(\mu_j)$. This also covers the case without boundary, since $M(\Sigma_h^0) = M(\Sigma_h^1)_0$.

Occasionally we will also use the notation $\mathcal{M}(\Sigma, G)$, in order to indicate the structure group. The decomposition into simple factors $G = G_1 \times \cdots \times G_s$ defines a decomposition of the loop group $LG = LG_1 \times \cdots \times LG_s$, and the moduli space is the direct product

$$\mathcal{M}(\Sigma, G) = \mathcal{M}(\Sigma, G_1) \times \cdots \times \mathcal{M}(\Sigma, G_s).$$

2.2. **Pre-quantization of the moduli space.** The space $\mathcal{M}(\Sigma)$ is pre-quantizable at integer level, that is if all k_i are integers (see e.g. Section 3.3. of [22] or [15]). For later use, we recall the construction of the pre-quantum line bundle. The central extension $\widehat{\mathcal{G}}(\Sigma)$ of $\mathcal{G}(\Sigma)$ is defined by the cocycle

$$c(g_1, g_2) = \exp(i\pi \int_\Sigma B(g_1^*\theta, g_2^*\overline{\theta})). \tag{3}$$

The group $\widehat{\mathcal{G}}(\Sigma)$ acts on the trivial line bundle over $\mathcal{A}(\Sigma)$ by

$$(g, z) \cdot (A, w) = \big(g \cdot A, \exp(-i\pi \int_\Sigma B(g^*\theta, A))zw\big). \tag{4}$$

The 1-form $a \mapsto \frac{1}{2}\int_\Sigma B(A,a)$ on $\mathcal{A}(\Sigma)$ defines an invariant pre-quantum connection. A trivialization of $\widehat{\mathcal{G}}(\Sigma)$ over the subgroup $\mathcal{G}_\partial(\Sigma)$ is given by the map

$$\alpha:\,\mathcal{G}_\partial(\Sigma)\to \mathrm{U}(1),\quad \alpha(g)=\exp(2\pi i\int_{\Sigma\times[0,1]}\overline{g}^*\eta). \tag{5}$$

Here η is the canonical 3-form on G, and $\overline{g}\in\mathcal{G}(\Sigma\times[0,1])$ is any extension such that $\overline{g}=g$ on $\Sigma\times\{0\}$ and $\overline{g}=e$ on $(\Sigma\times\{1\})\cup(\partial\Sigma\times[0,1])$. The map α is well-defined and satisfies the coboundary condition $\alpha(g_1g_2)=\alpha(g_1)\alpha(g_2)c(g_1,g_2)$. One defines the pre-quantum line bundle as a quotient $L(\Sigma)=(\mathcal{A}_{\mathrm{flat}}(\Sigma)\times\mathbb{C})/\mathcal{G}_\partial(\Sigma)$; it comes equipped with an action of $\widehat{LG}=\widehat{\mathcal{G}}(\Sigma)/\mathcal{G}_\partial(\Sigma)$.

We are thus in the setting of the fixed point formula ([FP], Theorem 4.3) which gives a formula for the Spin$_c$-index $\chi(\mathcal{M}(\Sigma)_\mu)$. To apply the fixed point formula we have to (i) describe the holonomy manifold $M(\Sigma):=\mathcal{M}(\Sigma)/\Omega G^r$, (ii) determine the fixed point manifolds for elements $(t_{\lambda_1},\ldots,t_{\lambda_r})$, and (iii) evaluate the fixed point data. These steps will be carried out in the subsequent sections.

2.3. **Holonomy manifolds.** The holonomy manifold $M(\Sigma):=\mathcal{M}(\Sigma)/\Omega G^r$ can be interpreted as the moduli space of flat connections

$$M(\Sigma)=\mathcal{A}_{\mathrm{flat}}(\Sigma)/\{g\in\mathcal{G}(\Sigma)|\,g(p_1)=\ldots=g(p_r)=e\}$$

where $p_1,\ldots,p_r$ are the base points on the boundary circles. The group-valued moment map $\Phi:\,M(\Sigma)\to G^r$ takes an equivalence class of flat connections to its holonomies around the boundary circles. The 2-form ω has the following explicit description (see [1, Section 9].) We begin with the case of a 2-holed sphere Σ_0^2. The surface Σ_0^2 is obtained from a 4-gon by identifying the sides according to the word $D_1AD_2A^{-1}$. Parallel transport along the paths A and $A^{-1}D_1$ defines a diffeomorphism

$$M(\Sigma_0^2)=G\times G.$$

The G^2-action is given by

$$(g_1,g_2)\cdot(a,b)=(g_1ag_2^{-1},\,g_2bg_1^{-1}). \tag{6}$$

The moment map is

$$\Phi(a,b)=(ab,a^{-1}b^{-1}) \tag{7}$$

and the 2-form is given by

$$\omega=\tfrac{1}{2}\big(B(a^*\theta,b^*\overline{\theta})+B(a^*\overline{\theta},b^*\theta)\big). \tag{8}$$

The holonomy manifolds for the general case $\Sigma=\Sigma_h^r$ are obtained from $M(\Sigma_0^2)$ by *fusion*, which we recall in Appendix B. First, the moduli space $M(\Sigma_1^1)$ for the 1-punctured torus is

$$M(\Sigma_1^1)=M(\Sigma_0^2)_{\mathrm{fus}}\cong G^2.$$

The G-action is conjugation on each factor and the moment map is the Lie group commutator $\Phi(a,b) = [a,b] = aba^{-1}b^{-1}$. The moduli space for the surface of genus h with 1 boundary component is an h-fold fusion product

$$M(\Sigma_h^1) = M(\Sigma_1^1) \circledast \ldots \circledast M(\Sigma_1^1) = G^{2h}.$$

G acts by conjugation on each factor, and the moment map is a product of Lie group commutators. The moduli space for the r-holed sphere Σ_0^r is an $(r-1)$-fold fusion product

$$M(\Sigma_0^r) = M(\Sigma_0^2) \circledast \ldots \circledast M(\Sigma_0^2) = G^{2(r-1)}$$

where we fuse with respect to the first G-factor for each G^2-space $M(\Sigma_0^2)$. Finally, the moduli space for Σ_h^r is

$$M(\Sigma_h^r) = M(\Sigma_h^1) \circledast M(\Sigma_0^r) = G^{2(h+r-1)}.$$

2.4. **The fixed point sets.** The fixed point sets for the action on the holonomy manifold are symplectic tori:

Proposition 2.1. *The fixed point set for the action of $(t_{\lambda_1}, \ldots, t_{\lambda_r})$ on $M(\Sigma_h^r) = G^{2(h+r-1)}$ is empty unless $\lambda_1 = \ldots = \lambda_r =: \lambda$, and*

$$M(\Sigma_h^r)^{(t_\lambda,\ldots,t_\lambda)} = F := T^{2(h+r-1)}.$$

Proof. Since $M(\Sigma_h^r)$ is obtained from a direct product of $h+r-1$ copies of $M(\Sigma_0^2)$ by passing to diagonal actions for some of the G-factors, it suffices to prove Proposition 2.1 for Σ_0^2. By (6), an element $(a,b) \in M(\Sigma_0^2)$ is fixed by $(t_{\lambda_1}, t_{\lambda_2})$ if and only if

$$t_{\lambda_1} = \mathrm{Ad}_a\, t_{\lambda_2}, \quad t_{\lambda_2} = \mathrm{Ad}_b\, t_{\lambda_1}. \tag{9}$$

Both t_{λ_1} and t_{λ_2} belong to the exponential of the alcove $\exp(\mathfrak{A})$. Since each conjugacy class meet $\exp(\mathfrak{A})$ only once, (9) holds if and only if $\lambda_1 = \lambda_2$. □

Notice that the fixed point set is independent of λ; in fact, F is fixed by the full diagonal torus $T \subset G^r$.

2.5. **Evaluation of the fixed point contributions.** Let $\Sigma = \Sigma_h^r$ and $\mu = (\mu_1, \ldots, \mu_r)$ with $\mu_j \in \Lambda_k^*$. By Theorem 4.3. of [FP], the Spin$_c$-index is given by the formula

$$\chi(\mathcal{M}(\Sigma)_\mu) = \frac{1}{(\#T_{k+c})^r} \sum_{\lambda \in \Lambda_k^*} |J(t_\lambda)|^{2r} \prod_{j=1}^r \chi_{\mu_j}(t_\lambda)^* \zeta_F(t_\lambda)^{1/2} \int_F \frac{\hat{A}(F) e^{\frac{1}{2}c_1(\mathcal{L}_F)}}{\mathcal{D}_{\mathbb{R}}(\nu_F, t_\lambda)}. \tag{10}$$

Here we abbreviated $(t_\lambda, \ldots, t_\lambda)$ to t_λ, viewing T as diagonally embedded into G^r. Since the normal bundle ν_F is T-equivariantly isomorphic to $(\mathfrak{g}/\mathfrak{t})^{2(h+r-1)}$, we have

$$D_{\mathbb{R}}(\nu_F, t_\lambda) = J(t_\lambda)^{2(h+r-1)} = (-1)^{(h+r-1)\#\mathfrak{R}} |J(t_\lambda)|^{2(h+r-1)}. \tag{11}$$

Furthermore, since F is a product of tori we have

$$\hat{A}(F) = 1. \tag{12}$$

It remains to work out the integral $\int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F))$ and to calculate the phase factor $\zeta_F(t_\lambda)^{1/2}$.

Proposition 2.2. *The integral of* $\exp(\frac{1}{2}c_1(\mathcal{L}_F))$ *over* F *equals* $(\#T_{k+c})^{h+r-1}$.

Proof. The line bundle $\mathcal{L} = L(\Sigma)^2 \otimes K^{-1}$ is $\widehat{LG^r}$-equivariant at levels $2(k+c), \ldots, 2(k+c)$. Since $\mathcal{M}(\Sigma)$ carries up to isomorphism a unique line bundle at every level [21, 3.12], it follows that $\mathcal{L}$ is the pre-quantum line bundle for the symplectic structure defined by $B_{2(k+c)}$. Hence $\mathcal{L}_F$ is a pre-quantum line bundle for the corresponding symplectic structure on F (cf. [FP], Subsection 4.4.3), and $\int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F))$ is the symplectic volume $\mathrm{Vol}_{B_{k+c}}(F)$ for the 2-form defined using B_{k+c}. We claim that the symplectic volume coincides with the Riemannian volume, which will complete the proof since $\mathrm{Vol}_{B_{k+c}}(T^2) = \#T_{k+c}$ by Lemma A.1 from Appendix A. By our description of $M(\Sigma_h^r)$ as a fusion product, the fixed point manifold $F = F(\Sigma_h^r)$ is obtained from the fixed point manifold $F(\Sigma_0^2)$ (viewed as a group valued Hamiltonian T^2-space) by fusion: $F(\Sigma_1^1) = F(\Sigma_0^2)_{\mathrm{fus}}$ and

$$F(\Sigma_h^r) = F(\Sigma_1^1) \circledast \cdots \circledast F(\Sigma_1^1) \circledast F(\Sigma_0^2) \circledast \cdots \circledast F(\Sigma_0^2),$$

with h factors $F(\Sigma_1^1)$ and $(r-1)$ factors $F(\Sigma_0^2)$. Lemma B.2 from Appendix B says that the symplectic volume of group valued Hamiltonian torus spaces does not change under fusion. Hence $\mathrm{Vol}_{B_{k+c}}(F(\Sigma_h^r)) = \mathrm{Vol}_{B_{k+c}}(F(\Sigma_0^2))^{h+r-1}$. Finally, the expression (8) for the 2-form on $M(\Sigma_0^2)$ shows that $\mathrm{Vol}_{B_{k+c}}(F(\Sigma_0^2))$ coincides with the Riemannian volume of T^2 with respect to B_{k+c}. □

Proposition 2.3. *The phase factor is given by* $\zeta_F(t_\lambda)^{1/2} = (-1)^{(h+r-1)\,\#\mathfrak{R}_+}$.

Proof. The point $m = (e, \ldots, e) \in F$ lies in identity level set of Φ, and its stabilizer in G^r is the image of the diagonal embedding of G. The 2-form ω restricts to a symplectic form on the tangent space $E = T_m M(\Sigma)$. By Equations (27) and (29) of [FP], $\zeta_F(t_\lambda)^{1/2}$ can be computed in terms of the symplectomorphism A of E defined by t_λ: Choose an A-invariant compatible complex structure on E to view A as a unitary transformation, and let $A^{1/2}$ be the unique square root having all its eigenvalues in the set $\{e^{i\phi} \mid 0 \le \phi < \pi\}$. Then $\zeta_F(t_\lambda)^{1/2} = \det(A^{1/2})$.

We first apply this recipe for the 2-holed sphere Σ_0^2, so that $E = T_m M(\Sigma_0^2) = \mathfrak{g} \oplus \mathfrak{g}$. Formula (8) shows that ω_m is the standard 2-form on $\mathfrak{g} \oplus \mathfrak{g}$, given by the inner product B. A compatible complex structure is given by the endomorphism $(\xi, \eta) \mapsto (-\eta, \xi)$. Thus, as a complex G-representation E is just the complexification $E = \mathfrak{g}^{\mathbb{C}}$. It follows that the eigenvalues of A (other than 1) come in complex conjugate pairs

$$e^{i\phi_j}, e^{-i\phi_j}, \quad 0 < \phi_j \le \pi/2,$$

and the corresponding eigenvalues of $A^{1/2}$ are $e^{i\phi_j/2}$ and $e^{i\pi - i\phi_j/2} = -e^{-i\phi_j/2}$. Hence

$$\zeta_F(t_\lambda)^{1/2} = (-1)^{\#\mathfrak{R}_+}.$$

Now consider the case $r \ge 1, h$ arbitrary. The tangent space is $T_m M(\Sigma_h^r) = (\mathfrak{g} \oplus \mathfrak{g})^{h+r-1}$, but because of the fusion terms the symplectic form is not the standard symplectic form defined by the inner product on $\mathfrak{g}$. However, by Appendix B, Lemma B.3 it is equivariantly and symplectically *isotopic* to the standard symplectic form. Since the phase factor $\zeta_F(t_\lambda)^{1/2}$ is a root of unity, it is invariant under equivariant symplectic isotopies, and we conclude as before that $\zeta_F(t_\lambda)^{1/2} = (-1)^{(h+r-1)\,\#\mathfrak{R}_+}$. □

2.6. **Verlinde formula.** From Equation (10) we obtain, using (11), (12) and Propositions 2.2 and 2.3,

Theorem 2.4 (Verlinde Formula). *Let G be a simply connected Lie group and k a given integral level. The* Spin$_c$*-index of the moduli space of flat connections on Σ_h^r at level k, with markings $\mu = (\mu_1, \ldots, \mu_r) \in (\Lambda_k^*)^r$ is given by the formula*

$$\chi(\mathcal{M}(\Sigma_h^r)_\mu) = (\#T_{k+c})^{h-1} \sum_{\lambda \in \Lambda_k^*} |J(t_\lambda)|^{2-2h} \chi_{\mu_1}(t_\lambda)^* \cdots \chi_{\mu_r}(t_\lambda)^*. \tag{13}$$

Remarks 2.5. (a) Theorem (2.4) also covers the case without boundary, since $M(\Sigma_h^0) = M(\Sigma_h^1, 0)$. One obtains

$$\chi(\mathcal{M}(\Sigma_h^0)) = (\#T_{k+c})^{h-1} \sum_{\lambda \in \Lambda_k^*} |J(t_\lambda)|^{2-2h}.$$

(b) For the two-holed sphere Σ_0^2, formula (13) simplifies by the orthogonality relations for level k characters, and gives $\chi(M(\Sigma_0^2)_{\mu_1,\mu_2}) = \delta_{\mu_1,*\mu_2}$.

(c) In Bismut-Labourie [9] the Spin$_c$-indices $\chi(\mathcal{M}(\Sigma_h^r)_\mu)$ are computed by direct application of the Kawasaki-Riemann-Roch formula to the reduced spaces. Their approach involves a description of all orbifold strata of the reduced space. The equality with the above sum over level k weights is non-trivial; it is established in [9] for sufficiently high level k.

(d) Theorem 2.4 gives a formula for a Spin$_c$-index rather than the dimension of a space of holomorphic sections. Vanishing results for higher cohomology groups have recently been proved by Teleman [28, Section 8].

3. Extension to non simply-connected groups

In this section we consider moduli spaces of flat connections for compact, connected semi-simple Lie groups that are not necessarily simply connected, for surfaces with one boundary component. The case of multiple boundary components will be considered elsewhere. Write $G' = G/\Gamma$, where G is simply connected and $\Gamma \subset Z(G)$ is a subgroup of the center $Z(G)$ of G. The covering $G \to G'$ identifies Γ with the fundamental group $\pi_1(G')$.

The main problem in dealing with non-simply connected groups is that the corresponding gauge groups are disconnected. This happens already for the loop group $LG' = \mathrm{Map}(S^1, G')$: The kernel of the natural map $LG' \to \pi_1(G') = \Gamma$ is the identity component L_0G' of LG', identifying the group of components with Γ.

The disconnectedness of gauge groups causes a number of difficulties: For example integrality of the level does not always guarantee the existence of a pre-quantization of the moduli space. In this section we will obtain sufficient conditions for the existence of a pre-quantization, and calculate the Spin_c-index of moduli spaces with prescribed holonomies around the boundaries for these cases.

3.1. **Gauge groups of surfaces for non-simply connected structure groups.** If $r \geq 1$, the surface Σ_h^r retracts onto a wedge of $2h+r-1$ circles. Hence every principal G'-bundle over Σ_h^r is trivial. On the other hand, principal G'-bundles over Σ_h^0 are classified by elements of Γ: The bundle corresponding to $\gamma \in \Gamma$ is obtained by gluing the trivial bundles over Σ_h^1 and over the disk Σ_0^1, using a loop $g' \in LG'$ representing γ as a transition function.

Let $\Sigma = \Sigma_h^1$. As before we consider the gauge group $\mathcal{G}'(\Sigma) = \mathrm{Map}(\Sigma, G')$ and the subgroup $\mathcal{G}'_\partial(\Sigma)$ of gauge transformation that are trivial on the boundary. To explain their relationship with the gauge groups $\mathcal{G}(\Sigma), \mathcal{G}_\partial(\Sigma)$ we consider the homomorphism

$$\mathcal{G}'(\Sigma) \to \Gamma^{2h} \tag{14}$$

which takes every gauge transformation $g' : \Sigma \to G'$ to the map induced on fundamental groups, viewed as an element of $\mathrm{Hom}(\pi_1(\Sigma), \pi_1(G')) = \mathrm{Hom}(\mathbb{Z}^{2h}, \Gamma) = \Gamma^{2h}$. Let $\mathcal{G}(\Sigma) \to \mathcal{G}'(\Sigma)$ be the map induced by the covering $G \to G'$.

Proposition 3.1. *The sequences*

$$1 \to \Gamma \to \mathcal{G}(\Sigma) \to \mathcal{G}'(\Sigma) \to \Gamma^{2h} \to 1 \tag{15}$$

and

$$1 \to \mathcal{G}_\partial(\Sigma) \to \mathcal{G}'_\partial(\Sigma) \to \Gamma^{2h} \to 1 \tag{16}$$

are exact. Restriction to the boundary $\mathcal{G}'(\Sigma) \to \mathcal{G}'(\partial\Sigma) \cong LG'$ takes values in L_0G', giving rise to an exact sequence

$$1 \to \mathcal{G}'_\partial(\Sigma) \to \mathcal{G}'(\Sigma) \to L_0G' \to 1. \tag{17}$$

Proof. We first show that the map $\mathcal{G}'_\partial(\Sigma) \to \Gamma^{2h}$ is surjective. Present Σ_h^0 as a quotient of a $4h$-gon P, with sides identified according to the word

$$C_1C_2C_1^{-1}C_2^{-1} \cdots C_{2h-1}C_{2h}C_{2h-1}^{-1}C_{2h}^{-1}.$$

Then $\Sigma = \Sigma_h^1$ is obtained as a similar quotient of P minus a disk in $\mathrm{int}(P)$. The sides C_j map to generators of $\pi_1(\Sigma)$, which we also denote C_j. Given $\gamma \in \Gamma^{2h}$, choose continuous maps $g'_j : C_j \to G$ such that $g'_j = e$ at the base point and such that the loop g'_j represents γ_j. Since $\pi_1(G')$ is abelian, the concatenation of loops

$$\prod_{j=1}^h [g'_{2j-1}, g'_{2j}] : \prod_{j=1}^h [C_{2j-1}, C_{2j}] \to G'$$

is homotopically trivial. Hence the maps g'_j extend to a continuous map $g' : \Sigma \to G'$ trivial on $\partial\Sigma$. By a C^0-small perturbation, g' can be changed to a smooth gauge transformation, which still vanishes on $\partial\Sigma$. We next show exactness of (16) at $\mathcal{G}'_\partial(\Sigma)$. Since G is simply connected, a necessary and sufficient condition for an element $g' \in \mathcal{G}'_\partial(\Sigma)$ to admit a lift $g \in \mathcal{G}_\partial(\Sigma)$ is that the induced maps on fundamental groups be trivial. Thus g' is in the image of the map $\mathcal{G}_\partial(\Sigma) \to \mathcal{G}'_\partial(\Sigma)$ if and only if it is in the kernel of the map $\mathcal{G}'_\partial(\Sigma) \to \Gamma^{2h}$. Injectivity of the map $\mathcal{G}_\partial(\Sigma) \to \mathcal{G}'_\partial(\Sigma)$ is obvious. This proves (16) and the proof of (15) is similar.

Finally we prove (17). Let $g' \in \mathcal{G}'(\Sigma)$. Since $\partial\Sigma$ is homotopic to $\prod_{j=1}^h [C_{2j-1}, C_{2j}]$, the restriction of g' to the boundary defines a contractible loop in G'. Hence $g'|_{\partial\Sigma} \in L_0 G'$. $\square$

3.2. **Moduli spaces of flat connections.** View $\mathcal{A}(\Sigma) = \underline{\Omega}^1(\Sigma, \mathfrak{g})$ as the space of G'-connections, and consider the action of the gauge group $\mathcal{G}'(\Sigma)$. As before, the action of the subgroup $\mathcal{G}'_\partial(\Sigma)$ is Hamiltonian, with moment map the curvature. The symplectic quotient is the moduli space of flat G'-connections up to based gauge equivalence,

$$\mathcal{M}'(\Sigma) = \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}'_\partial(\Sigma).$$

It carries a residual Hamiltonian action of $L_0 G' = \mathcal{G}'(\Sigma)/\mathcal{G}'_\partial(\Sigma)$, with moment map induced by the pullback of a connection to the boundary. Using the surjection $LG \to L_0 G'$, we will view $\mathcal{M}'(\Sigma)$ as a Hamiltonian LG-space where $\Gamma \subset LG$ acts trivially. We will also use the notation $\mathcal{M}(\Sigma, G') = \mathcal{M}'(\Sigma)$ to indicate the structure group.

The moduli space $\mathcal{M}(\Sigma)$ of flat G-connections is a finite covering of $\mathcal{M}'(\Sigma)$. Identify $\mathcal{G}_\partial(\Sigma)$ with the identity component of $\mathcal{G}'_\partial(\Sigma)_0$. By Proposition 3.1, there is an isomorphism

$$\mathcal{G}'(\Sigma)/\mathcal{G}'_\partial(\Sigma)_0 = \Gamma^{2h} \times \mathcal{G}'(\partial\Sigma)_0 \cong \Gamma^{2h} \times L_0 G'. \tag{18}$$

This shows that the action of LG on $\mathcal{M}(\Sigma) = \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_\partial(\Sigma)$ extends to an action of the direct product $\Gamma^{2h} \times LG$, and

$$\mathcal{M}'(\Sigma) = \mathcal{M}(\Sigma)/\Gamma^{2h}.$$

Similarly, the holonomy manifold $M'(\Sigma)$ of $\mathcal{M}'(\Sigma)$ is just

$$M'(\Sigma) = M(\Sigma)/\Gamma^{2h} = G^{2h}/\Gamma^{2h} = (G')^{2h}.$$

The product of commutators, $\Phi : M(\Sigma) = G^{2h} \to G$ is invariant under the action of Γ^{2h} and descends to the G-valued moment map $\Phi' : M'(\Sigma) \to G$.

For any $g' \in G'$, the moduli space of flat G'-connections with holonomy conjugate to g' is a disjoint union of symplectic quotients $M'(\Sigma)_g$ where g varies over all pre-images of g' in G. The reduced spaces at central elements $\gamma \in \Gamma \subset G$ may also be interpreted as moduli spaces of flat connections on the G'-bundle over Σ_h^0, with topological type given by γ.

The moduli space $\mathcal{M}(\Sigma, G')$ for the semi-simple group G' is a finite cover of a product of moduli spaces for simple groups. For $j = 1, \ldots, s$, let $\Gamma_j \subset G_j$ be the image of Γ under projection to the jth simple factor. Then G' covers the product of groups $G'_j = G_j/\Gamma_j$, and since $\mathcal{M}(\Sigma, \prod_{j=1}^s G'_j) = \prod_{j=1}^s \mathcal{M}(\Sigma, G'_j)$, one obtains a finite covering,

$$\mathcal{M}(\Sigma, G') \to \mathcal{M}(\Sigma, G'_1) \times \cdots \times \mathcal{M}(\Sigma, G'_s). \tag{19}$$

3.3. **Fixed point manifolds.** Since the G-action on $M'(\Sigma) = (G')^{2h}$ is the conjugation action on each factor, the fixed point set for elements $g \in G$ is the direct product of centralizers $G'_{g'}$ where g' is the image of g. In contrast to the simply connected case, these centralizers may now be disconnected. For example, if $G' = \mathrm{SO}(3)$, the centralizer of the group element given by rotation by π about the z-axis is $\mathrm{O}(2) \subset \mathrm{SO}(3)$.

Lemma 3.2. *Let t' be the image of a regular element $t \in T^{\mathrm{reg}}$, and $G'_{t'}$ its centralizer in G'. Then T' is a normal subgroup of $G'_{t'}$ and $G'_{t'}/T' = W_{t'}$, the stabilizer group of t' under the action of the Weyl group.*

Proof. The image $g' \in G'$ of an element $g \in G$ is fixed under t' if and only if $[t, g] \in \Gamma$. In this case, Ad_t fixes the maximal torus $\mathrm{Ad}_g(T)$ pointwise, hence $\mathrm{Ad}_g(T) = T$ since t is a regular element. Therefore $g \in N_G(T)$ and $g' \in N_{G'}(T')$. □

The stabilizer group $W_{t'}$ can be re-interpreted as follows. Define an injective group homomorphism

$$\varphi:\ Z(G) \to W,\ \gamma \mapsto w_\gamma \tag{20}$$

where $w_\gamma \in W$ is the unique element with $\gamma \exp(\mathfrak{A}) = w_\gamma(\exp(\mathfrak{A}))$. Since the action of $Z(G)$ on T commutes with the Weyl group action, it induces an action on the alcove $\mathfrak{A} = T/W$. The following is well-known:

Lemma 3.3. *For any $\xi \in \mathrm{int}(\mathfrak{A})$, the homomorphism φ restricts to an isomorphism from the stabilizer group Γ_ξ onto $W_{t'}$, where $t' = \exp_{G'}(\xi)$. In particular, $W_{t'}$ is abelian.*

Proof. We describe the inverse map. Let $t = \exp_G(\xi)$. Given $w \in W_{t'}$, the element $\gamma = t^{-1} w(t)$ lies in Γ. The equation $\gamma t = w(t)$ means, by definition of the action of Γ on $\mathfrak{A}$, that $\gamma \in \Gamma_\xi$. □

From Lemmas 3.2 and 3.3 we obtain the following description of the fixed point sets. For any $\gamma \in \Gamma^{2h}$, let $F_\gamma \subset M'(\Sigma) = (G')^{2h}$ be the pre-image of $\varphi(\gamma) \in W^{2h}$ under the homomorphism

$$(N_{G'}(T'))^{2h} \to (N_{G'}(T')/T')^{2h} = W^{2h}.$$

Then the fixed point set $M'(\Sigma)^t$ for $t = \exp(\xi)$, $\xi \in \mathrm{int}(\mathfrak{A})$ is the union,

$$M'(\Sigma)^t = \coprod_{\gamma \in \Gamma_\xi} F_\gamma. \tag{21}$$

Since the element $B_c^\sharp(\rho) \in \mathrm{int}(\mathfrak{A})$ is fixed under the action of Γ (see [9, Theorem 1.22]), all of the sub-manifolds F_γ arise as fixed point manifolds of $t = \exp(B_c^\sharp(\rho))$. In particular, ω pulls back to a symplectic form on each F_γ.

Proposition 3.4. *For all $\gamma \in \Gamma^{2h}$, the symplectic volume of F_γ is equal to the Riemannian volume of $(T')^{2h}$ with respect to the inner product B defining ω.*

Proof. We have $M'(\Sigma_h^1) = M'(\Sigma_1^1) \circledast \ldots \circledast M'(\Sigma_1^1)$, and the fixed point set is just a fusion product of the fixed point sets of the factors. Because of Lemma B.2 it is enough to consider the case $h = 1$. Thus $M'(\Sigma_1^1)$ is the G-valued Hamiltonian G-space $G' \times G'$ with G acting by conjugation, moment map $(a', b') \mapsto aba^{-1}b^{-1}$, and 2-form

$$\omega_{(a',b')} = \tfrac{1}{2}\big(B(a^*\theta, b^*\overline{\theta}) + B(a^*\overline{\theta}, b^*\theta) - B((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta})\big) \tag{22}$$

Using left-trivialization of the tangent bundle to identify $T_{(a',b')}M'(\Sigma_1^1) \cong \mathfrak{g} \oplus \mathfrak{g}$, and using the metric B to identify skew-symmetric 2-forms with skew-symmetric matrices, the 2-form ω is given at (a', b') by the block matrix,

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{23}$$

where C_{ij} are the following endomorphisms of $\mathfrak{g}$,

$$\begin{aligned} C_{11} &= \tfrac{1}{2}(\mathrm{Ad}_b - \mathrm{Ad}_{b^{-1}}), \\ C_{12} &= \tfrac{1}{2}(-1 + \mathrm{Ad}_b + \mathrm{Ad}_{a^{-1}} + \mathrm{Ad}_{ba^{-1}}), \\ C_{21} &= \tfrac{1}{2}(1 - \mathrm{Ad}_a - \mathrm{Ad}_{b^{-1}} - \mathrm{Ad}_{ab^{-1}}), \\ C_{22} &= \tfrac{1}{2}(\mathrm{Ad}_{a^{-1}} - \mathrm{Ad}_a). \end{aligned}$$

Suppose now that $F \subset G^2$ is the fixed point manifold labeled by $(\gamma_1, \gamma_2) \in \Gamma^2$. The 2-form ω_F on F is given at any point $(a', b') \in F$ by the endomorphism $C^T = C|_{t\times t}$, with $\mathrm{Ad}_a, \mathrm{Ad}_b$ becoming the Weyl group action of $w_1 = \varphi(\gamma_1)$, $w_2 = \varphi(\gamma_2)$ on $\mathfrak{t}$. Using that w_1, w_2 commute one verifies that

$$C_{11}^T C_{22}^T - C_{12}^T C_{21}^T = 1,$$

showing $\det(C^T) = 1$. Since the top degree part of $\exp(\omega_F)$ is equal to the standard volume form on $(T')^2$, times the Pfaffian of C^T, this shows $\mathrm{Vol}(F) = \mathrm{Vol}(T')^2$. □

Later we will need the following remarkable fact.

Lemma 3.5. *The identity level set $(\Phi')^{-1}(e)$ intersects each of the sub-manifolds F_γ.*

Proof. It suffices to consider the case of the 1-punctured torus $\Sigma = \Sigma_1^1$. Let $\gamma = (\gamma_1, \gamma_2)$ be given. By Proposition C.1 from Appendix C, it is possible to choose commuting lifts $g_1, g_2 \in N_G(T)$ of $w_1 = \varphi(\gamma_1),\ w_2 = \varphi(\gamma_2)$. Then $(g_1', g_2') \in F_\gamma$ and $\Phi'(g_1', g_2') = [g_1, g_2] = e$. □

4. Pre-quantization of $\mathcal{M}'(\Sigma)$

The purpose of this section is to prove the following theorem.

Theorem 4.1. (a) *If $L' \to \mathcal{M}'(\Sigma)$ is an $\widehat{LG}$-equivariant line bundle at level l, then L' admits a connection whose curvature is the two-form defined by B_l.*

(b) *Any two $\widehat{LG}$-equivariant level l line bundles differ by the pull-back of a flat line bundle on $(G')^{2h}$ defined by a character of Γ^{2h}.*
(c) *$\mathcal{M}'(\Sigma)$ admits a level $l = (l_1, \ldots, l_s)$ line bundle if each l_j is a multiple of the greatest common divisor of $2c_j$ and $\#\Gamma_j^2$.*
(d) *For each such l, there is a unique level l line bundle with the property that for all $t \in T^{\mathrm{reg}}$, and all fixed points $\hat{m}' \in \mathcal{M}'(\Sigma)^t$ in the zero level set of the moment map, the action of t on the fiber at $\hat{m}'$ is trivial.*

Example 4.2. We consider the adjoint groups $G' = G/Z(G)$ for the simple groups. From the table in Appendix A we obtain the following smallest positive levels l_0 for which our criterion guarantees a level l_0 line bundle over $\mathcal{M}'(\Sigma)$.
A'_N : $l_0 = N + 1$ for N even, and $l_0 = 2(N + 1)$ for N odd.
B'_N : $l_0 = 2$.
C'_N : $l_0 = 2$ for N even, $l_0 = 4$ for N odd.
D'_N : $l_0 = 4$ for N even, $l_0 = 16$ for $N = 5, 9, \ldots$, $l_0 = 8$ for $N = 7, 11, \ldots$.
E'_6 : $l_0 = 3$.
E'_7 : $l_0 = 4$.

The condition for the adjoint group of $\mathrm{SU}(n) = A_{n+1}$ coincides with the condition from Beauville-Laszlo-Sorger [6, 8]. Recent results of Fuchs–Schweigert [16] (also Teleman [28]) suggest that these conditions are not optimal.

Proof of Theorem 4.1. Recall that $\mathcal{M}(\Sigma)$ carries a unique equivariant line bundle $L^{(l)}(\Sigma)$ at level l [21, Proposition 3.12]. Moreover, this line bundle carries a connection $\nabla^{(l)}$ whose curvature is the two-form defined by B_l. The pull-back of L' to $\mathcal{M}(\Sigma)$ is isomorphic to $L^{(l)}(\Sigma)$. The average of $\nabla^{(l)}$ over Γ^{2h} descends to a connection on L'.

For part (b), we may assume L' is a line bundle at level $l = 0$. The quotient $L'/\Omega G$ is a G-equivariant line bundle over $M'(\Sigma) = (G')^{2h}$. Since $H^2((G')^{2h}, \mathbb{R}) = H^2_G((G')^{2h}, \mathbb{R}) = 0$, every G-line bundle over $M'(\Sigma)$ is isomorphic to a flat line bundle defined by a character of Γ^{2h} (see e.g. [23]).

The discussion in Subsection 4.1 (resp. 4.2) shows that there exists a level l_j line bundle over $\mathcal{M}(\Sigma, G'_j)$ if l_j is a multiple of $2c_j$ (resp. $\#\Gamma_j^2$) with the property given in part (d) of the theorem. The theorem follows by taking tensor products and pull-backs under the map (19). □

4.1. **Canonical bundles.** The anti-canonical line bundles over each factor $\mathcal{M}(\Sigma, G'_j)$ in the decomposition (19) are at level $2c_j$. It remains to show that for any $t \in T^{\mathrm{reg}}$ and all fixed points $\hat{m}' \in \mathcal{M}'(\Sigma, G_j)^t$ in the zero level set of the moment map , the eigenvalue $\kappa(\hat{m}', t)$ for the action of t on the fiber at $\hat{m}'$ is trivial. We will prove a stronger statement, which we will need later in the fixed point theorem. Recall from [FP], Section 4.4 the definition of the square root $\kappa(\hat{m}', t)^{1/2}$.

Proposition 4.3. *The square root for the eigenvalue of t on the fiber of the canonical line bundle at $\hat{m}'$ is given by $\kappa(\hat{m}', t)^{1/2} = (-1)^{h\#\mathfrak{R}_+}$.*

Proof. Let $m' \in (\Phi')^{-1}(e)$ be the image of $\hat{m}'$ in $M'(\Sigma)^t$, and F_γ the connected component containing m'. By (29) of [FP], the square root $\kappa(\hat{m}', t)^{1/2}$ coincides with the square root of the eigenvalue of the action of t on the symplectic vector space $T_{m'}M'(\Sigma)$.

As in the simply connected case, it is enough to consider the case of the 1-punctured torus $\Sigma = \Sigma_1^1$. We adopt the notation and definitions (22),(23) from the proof of Proposition 3.4.

Suppose $\mathfrak{k} \subset \mathfrak{g}$ is invariant under both of the commuting operators Ad_a and Ad_b. From the formula for $\omega_{(a',b')}$ it follows that

$$\mathfrak{g} \oplus \mathfrak{g} = (\mathfrak{k} \oplus \mathfrak{k}) \oplus (\mathfrak{k}^\perp \oplus \mathfrak{k}^\perp)$$

as an ω-orthogonal direct sum. Take $\mathfrak{k}$ to be the direct sum of joint eigenspaces for $\mathrm{Ad}_a, \mathrm{Ad}_b$ with eigenvalues $(-1, 1)$, $(-1, -1)$ or $(1, -1)$. Then

$$C_{11}|_\mathfrak{k} = C_{22}|_\mathfrak{k} = 0, \quad C_{12}|_\mathfrak{k} = -\mathrm{Id} = -C_{21}|_\mathfrak{k}.$$

Hence ω restricts to *minus* the standard symplectic form on $\mathfrak{k} \oplus \mathfrak{k}$. Therefore the square root for the action on the canonical bundle $\det_\mathbb{C}^{-1}(\mathfrak{k} \oplus \mathfrak{k})$ is $(-1)^{(\dim \mathfrak{k} - \dim(\mathfrak{k})^t)/2}$.

Consider on the other hand the $\mathfrak{k}^\perp \subset \mathfrak{g}$ contribution. We claim that the symplectic form C on $\mathfrak{k}^\perp \oplus \mathfrak{k}^\perp$ is homotopic to the standard symplectic form $C_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$, by a homotopy of symplectic forms which are invariant under the action of t. This will imply that the square root for the action on $\det_\mathbb{C}^{-1}(\mathfrak{k}^\perp \oplus \mathfrak{k}^\perp)$ is $(-1)^{(\dim \mathfrak{k}^\perp - \dim(\mathfrak{k}^\perp)^t)/2}$, and finally

$$\kappa(\hat{m}', t)^{1/2} = (-1)^{(\dim \mathfrak{g} - \dim \mathfrak{t})/2} = (-1)^{\#\mathfrak{R}_+}.$$

For the rest of this paragraph we consider the restriction $C^\perp$ of C to $\mathfrak{k}^\perp \oplus \mathfrak{k}^\perp$. For $s \in [0, 1]$ let $C_s^\perp = (1 - s)C_0^\perp + sC^\perp$. It suffices to show that $\det(C_s^\perp) \neq 0$ for $s \in [0, 1]$. We calculate

$$\begin{aligned} \det(C_s^\perp) &= \det \begin{pmatrix} sC_{11}^\perp & 1 + s(C_{12}^\perp - 1) \\ -1 + s(C_{21}^\perp + 1) & sC_{22}^\perp \end{pmatrix} \\ &= \det \left(s^2(C_{11}^\perp C_{22}^\perp - C_{12}^\perp C_{21}^\perp - C_{12}^\perp + C_{21}^\perp + 1) - s(C_{21}^\perp - C_{12}^\perp + 2) + 1 \right) \\ &= \det \left((s^2 - s)(2 + (C_{21}^\perp - C_{12}^\perp)) + 1 \right) \end{aligned}$$

For $s \in [0, 1]$ one has $0 \geq s^2 - s \geq -\frac{1}{4}$. Hence it suffices to show that all eigenvalues of the symmetric operator $C_{21}^\perp - C_{12}^\perp$ are strictly smaller than 2. But

$$C_{21}^\perp - C_{12}^\perp = 1 - \tfrac{1}{2}(\mathrm{Ad}_a + \mathrm{Ad}_{a^{-1}} + \mathrm{Ad}_b + \mathrm{Ad}_{b^{-1}} + \mathrm{Ad}_{a\,b^{-1}} + \mathrm{Ad}_{b\,a^{-1}}).$$

On the joint eigenspace of $\mathrm{Ad}_a, \mathrm{Ad}_b$ with eigenvalue $(1, 1)$, this is strictly negative. On the orthogonal complement of the eigenspaces for eigenvalue pairs $(-1, -1), (-1, 1), (1, -1)$ and $(1, 1)$, the operators $\mathrm{Ad}_a, \mathrm{Ad}_b$ are represented by 2-dimensional rotations by angles ψ_a, ψ_b not both of which are multiples of π. On any such 2-plane, the operator $C_{21}^\perp - C_{12}^\perp$ becomes $(1 - \cos(\psi_a) - \cos(\psi_b) - \cos(\psi_a - \psi_b)) \,\mathrm{Id}$. The claim follows since $1 - \cos(\psi_a) - \cos(\psi_b) - \cos(\psi_a - \psi_b) < 2$. ∎

4.2. **Central extension of the gauge group $\mathcal{G}'(\Sigma)$.** In order to define a pre-quantum bundle over $\mathcal{M}(\Sigma, G')$ at level $\#\Gamma^2$, we imitate the construction from the simply-connected case. The definition (3) of the cocycle carries over to the non-simply connected case, and defines a central extension $\widehat{\mathcal{G}}'(\Sigma)$ of the gauge group.

Proposition 4.4. *Suppose each k_j is a multiple of $\#\Gamma^2$. Then there is a canonical trivialization of the central extension $\widehat{\mathcal{G}}'(\Sigma)$ over $\mathcal{G}'_\partial(\Sigma)$.*

Proof. Let $\overline{\Sigma}$ be the surface obtained from Σ by capping off the boundary component. Let $\mathcal{G}'_c(\Sigma) \subset \mathcal{G}'_\partial(\Sigma)$ denote the kernel of the restriction map $\mathcal{G}'(\overline{\Sigma}) \to \mathcal{G}'(\overline{\Sigma} - \Sigma)$. As explained in [22, p. 431], it suffices to show that the cocycle is trivial over the subgroup $\mathcal{G}'_c(\Sigma)$ of $\mathcal{G}'_\partial(\Sigma)$. We would like to define a map $\alpha : \mathcal{G}'_c(\Sigma) \to \mathrm{U}(1)$ with coboundary condition

$$\alpha(g_1' g_2') = \alpha(g_1')\alpha(g_2')c(g_1', g_2') \tag{24}$$

Given $g' \in \mathcal{G}'_c(\Sigma)$, let $\gamma \in \Gamma^{2h} \cong \mathrm{Hom}(\pi_1(\Sigma), \Gamma)$ be its image under the map (14). The element γ defines a covering Σ^γ of Σ by Γ, which is a possibly disconnected surface with $\#\Gamma$ boundary components. Choose a base point of Σ^γ mapping to the base point of Σ, and let $\pi^\gamma : \Sigma^\gamma \to \Sigma$ be the covering projection. The pull-back $(\pi^\gamma)^* g'$ admits a unique lift $g \in \mathcal{G}(\Sigma^\gamma)$, with $g = e$ at the base point and $g(\lambda \cdot x) = \lambda g(x)$ for all $\lambda \in \Gamma$, $x \in \Sigma^\gamma$. Note that g is constant along each of the boundary circles of Σ^γ.

The covering $\Sigma^\gamma \to \Sigma$ extends to a covering $\overline{\Sigma}^\gamma \to \overline{\Sigma}$ over the capped-off surface. Extend $g \in \mathcal{G}(\Sigma^\gamma)$ first to $\overline{\Sigma}^\gamma$ by the constant extension on the capping disks, and then further to a map $\overline{g} \in \mathcal{G}(\Sigma^\gamma \times [0, 1])$, in such a way that the extension is trivial on $\overline{\Sigma}^\gamma \times \{1\}$. Define

$$\alpha(g') = \exp(2\pi i \frac{1}{\#\Gamma} \int_{\overline{\Sigma}^\gamma \times [0,1]} \overline{g}^* \eta). \tag{25}$$

Since

$$\frac{1}{\#\Gamma^2}[\eta] \in H^2(G, \mathbb{Z}) \tag{26}$$

by our assumption on the level, the definition (25) is independent of the choice of $\overline{g}$. It remains to verify the coboundary condition. Let $g_1', g_2' \in \mathcal{G}'_c(\Sigma)$ and $\gamma_1, \gamma_2 \in \Gamma^{2h}$ their images. Define $\overline{\Sigma}^{\gamma_1,\gamma_2}$ as the fibered product of $\overline{\Sigma}^{\gamma_1}$ and $\overline{\Sigma}^{\gamma_2}$. Let π_1, π_2, π_{12} denote the projections onto $\overline{\Sigma}^{\gamma_1}, \overline{\Sigma}^{\gamma_2}, \overline{\Sigma}^{\gamma_1\gamma_2}$ respectively. Let $g_{12}' = g_1' g_2' \in \mathcal{G}'_c(\Sigma)$. We have canonical lifts $g_1 \in \mathcal{G}(\Sigma^{\gamma_1}), g_2 \in \mathcal{G}(\Sigma^{\gamma_2})$ and $g_{12} \in \mathcal{G}(\Sigma^{\gamma_1\gamma_2})$. Choose extensions $\overline{g}_1, \overline{g}_2$ and $\overline{g_{12}}$ as above. Then both $\pi_{12}^* \overline{g_{12}}$ and $\pi_1^* \overline{g_1}\ \pi_2^* \overline{g_2}$ are equal on $\overline{\Sigma}^{\gamma_1,\gamma_2} \times \{0\}$. Using (26),

$$\begin{aligned}
\frac{1}{\#\Gamma} \int_{\overline{\Sigma}^{\gamma_1\gamma_2} \times [0,1]} \overline{g_{12}}^* \eta &= \frac{1}{\#\Gamma^2} \int_{\Sigma^{\gamma_1,\gamma_2} \times [0,1]} (\pi_{12}^* \overline{g_{12}})^* \eta \\
&= \frac{1}{\#\Gamma^2} \int_{\Sigma^{\gamma_1,\gamma_2} \times [0,1]} (\pi_1^* \overline{g_1}\ \pi_2^* \overline{g_2})^* \eta \mod \mathbb{Z}
\end{aligned}$$

For the last term we compute, using the property of the 3-form η under group multiplication $\mathrm{Mult}_G : G \times G \to G$,

$$\mathrm{Mult}_G^* \eta = \mathrm{pr}_1^* \eta + \mathrm{pr}_2^* \eta + \mathrm{d}\big(\tfrac{1}{2}B(\mathrm{pr}_1^* \theta, \mathrm{pr}_2^* \overline{\theta})\big)$$

(where $\mathrm{pr}_j : G \times G \to G$ are projections to the respective G-factor),

$$\begin{aligned}
&\frac{1}{\#\Gamma^2}\int_{\Sigma^{\gamma_1,\gamma_2}\times[0,1]} (\pi_1^*\overline{g_1}\ \pi_2^*\overline{g_2})^*\eta \\
&= \frac{1}{\#\Gamma^2}\int_{\Sigma^{\gamma_1,\gamma_2}\times[0,1]} \big(\pi_1^*\overline{g_1}^*\eta + \pi_2^*\overline{g_2}^*\eta + \tfrac{1}{2}\mathrm{d}B(\pi_1^*\overline{g_1}^*\theta, \pi_2^*\overline{g_2}^*\overline{\theta})\big) \\
&= \frac{1}{\#\Gamma}\int_{\Sigma^{\gamma_1}\times[0,1]} \pi_1^*\overline{g_1}^*\eta + \frac{1}{\#\Gamma}\int_{\Sigma^{\gamma_2}\times[0,1]} \pi_2^*\overline{g_2}^*\eta + \tfrac{1}{2}\int_\Sigma B((g_1')^*\theta, (g_2')^*\overline{\theta}).
\end{aligned}$$

This shows that the cocycle condition holds and completes the proof. □

As a consequence, if k is a multiple of $\#\Gamma^2$ the moduli space $\mathcal{M}'(\Sigma)$ carries a line bundle

$$L'(\Sigma) = (\mathcal{A}_{\mathrm{flat}}(\Sigma) \times \mathbb{C})/\mathcal{G}'_\partial(\Sigma)$$

equipped with an action of $\widehat{\mathcal{G}}'(\Sigma)/\mathcal{G}'_\partial(\Sigma)$ where the central circle acts with weight 1. The map $LG \to L_0G' = \mathcal{G}'(\Sigma)/\mathcal{G}'_\partial(\Sigma)$ induces a map from the central extension at level k, $\widehat{LG} \to \widehat{\mathcal{G}}'(\Sigma)/\mathcal{G}'_\partial(\Sigma)$ which restricts to the identity map on the central circle and has kernel Γ. The upshot is that $L'(\Sigma)$ is a level k line bundle over $\mathcal{M}'(\Sigma)$, where $\Gamma \subset G \subset \widehat{LG}$ acts trivially. It remains to verify property (d) of Theorem 4.1.

Lemma 4.5. *Let $t \in T^{\mathrm{reg}} \subset LG$ a regular element, and $\hat{m}' \in \mathcal{M}'(\Sigma)$ a fixed point for the action of t in the zero level set of the moment map. Then t acts trivially on the fiber $L'(\Sigma)_{\hat{m}'}$.*

Proof. If t is contained in the identity component of the stabilizer $LG_{\hat{m}'}$, then the formula follows from the pre-quantum condition.

Our strategy is to reduce to this case, using finite covers. Since t fixes $\hat{m}'$, there exists a flat connection A mapping to $\hat{m}'$ and a gauge transformation $g' \in \mathcal{G}'(\Sigma)$ restricting to t' on the boundary such that $g' \cdot A = A$. The eigenvalue for the action of t on $L'(\Sigma)$ is equal to the eigenvalue for the action of g' on the fiber $\{A\} \times \mathbb{C}$ of the level k pre-quantum line bundle over $\mathcal{A}(\Sigma)$.

Let $\gamma \in \Gamma^{2h}$ be the image of g' under the map $\mathcal{G}'(\Sigma) \to \Gamma^{2h}$. As in the proof of Proposition 4.4 let $\pi^\gamma : \Sigma^\gamma \to \Sigma$ the covering defined by γ, and $g \in \mathcal{G}(\Sigma^\gamma)$ be the lift of the pull-back $(\pi^\gamma)^* g'$. Clearly g fixes $A^\gamma := (\pi^\gamma)^* A$.

The pull-back map $\mathcal{G}'(\Sigma) \to \mathcal{G}'(\Sigma^\gamma)$ lifts to the central extensions if one changes the level. Indeed, the cocycle for the central extension at level k pulls back to the cocycle for the extension at level $k/\#\Gamma$. The map is compatible with the given trivializations over $\mathcal{G}'_\partial(\Sigma)$ and $\mathcal{G}'_\partial(\Sigma^\gamma)$, hence they define a commutative diagram

$$\begin{tikzcd} \mathcal{G}'_\partial(\Sigma) \arrow[r] \arrow[d] & \mathcal{G}'_\partial(\Sigma^\gamma) \arrow[d] \\ \widehat{\mathcal{G}}'(\Sigma)^{(k)} \arrow[r] & \widehat{\mathcal{G}}'(\Sigma^\gamma)^{(\frac{k}{\#\Gamma})} \end{tikzcd} \tag{27}$$

It follows that the eigenvalue of g' on the fiber $\{A\} \times \mathbb{C}$ is equal to that of g on the fiber $\{A^\gamma\} \times \mathbb{C}$. By Lemma 4.6 below, g is contained in the identity component of $\mathcal{G}(\Sigma^\gamma)$. Hence its action on the line-bundle is determined by the moment map (Kostant's formula), and hence is trivial since A^γ pulls back to the zero connection on $\partial\Sigma$. □

In the proof we used the following Lemma.

Lemma 4.6. *Let Σ_h^r be any oriented surface (possibly with boundary), and $g \in \mathcal{G}(\Sigma_h^r)$, $A \in \mathcal{A}_{\mathrm{flat}}(\Sigma_h^r)$ with $g \cdot A = A$. Suppose that for some point $x \in \Sigma_h^r$, $g(x)$ is a regular element. Then g is contained in the identity component of the stabilizer $\mathcal{G}(\Sigma_h^r)_A$.*

Proof. It is well-known that evaluation at x, $\mathcal{G}(\Sigma_h^r) \to G$ restricts to an injective map $\mathcal{G}(\Sigma_h^r)_A \hookrightarrow G$. The image G_A of this map is the centralizer of the image Hol_A of the homomorphism $\pi_1(\Sigma_h^r, x) \to G$ defined by parallel transport using A. Since $\mathrm{Hol}_A \subset G_{g(x)}$ and $G_{g(x)}$ is a torus, $G_{g(x)} \subset G_A$. It follows that g is contained in the identity component of $\mathcal{G}(\Sigma_h^r)_A$. □

4.3. **The phase factor.** Using Theorem 4.1 and Proposition 4.3 we compute the phase factor appearing in the fixed point formula.

Proposition 4.7. *Let $k = (k_1, \ldots, k_s)$, where each k_j is a positive multiple of the greatest common divisor of c_j and $\#\Gamma_j^2$. Let $L'(\Sigma) \to \mathcal{M}'(\Sigma)$ be the pre-quantum line bundle at level k corresponding to a character $\phi: \Gamma^{2h} \to U(1)$ (cf. Theorem 4.1). Let $t \in T^{\mathrm{reg}}$, and $F_\gamma \subset M'(\Sigma)$ any fixed point component. The phase factor $\zeta_{F_\gamma}(t)^{1/2}$ for the action on the $\mathcal{L}_{F_\gamma}$ is given by*

$$\zeta_{F_\gamma}(t)^{1/2} = \phi(\gamma)(-1)^{h\#\mathfrak{R}_+}.$$

Proof. Let $m' \in F_\gamma \cap \Phi^{-1}(e)$, and $\hat{m}' \in \Phi^{-1}(0)$ the unique element projecting to m'. By Theorem 4.1, the weight for the action of t on $L'(\Sigma)_{\hat{m}'}$ is $\phi(\gamma)$. On the other hand, by Proposition 4.3, $\kappa(\hat{m}', t)^{1/2} = (-1)^{h\#\mathfrak{R}_+}$, which completes the proof. □

5. Verlinde formulas for non simply-connected groups

In this section we apply the fixed point formula to the Hamiltonian LG-manifold $\mathcal{M}'(\Sigma) = \mathcal{M}(\Sigma, G')$. The main result of the paper is

Theorem 5.1. *Let $k = (k_1, \ldots, k_s)$, where each k_j is a positive multiple of the greatest common divisor of c_j and $\#\Gamma_j^2$. Let $L'(\Sigma) \to \mathcal{M}'(\Sigma)$ be the pre-quantum line bundle at* *level k corresponding to a character $\phi \in \mathrm{Hom}(\Gamma^{2h}, \mathrm{U}(1))$, as in Theorem 4.1. For any $\mu \in \Lambda_k^*$, the* Spin_c*-index of the symplectic quotient $\mathcal{M}'(\Sigma)_\mu$ at μ is given by the formula,*

$$\chi(\mathcal{M}'(\Sigma)_\mu) = \frac{(\#T_{k+c})^{h-1}}{\#\Gamma^{2h}} \sum_{\lambda \in \Lambda_k^*} \epsilon(\phi, \lambda) \; \frac{\#\Gamma_\lambda^{2h}}{|J(t_\lambda)|^{2h-2}} \; \chi_\mu(t_\lambda)^*. \tag{28}$$

Here $\epsilon(\phi, \lambda) = 1$ if ϕ restricts to the trivial homomorphism on Γ_λ^{2h}, and 0 otherwise.

Proof. Recall $t_\lambda = \exp(B_{k+c}^\sharp(\lambda+\rho))$. Since $B_c^\sharp(\rho)$ is fixed by Γ, the stabilizer of $B_{k+c}^\sharp(\lambda+\rho)$ is equal to the stabilizer of λ, using the embedding $\Lambda_k^* \to \mathfrak{A}$ induced by B_k. By (21), the fixed point components for the action of t_λ on $M'(\Sigma)$ are the sub-manifolds F_γ with $\gamma \in \Gamma_\lambda^{2h}$. Since F_γ is diffeomorphic to T^{2h}, we have $\hat{A}(F_\gamma) = 1$. The normal bundle ν_{F_γ} is t_λ-equivariantly isomorphic to the constant bundle with fiber $(\mathfrak{g}/\mathfrak{t})^{2h}$. Indeed, left translation by any $m = (g_1', \ldots, g_{2h}') \in F_\gamma$ on $(G')^{2h}$ commutes with the action of t_λ, and induces a t_λ-equivariant isomorphism of $\nu_{F_\gamma}(m)$ with $(\mathfrak{g}/\mathfrak{t})^{2h}$. Therefore,

$$\mathcal{D}_{\mathbb{R}}(\nu_{F_\gamma}, t_\lambda) = J(t_\lambda)^{2h} = (-1)^{h\#\mathfrak{R}_+} \, |J(t_\lambda)|^{2h}. \tag{29}$$

Since the line bundle $\mathcal{L}$ is pre-quantum at level $2(k+c)$, the integral $\int_{F_\gamma} \exp(\frac{1}{2}c_1(\mathcal{L}_F))$ is equal to the symplectic volume with respect to the inner product B_{k+c}. By Proposition 3.4 the symplectic volume is equal to the Riemannian volume. Since $T' = T/\Gamma$, this shows

$$\int_{F_\gamma} \exp(\tfrac{1}{2}c_1(\mathcal{L}_{F_\gamma})) = \frac{\mathrm{Vol}_{B_{k+c}}(T^{2h})}{(\#\Gamma)^{2h}} = \frac{(\#T_{k+c})^h}{(\#\Gamma)^{2h}}. \tag{30}$$

Miraculously, the contribution is independent of γ. The fixed point formula hence gives the following expression for the Spin_c-index of a symplectic quotient $\mathcal{M}'(\Sigma)_\mu$, for $\mu \in \Lambda_k^*$:

$$\chi(\mathcal{M}'(\Sigma)_\mu) = \frac{(\#T_{k+c})^{h-1}}{(\#\Gamma)^{2h}} \sum_{\lambda \in \Lambda_k^*} \frac{\chi_\mu(t_\lambda)^*}{|J(t_\lambda)|^{2h-2}} \; (-1)^{h\#\mathfrak{R}_+} \sum_{\gamma \in \Gamma_\lambda^{2h}} \zeta_{F_\gamma}(t_\lambda)^{1/2}. \tag{31}$$

By the computation of the phase factor in Proposition 4.7,

$$\sum_{\gamma \in \Gamma_\lambda^{2h}} \zeta_{F_\gamma}(t_\lambda)^{1/2} = (-1)^{h\#\mathfrak{R}_+} \sum_{\gamma \in \Gamma_\lambda^{2h}} \phi(\gamma) = (-1)^{h\#\mathfrak{R}_+} \epsilon(\phi, \lambda) \; \#\Gamma_\lambda^{2h}$$

which completes the proof. ☐

It is remarkable that the right-hand side of (28) always produces an integer, under the given assumptions on k.

Remark 5.2 (Contribution from the exceptional element). Suppose G is simple. As mentioned above, the element $\xi_0 = B_c^\sharp(\rho)$ is always a fixed point for the action of Γ. The element $\lambda_0 = B_k^\flat(\xi_0) = \frac{k}{c}\rho$ is a weight if and only if $c|k$. By [9, Proposition 1.10], we have $|J(\exp\xi_0)|^2 = \#T_c = (1+\frac{k}{c})^{-\dim T}\#T_{k+c}$. Hence the contribution of λ_0 is

$$(1+\frac{k}{c})^{(h-1)\dim T}\epsilon(\phi,\lambda_0)\chi_\mu(t_{\lambda_0})^*.$$

Since $\Gamma_{\lambda_0}=\Gamma$, the factor $\epsilon(\phi,\lambda_0)$ is 1 if ϕ is trivial and 0 if ϕ is non-trivial. A theorem of Kostant [19, Theorem 3.1] asserts that the character value $\chi_\mu(t_{\lambda_0})$ is either $-1,0$, or 1. More precisely, Kostant proves [19, Lemma 3.6] that

$$\chi_\mu(t_{\lambda_0}) = \begin{cases} (-1)^{l(w)} & \text{if } \exists w\in W:\ w(\mu+\rho)-\rho\in\overline{R}' \\ 0 & \text{otherwise}\end{cases}$$

where $\overline{R}'\subset\overline{R}$ is the lattice generated by all $cm_\alpha\alpha$, where $m_\alpha = ||\alpha_0||^2/||\alpha||^2\in\{1,2,3\}$. In particular, for simply laced groups $\overline{R}'=c\overline{R}$.

5.1. **The case $G'=\mathrm{PSU}(p)$.** Let us specialize the above formulas to $G=\mathrm{SU}(p)$ and $\Gamma=Z(G)=\mathbb{Z}_p$ so that $G'=\mathrm{PSU}(p)$. In this case Theorem 5.1 reduces to the formulas of Pantev [24] ($p=2$), and Beauville [6] ($p\geq 3$ prime), as follows. Recall $c=p$ and that for p odd, $\mathcal{M}'(\Sigma)$ is pre-quantizable for k any multiple of p. The alcove $\mathfrak{A}$ is a simplex with p vertices, and $\Gamma=\mathbb{Z}_p$ acts by rotation inducing a cyclic permutation of the vertices. If p is prime, the only point of $\mathfrak{A}$ with non-trivial stabilizer is the center $\xi_0=B_c^\sharp(\rho)$ of the alcove. By Remark 5.2, the fixed point contribution for the exceptional element is

$$(1+\frac{k}{p})^{(p-1)(h-1)}\epsilon(\phi,\lambda_0)\ \chi_\mu(t_{\lambda_0})^*.$$

Let $\chi(\mathcal{M}(\Sigma)_\mu)$ be the index for the moduli space of G-connections. Its fixed point contributions are exactly as for $M'(\Sigma)_\mu$, except for the overall factor $(\#\Gamma)^{-2h}=p^{-2h}$ and a different weight for the exceptional element t_{λ_0}. This shows,

$$\chi(\mathcal{M}'(\Sigma)_\mu) = p^{-2h}\chi(\mathcal{M}(\Sigma)_\mu) + (\epsilon(\phi,\lambda_0)-p^{-2h})(1+\frac{k}{p})^{(p-1)(h-1)}\chi_\mu(t_{\lambda_0})^*,$$

which (for $\mu=0$ and $\phi=1$) is exactly the formula given by Beauville.

5.2. **The sum over components.** Let $\mu\in\mathfrak{A}$, and $\mathcal{C}'_\mu\subset G'$ the conjugacy class of $\exp_{G'}(\mu)$. The moduli space $\mathcal{M}'(\Sigma,\mathcal{C}'_\mu)$ of flat G'-connections on $\Sigma=\Sigma_h^1$, with holonomy around the boundary in $\mathcal{C}'_\mu$ is a disjoint union

$$\mathcal{M}'(\Sigma,\mathcal{C}'_\mu) = \coprod_{\gamma\in\Gamma/\Gamma_\mu}\mathcal{M}'(\Sigma)_{\gamma\mu}.$$

(For $\mu=0$, the components may be identified with flat G'-bundles over the closed surface Σ_h^0.) The action of Γ on $\mathfrak{A}$ preserves the set $\Lambda_k^*\subset\mathfrak{A}$ of level k weights. If $\mu\in\Lambda_k^*$, the Spin_c-indices for the space $\mathcal{M}'((\Sigma)_{\gamma\mu})$ can be computed using Theorem 5.1. The sum over Γ/Γ_μ can be carried out using the following transformation property of level k characters (see Bismut-Labourie [9, Theorem 1.33])

$$\chi_{\gamma\mu}(t_\lambda) = \gamma^\lambda \chi_\mu(t_\lambda) \quad \mu, \lambda \in \Lambda_k^*, \ \gamma \in \Gamma \tag{32}$$

where γ^λ is defined via the embedding $\Gamma \to T$. We find,

$$\sum_{\gamma \in \Gamma/\Gamma_\mu} \chi_{\gamma\mu}(t_\lambda)^* = \frac{1}{\#\Gamma_\mu} \chi_\mu(t_\lambda)^* \sum_{\gamma \in \Gamma} \gamma^\lambda.$$

The sum $\sum_{\gamma\in\Gamma} \gamma^\lambda$ is equal to $\#\Gamma$ if λ is a weight of $T' = T/\Gamma$, and 0 otherwise. Let $\Lambda_k'^* \subset \Lambda_k^*$ denote those level k weights which are weights for T'. For any $\lambda \in \Lambda_k'^*$, the character χ_λ descends to T'. Therefore

$$\chi(\mathcal{M}'(\Sigma, \mathcal{C}'_\mu)) = \frac{\#T_{k+c}^{h-1}}{\#\Gamma_\mu \#\Gamma^{2h-1}} \sum_{\lambda \in \Lambda_k'^*} \epsilon(\phi, \lambda) \frac{\#\Gamma_\lambda^{2h}}{|J(t_\lambda)|^{2h-2}} \chi_\mu(t_\lambda)^*.$$

Finally, since

$$\chi_\mu(t_\lambda) = \chi_\mu(w_\gamma(t_\lambda)) = \chi_\mu(\gamma t_\lambda),$$

and using that the factor $\epsilon(\phi, \lambda) \in \{0, 1\}$ is invariant under the action of Γ, we can re-write the result as a sum over $\Lambda_k'^*/\Gamma$:

Theorem 5.3. *Let $\mu \in \Lambda_k^*$ and $\mathcal{M}'(\Sigma, \mathcal{C}'_\mu)$ the moduli space of flat G'-connections on Σ_1^h, with holonomy around the boundary in the conjugacy class $\mathcal{C}'_\mu \subset G'$. The* Spin$_c$*-index of $\mathcal{M}'(\Sigma, \mathcal{C}'_\mu)$ is given by*

$$\chi(\mathcal{M}'(\Sigma, \mathcal{C}'_\mu)) = \frac{\#T_{k+c}^{h-1}}{\#\Gamma_\mu \#\Gamma^{2h-2}} \sum_{\lambda \in \Lambda_k'^*/\Gamma} \epsilon(\phi, \lambda) \frac{\#\Gamma_\lambda^{2h-1}}{|J(t_\lambda)|^{2h-2}} \chi_\mu(t_\lambda)^*.$$

Appendix A. The cardinality of T_l

Suppose G is simple and simply connected. For a proof of the following fact, see e.g. Beauville [5, Remark 9.9], Bismut-Labourie [9, Prop. 1.2, 1.3].

Lemma A.1. *For any $l > 0$ the number of elements in T_l is given by*

$$\#T_l = l^{rank(G)} \operatorname{Vol}(T)^2 = l^{rank(G)} \#Z(G) \ \#(\overline{\mathfrak{R}}/\overline{\mathfrak{R}}_{long}).$$

Here $\operatorname{Vol}(T)$ is the Riemannian volume of T computed with respect to the basic inner product, $\overline{\mathfrak{R}}$ is the root lattice and $\overline{\mathfrak{R}}_{long}$ the sub-lattice generated by the long roots.

The table below gives the centers $Z(G)$, dual Coxeter numbers c, and the index of the long-root lattice $\overline{\mathfrak{R}}_{long}$ inside the root lattice $\overline{\mathfrak{R}}$ for all simple, simply connected compact

Lie groups G. All of this information can be read off from tables in Bourbaki [11].

G	A_N	B_N	C_N	D_N	E_6	E_7	E_8	F_4	G_2
c	$N+1$	$2N-1$	$N+1$	$2N-2$	12	18	30	9	4
$Z(G)$	$\mathbb{Z}_{N+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$ (N odd) $\mathbb{Z}_2\times\mathbb{Z}_2$ (N even)	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0	0	0
$\#(\overline{\mathfrak{R}}/\overline{\mathfrak{R}}_{long})$	1	2	2^{N-1}	1	1	1	1	4	3

Appendix B. Fusion of group valued Hamiltonian G-spaces

In this appendix we collect some facts about fusion of Hamiltonian G-spaces with group-valued moment maps.

Theorem B.1. [1, Theorem 6.1] *Let G, H be compact Lie groups, and $(M, \omega, (\Phi_1, \Phi_2, \Psi))$ a group valued Hamiltonian $G\times G\times H$-manifold. Let M_{fus} be the same manifold with diagonal $G\times H$-action, $\Phi_{\text{fus}} = \Phi_1\Phi_2$, and $\omega_{\text{fus}} = \omega - \frac{1}{2}B(\Phi_1^*\theta, \Phi_2^*\overline{\theta})$. Then $(M_{\text{fus}}, \omega_{\text{fus}}, (\Phi_{\text{fus}}, \Psi))$ is a group valued Hamiltonian $G \times H$-manifold.*

The correction term $\frac{1}{2}B(\Phi_1^*\theta, \Phi_2^*\overline{\theta})$ will be loosely referred to as the "fusion term". If $M = M_1 \times M_2$ is a product of two $G \times H_i$-valued Hamiltonian spaces, we also write $M_1 \circledast M_2 := (M_1 \times M_2)_{\text{fus}}$.

Recall that if G is a torus, a space with G-valued moment map is just a symplectic manifold with a multi-valued moment map in the usual sense. Fusion of such spaces changes the symplectic form, but not the volume:

Lemma B.2. *Suppose T is a torus, that $(M, \omega, (\Phi_1, \Phi_2))$ a compact group valued Hamiltonian $T \times T$-space, and $(M_{\text{fus}}, \omega_{\text{fus}}, (\Phi_{\text{fus}}))$ is the group valued Hamiltonian T-space obtained by fusion. Then the symplectic volumes of M and M_{fus} are the same.*

Proof. This is a special case of a result for non-abelian groups proved in [2]. In the abelian case, the following much simpler argument is available. Notice that M with diagonal T-action has moment map $\Phi_{\text{fus}} = \Phi_1\Phi_2$ not only for the fusion form ω_{fus} but also for the original symplectic 2-form ω. Suppose $t \in T$ is a (weakly) regular value of Φ_{fus}, so that $(\Phi_{\text{fus}})^{-1}(t)$ is a smooth sub-manifold and $M_t = (\Phi_{\text{fus}})^{-1}(t)/T$ is an orbifold. Since the pull-back of the 2-form $\frac{1}{2}B(\Phi_1^*\theta, \Phi_2^*\theta)$ to $(\Phi_{\text{fus}})^{-1}(t)$ vanishes, the reduced symplectic forms are the same: $\omega_t = (\omega_{\text{fus}})_t$. It follows that the two Duistermaat-Heckman measures $\mathfrak{m} = \frac{1}{n!}(\Phi_{\text{fus}})_*(|\omega^n|)$ and $\mathfrak{m}_{\text{fus}} = \frac{1}{n!}(\Phi_{\text{fus}})_*(|\omega_{\text{fus}}^n|)$ coincide. Since the symplectic volume is the integral of the Duistermaat-Heckman measure, the proof is complete. $\square$

For any group-valued Hamiltonian G-space, the 2-form ω is non-degenerate on the tangent space at any point in the identity level set. The following Lemma shows that fusion does not change the isotopy class of this symplectic structure. Its proof relies on the notion of *exponential* of a Hamiltonian space [1]: Let $\varpi \in \Omega^2(\mathfrak{g})$ be the image of $\exp^*\eta$ under the homotopy operator $\Omega^\star(\mathfrak{g}) \to \Omega^{\star-1}(\mathfrak{g})$. Then if (M, ω_0, Φ_0) is a

Hamiltonian G-space in the usual sense, with $\Phi_0(M)$ contained in a sufficiently small neighborhood of 0, then (M,ω,Φ) with $\omega=\omega_0+\Phi_0^*\varpi$ and $\Phi=\exp(\Phi_0)$ is a group valued Hamiltonian G-space. Conversely, if (M,ω,Φ) is a group-valued Hamiltonian G-space, any small neighborhood of $\Phi^{-1}(e)$ is obtained in this way. The 2-form ϖ vanishes at 0, so that $\omega_m=(\omega_0)_m$ for points in the zero level set.

Lemma B.3. *Let $(M,\omega,(\Phi_1,\Phi_2,\Psi))$ be a group valued Hamiltonian $G\times G\times H$-space, and $(M_{\mathrm{fus}},\omega_{\mathrm{fus}},(\Phi_{\mathrm{fus}},\Psi))$ its fusion. Let $m\in M$ be a point in the identity level set of (Φ_1,Φ_2,Ψ). The symplectic 2-forms $\omega|_m$ and $\omega_{\mathrm{fus}}|_m$ on T_mM are isotopic through a path of symplectic forms, invariant under the stabilizer group $(G\times H)_m$.*

Proof. We may assume that M is the exponential of a Hamiltonian $G\times G\times H$-space $(M,\omega_0,(\Phi_{0,1},\Phi_{0,2},\Psi_0))$. Rescaling by $s>0$, we obtain a family of Hamiltonian spaces $(M,s\omega_0,(s\Phi_{0,1},s\Phi_{0,2},s\Psi_0))$, together with their exponentials. Let ω^s_{fus} be the corresponding fusion forms. We claim that $s^{-1}\omega^s_{\mathrm{fus}}|_m$ give the required isotopy of symplectic forms. Indeed, each $\omega^s_{\mathrm{fus}}|_m$ is symplectic, and for $s\to 0$,

$$\omega^s_{\mathrm{fus}}|_m=s\omega_0|_m-\tfrac12 B(\exp(s\Phi_{0,1})^*\theta,\exp(s\Phi_{0,2})^*\overline{\theta})|_m=s\omega_0|_m+O(s^2)$$

showing that $\lim_{s\to 0}s^{-1}\omega^s_{\mathrm{fus}}|_m=\omega_0|_m$. $\square$

Appendix C. Lifts of Weyl group elements

In this section we prove the following result about the embedding $\varphi:\ Z(G)\to W$ introduced earlier:

Proposition C.1. *For any simply connected Lie group G and any two elements $\gamma_1,\gamma_2\in Z(G)$ of the center, the Weyl group elements $w_j=\varphi(\gamma_j)\in W=N_G(T)/T$ lift to commuting elements $g_1,g_2\in N_G(T)$.*

Proof. Decomposing into irreducible factors we may assume G is simple. For any simple group other than $G=D_N$ with N even, the center $Z(G)$ is a cyclic group and the claim follows by choosing a lift of the generator. It remains to consider the case $G=D_N=\mathrm{Spin}(2N)$, N even which has center $\mathbb{Z}_2\times\mathbb{Z}_2$. We use the usual presentation [11] of the root system of D_N as the set of vectors $\pm\epsilon_i\pm\epsilon_j,\ i\neq j$ in $\mathfrak{t}^*\cong\mathbb{R}^N$, where ϵ_i are the standard basis vectors of $\mathbb{R}^N$. The basic inner product on D_N is the standard inner product on $\mathbb{R}^N$, and will be used to identify $\mathfrak{t}^*\cong\mathfrak{t}$. We choose simple roots $\alpha_j=\epsilon_j-\epsilon_{j+1},\ j=1,\ldots,N-1$, and $\alpha_N=\epsilon_{N-1}+\epsilon_N$. Then $\alpha_{\max}=\epsilon_1+\epsilon_2$ is the highest root, and the fundamental alcove $\mathfrak{A}$ is the polytope defined by $\xi_1\ge\xi_2\ge\ldots\ge\xi_{N-1}\ge|\xi_N|$ and $\xi_1+\xi_2\le 1$. The following four vertices of $\mathfrak{A}$ exponentiate to the central elements of G:

$$\zeta_0=(0,\ldots,0),\ \zeta_1=(1,0,\ldots,0),\ \zeta_2=\tfrac12(1,\ldots,1),\ \zeta_3=\tfrac12(1,\ldots,1,-1).$$

To describe the homomorphism (20), consider the exceptional element

$$\rho/c=\frac{1}{2N-2}(N-1,N-2,N-3,\ldots,1,0).$$

We have

$$\rho/c + \zeta_1 = \frac{1}{2N-2}(3N-3, N-2, N-3, \dots, 1, 0) = w_1\rho/c + \lambda_1,$$

where $\lambda_1 = 2\epsilon_1 \in \overline{\mathfrak{R}} = \Lambda$, and $w_1 = \varphi(\exp(\zeta_1)) \in W$ is the Weyl group element,

$$w_1(\xi_1, \dots, \xi_N) = (-\xi_1, \xi_2, \dots, \xi_{N-1}, -\xi_N).$$

Similarly

$$\rho/c + \zeta_2 = \frac{1}{2N-2}(2N-2,\ 2N-3, \dots,\ N-2,\ N-1) = w_2\rho/c + \lambda_2,$$

with $\lambda_2 = \sum_{j=1}^{N} \epsilon_j \in \Lambda$ (since N is even) and $w_2 = \varphi(\exp\zeta_2)$ given as

$$w_2(\xi_1, \dots, \xi_N) = (-\xi_N, -\xi_{N-1}, \dots, -\xi_2, -\xi_1).$$

We construct lifts g_1, g_2 of w_1, w_2 in two stages. First we lift to commuting elements $g_1', g_2' \in \mathrm{SO}(2N)$ given in terms of their action on $\mathbb{R}^{2N}$ as follows,

$$\begin{aligned} g_1'(x_1, \dots, x_{2N}) &= (x_1, -x_2, x_3, x_4, \dots, x_{2N-3}, x_{2N-2}, x_{2N-1}, -x_{2N}), \\ g_2'(x_1, \dots, x_{2N}) &= (x_{2N-1}, -x_{2N}, x_{2N-3}, -x_{2N-2}, \dots, x_3, -x_4, -x_1, x_2). \end{aligned}$$

We claim that any two lifts g_1, g_2 to $\mathrm{Spin}(2N)$ still commute. The transformations g_1', g_2' restrict to rotations on the $2N-2$-dimensional subspace $E = \{x|\ x_2 = x_{2N} = 0\}$ and on its orthogonal complement. The restrictions of g_j' to the subspace E lift to commuting transformations of $\mathrm{Spin}(E)$ since $g_1'|E$ is the identity. The restrictions to $E^\perp$ lift to commuting transformations of $\mathrm{Spin}(E^\perp)$ since $E^\perp$ is two-dimensional and $\mathrm{Spin}(2) = \mathrm{U}(1)$ is abelian. $\square$

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