WITTEN'S FORMULAS FOR INTERSECTION PAIRINGS ON MODULI SPACES OF FLAT G-BUNDLES

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Abstract. In a 1992 paper [41], Witten gave a formula for the intersection pairings of the moduli space of flat G-bundles over an oriented surface, possibly with markings. In this paper, we give a general proof of Witten's formula, for arbitrary compact, simple groups, and any markings for which the moduli space has at most orbifold singularities.

Contents

1. Introduction

Let Σ be a compact, connected, oriented surface of genus s with $r \geq 1$ boundary components, and G a compact, connected Lie group. Given conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ in G, let

$$\mathcal{M}(\Sigma; \mathcal{C}_1, \ldots, \mathcal{C}_r) \tag{1}$$

denote the moduli space of flat G-bundles over Σ, with holonomy around the jth boundary component in the prescribed conjugacy class $\mathcal{C}_j$. For 'generic' conjugacy classes $\mathcal{C}_j$, the moduli space has the structure of a smooth, compact, connected orbifold.

Of particular interest to algebraic geometry is the case $G = \mathrm{SU}(n)$, $r = 1$, and $\mathcal{C} = \{c\}$ the conjugacy class consisting of a generator of the center of $\mathrm{SU}(n)$. That is, $c = \exp(2\pi i \frac{d}{n})\, I$ where d and n are coprime. In this case, the moduli space $\mathcal{M}(\Sigma; c)$ is *smooth*, and the Narasimhan-Seshadri theorem [33] identifies the

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moduli space with a moduli space of stable holomorphic vector bundles of rank n and degree d over a Riemann surface of genus s. In [21] Harder-Narasimhan used methods from number theory to calculate the Poincaré polynomial of the space $\mathcal{M}(\Sigma; c)$. In 1984, Atiyah-Bott [5] gave a more geometric computation of the Poincaré polynomial, based on the Morse theory of the Yang-Mills functional. Furthermore, they constructed classes

$$\begin{aligned} \mathfrak{a}^p &\in H^{2d}(\mathcal{M}(\Sigma; c)), \\ \mathfrak{b}_i^p &\in H^{2d-1}(\mathcal{M}(\Sigma; c)), \quad i = 1, \ldots, 2h \\ \mathfrak{f}^p &\in H^{2d-2}(\mathcal{M}(\Sigma; c)), \end{aligned}$$

for any invariant polynomial $p \in \mathrm{Pol}^d(\mathfrak{su}(n))^G$, which generate the cohomology ring as p ranges over the set of polynomials $p(\xi) = \mathrm{tr}(\xi^k)$, $k = 2, \ldots, n$. Beauville [7] gave an alternative construction of the Atiyah-Bott classes, and Biswas-Raghavendra [10] obtained generators for moduli spaces of parabolic bundles. See also Racanière [35].

In 1992 Witten [41] proposed general formulas computing all intersection pairings between Atiyah-Bott classes, generalizing results of Thaddeus [39] for the $n = 2$ case. Witten's formulas were confirmed a few years later by Jeffrey-Kirwan [26]. Their main result expressed the intersection pairings in terms of iterated residues; the equivalence to Witten's version (as a sum over irreducible representations) was obtained using results of Szenes [38] (see Brion-Vergne [13, 14] for further developments in this direction). In Earl-Kirwan [19], these results were used to give explicit formulas for the relations in the cohomology ring.

The formulas for intersection pairings in [41] were stated not only for the group $G = \mathrm{SU}(n)$, but for arbitrary compact, simply connected Lie groups. The aim of the present paper is to give a proof of Witten's formula in this generality, for any collection of conjugacy classes $\mathcal{C}_j$ for which the moduli space has at most orbifold singularities.

It was observed by Atiyah-Bott that the moduli spaces $\mathcal{M}(\Sigma; c)$ have a natural structure as an infinite-dimensional symplectic quotient for the gauge group action on the space of connections on Σ. Witten obtained his formulas by an application of equivariant localization techniques to this infinite-dimensional setting. Jeffrey-Kirwan worked with a different expression of the moduli space $\mathcal{M}(\Sigma; c)$ as a finite-dimensional symplectic quotient $M/\!\!/G$, however the symplectic manifold M is both singular and non-compact. This led to technical difficulties, which prevented the generalization of this approach to other groups and holonomies. A second problem was that Atiyah-Bott's result for generators of the cohomology ring, and its subsequent generalization by Biswas-Raghavendra, was not established for general compact simple Lie groups.

In this paper, we will calculate the intersection pairings using localization on smooth, compact, finite-dimensional manifolds. This calculation is based on a more general localization formula for Hamiltonian G-spaces with *group-valued* moment maps $\Phi : M \to G$, as introduced in [1]. In the case at hand, $\mathcal{M}(\Sigma; \mathcal{C}_1, \ldots, \mathcal{C}_r)$ is expressed as a symplectic quotient $M/\!\!/G = \Phi^{-1}(e)/G$ where $M = G^{2s} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$, with G acting by conjugation and moment map

$$\Phi(a_1, b_1, \ldots, a_s, b_s, u_1, \ldots, u_r) = \prod_{i=1}^{s}[a_i, b_i] \prod_{j=1}^{r} u_j$$

where $[a, b] = aba^{-1}b^{-1}$ is the Lie group commutator. As for usual $\mathfrak{g}^*$-moment maps, there is a localization formula [3], expressing intersection pairings of classes in the image of the ring homomorphism $H_G(M) \to H(M/\!/G)$ in terms of fixed point data on M. Unfortunately, in contrast to the $\mathfrak{g}^*$-valued case this homomorphism need not be surjective. For instance, it turns out that for the moduli space $\mathcal{M}(\Sigma; c)$, the Atiyah-Bott classes $\mathfrak{a}^p$, $\mathfrak{b}_i^p$ are in the image, but not in general the classes $\mathfrak{f}^p$.

The missing generators can be recovered as follows. For any invariant polynomial p of degree $d > 0$, transgression defines a closed equivariant differential form $\eta_G^p \in \Omega_G^{2d-1}(G)$ on G. Suppose $\omega^p \in \Omega_G^{2d-2}(M)$ is an equivariant form with $\mathrm{d}_G\omega^p = \Phi^*\eta_G^p$. Then the pull-back of ω^p to the level set $\Phi_G^{-1}(e)$ is equivariantly closed, hence defines a class in $H_G(\Phi_G^{-1}(e)) \cong H(M/\!/G)$. A recent result of Bott-Tolman-Weitsman [12] asserts that classes of this type, together with the image of the Kirwan map, generate the cohomology ring of $M/\!/G$ provided *all* forms $\Phi^*\eta_G^p$ are equivariantly exact. In Section 7 we will show that this condition holds for the moduli space examples, by explicit construction of the forms ω^p for these cases. That is, one has canonical generators for the cohomology ring of $\mathcal{M}(\Sigma; \mathcal{C}_1, \ldots, \mathcal{C}_r)$ in full generality.

The main result of this paper is a localization formula for intersection pairings of the Bott-Tolman-Weitsman classes on $M/\!/G$ with classes in the image of the Kirwan map. In the moduli space setting, the evaluation of the fixed point data is fairly straightforward and immediately leads to Witten-type formulas.

The organization of the paper is as follows. In Section 2 we review G-valued moment maps, and in Section 3 we show how to relate G-valued moment maps to standard $\mathfrak{g}^*$-moment maps (linearization) or to T-valued moment maps (Abelianization). Section 4 is a review of Duistermaat-Heckman theory for $\mathfrak{g}^*$-valued moment maps. The somewhat unusual perspective taken here is that Duistermaat-Heckman measures are cohomology classes for a 'twisted' equivariant differential on $\mathfrak{g}^*$. We explain that the DH-distributions can be equally well defined as distributions on $\mathfrak{t}^*$, and give a geometric interpretation. These ideas are put to use in Section 5, in order to define DH-distributions for group-valued moment maps. Again one encounters a certain twisted differential, this time on the group G rather than on $\mathfrak{g}^*$. We associate to any cocycle for this differential an invariant distribution on G, by first Abelianizing the problem. We find that on one hand, the resulting DH-distributions encode intersection pairings on symplectic quotients, and on the other hand are calculated by localization. In Section 6, we use a similar approach to study more complicated twistings, required to deal with the Bott-Tolman-Weitsman classes $[\omega_{\mathrm{red}}^p]$. As it turns out, Witten's 'change of variables' [41] becomes very natural in this context. Finally, in Section 7 we apply our localization formula to the moduli space examples.

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Notation. Throughout this paper, G will be a compact, connected Lie group. We fix an invariant inner product $\cdot$ on the Lie algebra $\mathfrak{g}$, which we will often use to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. Choose a maximal torus T of G, with Lie algebra $\mathfrak{t}$, and let $\mathfrak{p} = \mathfrak{t}^\perp$ be the unique T-invariant complement of $\mathfrak{t}$ in $\mathfrak{g}$. The Weyl group of (G, T) will be denoted $W = N_G(T)/T$.

The integral lattice $\Lambda \subset \mathfrak{t}$ is the kernel of the exponential map $\exp : \mathfrak{t} \to T$, and the (real) weight lattice is its dual, $\Lambda^* = \{\lambda \in \mathfrak{t}^* | \, \langle \lambda, \xi \rangle \in \mathbb{Z} \; \forall \xi \in \Lambda\}$. Weights $\lambda \in \Lambda^*$ parametrize homomorphims $\epsilon_\lambda : \; T \to \mathrm{U}(1)$ where

$$\epsilon_\lambda(\exp \xi) = e^{2\pi i \lambda \cdot \xi}, \quad \xi \in \mathfrak{t}.$$

Let $\mathfrak{R} \subset \Lambda^*$ be the set of (real) roots. Fix a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$, and let

$$\mathfrak{R}_+ = \{\alpha \in \mathfrak{R} | \, \langle \alpha, \xi \rangle \geq 0 \text{ for all } \xi \in \mathfrak{t}_+\}$$

be the corresponding set of positive roots. The cardinality of the set of positive roots will be denoted by $n_+ = \#\mathfrak{R}_+$. Recall that the choice of $\mathfrak{t}_+$ defines a unique T-invariant complex structure on $\mathfrak{p}$, in such a way that $\mathfrak{R}_+$ are the weights for the T-action. In particular, this defines an orientation on $\mathfrak{p}$.

Let the homogeneous space G/T be equipped with the G-invariant Riemannian metric and orientation induced from $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{t}$. The Riemannian volume of G/T is given by the formula [8, Corollary 7.27]

$$\mathrm{vol}_{G/T} = \Big(\prod_{\alpha \in \mathfrak{R}_+} 2\pi \alpha \cdot \rho \Big)^{-1}, \tag{2}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha$. More generally, if $K \subset G$ is a connected closed subgroup containing T, the homogeneous space G/K carries a Riemannian metric and orientation induced from $\mathfrak{k}^\perp$, and the Riemannian volume $\mathrm{vol}_{G/K}$ is given by a formula similar to (2), but with a product over only those $\alpha \in \mathfrak{R}_+$ that are not roots of K.

Let the $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be unique Weyl chamber in $\mathfrak{t}^*$ containing ρ. (Clearly, the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ given by the inner product identifies $\mathfrak{t}_+ \cong \mathfrak{t}^*_+$.) For any $\mu \in \mathfrak{t}^*_+$, the symplectic volume of the (co-)adjoint orbit $G.\mu$ is given by

$$\mathrm{Vol}(G.\mu) = \Big(\prod_{\alpha \cdot \mu > 0} 2\pi \alpha \cdot \mu \Big) \; \mathrm{vol}_{G/G_\mu}, \tag{3}$$

where vol_{G/G_μ} is the Riemannian volume corresponding to the metric on $\mathfrak{g}_\mu^\perp$ induced from $\mathfrak{g}$.

For any $\lambda \in \Lambda^*_+ := \Lambda^* \cap \mathfrak{t}_+$ we denote by V_λ the irreducible representation of highest weight λ, and by χ_λ its character.

2. Group-valued moment maps

2.1. q-Hamiltonian G-spaces. In this Section we recall the concept of a group-valued moment map introduced in [1].

Recall that for any G-action on a manifold M, the equivariant cohomology $H_G(M)$ may be computed using the Cartan complex $(\Omega_G(M), \mathrm{d}_G)$, where $\Omega_G(M) = (\mathrm{Pol}(\mathfrak{g}) \otimes \Omega(M))^G$ is the algebra of G-equivariant polynomial maps $\mathfrak{g} \to \Omega(M)$ and d_G is the equivariant differential, $(\mathrm{d}_G\beta)(\xi) = (\mathrm{d} - \iota(\xi_M))\beta(\xi)$. See Appendix B.1. Letting G act on itself by conjugation, we define an equivariant 3-form $\eta_G \in \Omega^3_G(G)$ by

$$\eta_G(\xi) = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] - \frac{1}{2}(\theta^L + \theta^R) \cdot \xi$$

where $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ are the left-invariant, right-invariant Maurer-Cartan forms. It is easily verified that η_G is an equivariant cocycle, i.e. $\mathrm{d}_G\eta_G = 0$. A *Hamiltonian G-space with group-valued moment map* (in short, a *q-Hamiltonian G-space*) is a triple (M, ω, Φ) consisting of a G-manifold, an equivariant map $\Phi : M \to G$, and an invariant 2-form $\omega \in \Omega^2(M)$ satisfying the *moment map condition*

$$\mathrm{d}_G\omega = \Phi^*\eta_G$$

and the *minimal degeneracy condition*

$$\ker(\omega_x) = \{\xi_M(x) |\ \mathrm{Ad}_{\Phi(x)}\,\xi = -\xi\}, \quad x \in M.$$

Sometimes we will omit the minimal degeneracy condition, in which case we refer to (M, ω, Φ) as a *degenerate* q-Hamiltonian G-space. We list some examples of non-degenerate q-Hamiltonian G-spaces, with references for further details:

(a) Conjugacy classes $\mathcal{C} \subset G$, with moment map the inclusion [1],
(b) G^2, with G acting by conjugation on each factor and moment map $(a, b) \mapsto aba^{-1}b^{-1}$ the Lie group commutator [1],
(c) For any symmetric space $X = G/K$ of G, the space X^2, with moment the product of the natural inclusions $X \to G$ [4],
(d) even dimensional spheres S^{2n}, viewed as compactifications of a ball $B \subset \mathbb{C}^n$, and with $G = \mathrm{U}(n)$-action induced from the defining representation on $\mathbb{C}^n$ (see [4, 22] for $n = 2$, the generalization to higher rank was recently obtained by Hurtubise-Jeffrey-Sjamaar [23]).

There is a symplectic reduction procedure for q-Hamiltonian G-spaces, similar to the usual Marsden-Weinstein reduction for $\mathfrak{g}^*$-valued moment maps: If the group unit $e \in G$ is a regular value of the moment map, then G acts locally freely on the level set $\Phi^{-1}(e)$, the pull-back of ω is G-basic, and the induced 2-form ω_{red} on $\Phi^{-1}(e)/G$ is *symplectic.* We will refer to

$$M/\!\!/G = \Phi^{-1}(e)/G$$

as the *symplectic quotient* of (M, ω, Φ). More generally, one defines symplectic quotients $M_g = \Phi^{-1}(g)/G_g$ (where G_g is the centralizer of g) at other regular values of the moment map.

It was shown in [1] that the moduli space (1) of flat G-bundles may be written as a symplectic quotient

$$\mathcal{M}(\Sigma;\mathcal{C}_1,\ldots,\mathcal{C}_r)=M/\!/G$$

of a q-Hamiltonian G-space (M,ω,Φ). Here

$$M=G^{2s}\times\mathcal{C}_1\times\cdots\times\mathcal{C}_r, \tag{4}$$

with G acting by conjugation on each factor, and moment map

$$\Phi(a_1,b_1,\ldots,a_s,b_s,u_1,\ldots,u_r)=\prod_{i=1}^{s}[a_i,b_i]\prod_{j=1}^{r}u_j. \tag{5}$$

Here $[a,b]\equiv aba^{-1}b^{-1}$ is the group commutator. The 2-form ω on M is given by an explicit, but somewhat complicated formula spelled out in [1] (see also Section 7 below).

In general, q-Hamiltonian G-spaces need not be orientable – a counterexample is $M=\mathbb{R}P(2)$ as a conjugacy class for $G=\mathrm{SO}(3)$. However, this problem does not arise if G is simply connected, or more generally if the half-sum of positive roots, ρ, is in the weight lattice Λ^*. For any differential form $\beta\in\Omega(M)$, let $\beta^{[k]}$ denotes its component in $\Omega^k(M)$.

Lemma 2.1. [4] *Suppose ρ is a weight of G. Let (M,ω,Φ) be a q-Hamiltonian G-space. Then M carries a canonical volume form Γ with the property*

$$(\exp\omega)^{[\dim M]}=\frac{\Phi^*\chi_\rho}{\dim V_\rho}\,\Gamma. \tag{6}$$

In particular, M is orientable.

Remark 2.2. Suppose that M is connected and that $\Phi^*\chi_\rho$ does not identically vanish on M. Then (6) may be used as a definition of the volume form Γ. In particular, this is the case if $\Phi^{-1}(e)\neq\emptyset$. Note that for all $x\in\Phi^{-1}(e)$, the 2-form ω_x on T_xM is non-degenerate, and that the symplectic orientation on T_xM coincides with the orientation given by Γ.

Example 2.3. For a conjugacy class $\mathcal{C}=G/G_g$, the volume form may be described (up to sign) as the Riemannian volume form for the homogeneous space G/G_g, times $|\det_{\mathfrak{g}_g^\perp}(\mathrm{Ad}_g-1)|^{1/2}$. Suppose G is simple and simply connected, and let $\mathfrak{A}\subset\mathfrak{t}$ be the fundamental alcove, i.e. the subset of $\mathfrak{t}_+$ cut out by the inequality $\alpha_{\max}\cdot\mu\le 1$ where $\alpha_{\max}$ is the highest root. Recall that $\mathfrak{A}$ parametrizes the set of conjugacy classes in G, in the sense that every conjugacy class contains a unique element $\exp(\mu)$ with $\mu\in\mathfrak{A}$. The volume of the conjugacy class $G.\exp\mu$ is given by the formula [4],

$$\mathrm{Vol}(G.\exp\mu)=\Big(\prod_{\alpha\in\mathfrak{R}_+,\ \alpha\cdot\mu\notin\mathbb{Z}}2\sin\pi\alpha\cdot\mu\Big)\ \mathrm{vol}_{G/G_{\exp\mu}} \tag{7}$$

compare with (3). The orientation on the conjugacy class $G.\exp\mu$ differs from the orientation of the homogeneous space $G/G_{\exp\mu}$ by a sign, $(-1)^{2\rho_K\cdot\mu}$, where $2\rho_K$ is the sum of positive roots of $K = G_g$, i.e. the sum over all those roots $\alpha \in \mathfrak{R}_+$ with $\alpha \cdot \mu \in \mathbb{Z}$.

2.2. **Bott-Tolman-Weitsman theorem.** Suppose (M, ω, Φ) is a compact, connected, q-Hamiltonian G-space, and that e is a regular value of the moment map. Pull-back to the identity level set defines a *Kirwan map*

$$H_G^\bullet(M) \to H_G^\bullet(\Phi^{-1}(e)) = H^\bullet(M/\!\!/G).$$

In contrast to ordinary Hamiltonian G-spaces, this map is not onto, in general.

Example 2.4. Let $G = \mathrm{SU}(2)$, $\mathcal{C} = \{c\}$ the conjugacy class consisting of the non-trivial central element. For $s \geq 2$ consider $M = G^{2s} \times \mathcal{C}$, with moment map Φ as in (5). It is easy to see that away from $\Phi^{-1}(c)$, the G-action has constant stabilizer equal to the center $Z(G)$. In particular, $G/Z(G)$ acts freely on $\Phi^{-1}(e)$, and therefore $M/\!\!/G$ is a smooth symplectic manifold of dimension $(2s-2)\dim G > 0$. The symplectic form defines a non-trivial class in $H^2(M/\!\!/G)$. On the other hand $H_G^2(M) = 0$. This shows that the Kirwan map cannot be surjective in this example.

We now explain how to obtain a set of generators of $H(M/\!\!/G)$ in the general case. As it turns out, it is necessary to take the topology of G (as a target of the moment map) into account. Recall that generators of $(\wedge\mathfrak{g}^*)^G \cong H(G)$ are obtained as images of the *transgression map* $\mathrm{Pol}^\bullet(\mathfrak{g})^G \to (\wedge^{2\bullet-1}\mathfrak{g}^*)^G$. The transgression construction can be made equivariant for the conjugation action: That is, there is a canonical linear map

$$\mathrm{Pol}^\bullet(\mathfrak{g})^G \to \Omega_G^{2\bullet-1}(G),\ p \mapsto \eta_G^p$$

such that the forms η_G^p are closed, and their classes $[\eta_G^p]$ generate $H_G(G)$ as an algebra over $H_G(\mathrm{pt})$. (In fact, one already obtains a set of generators if one restricts to the subspace $P \subset \mathrm{Pol}^\bullet(\mathfrak{g})^G$ spanned by primitive generators for the algebra $\mathrm{Pol}^\bullet(\mathfrak{g})^G$.)

An explicit formula for the forms η_G^p is worked out in Jeffrey's paper [24]. For any $p \in \mathrm{Pol}(\mathfrak{g})$ let $\xi \mapsto p'(\xi) \in \mathfrak{g}$ be its gradient, defined by $p'(\xi) \cdot \zeta = \frac{d}{dt}\Big|_{t=0} p(\xi + t\zeta)$. Then

$$\eta_G^p(\xi) = -\theta^L \cdot \int_0^1 \mathrm{d}t\ p'\Big((1-t)\xi + \mathrm{Ad}_{g^{-1}}(\xi) - \frac{t(1-t)}{2}[\theta^L, \theta^L]\Big). \tag{8}$$

We will review this derivation of this formula in the Appendix, Section B.8.

Remarks 2.5. (a) Note that if $G = T$, the formula for the equivariant forms η_G^p simplifies to

$$\eta_T^p(\xi) = -\theta_T \cdot p'(\xi),$$

where θ_T is the Maurer-Cartan form for T.

(b) For the quadratic polynomial $p(\xi) = \frac{1}{2}\xi \cdot \xi$, the form η_G^p coincides with the form η_G considered above.

Suppose that (M, ω, Φ) is a q-Hamiltonian G-space. We will refer to equivariant forms $\omega^p \in \Omega_G^{2d-2}(M)$ with

$$\mathrm{d}_G \omega^p = \Phi^* \eta_G^p. \tag{9}$$

as *higher q-Hamiltonian forms*. Since η_G^p has odd degree, its pull-back to the group unit $e \in G$ vanishes and therefore $\iota_{\Phi^{-1}(e)}^* \omega^p$ is closed. Hence $[\iota_{\Phi^{-1}(e)}^* \omega^p] \in H_G(\Phi^{-1}(e))$ is defined, and descends to an ordinary cohomology class $[\omega_{\mathrm{red}}^p]$ on the symplectic quotient. After choosing a principal connection on $\Phi^{-1}(e)$, Cartan's theorem (see Appendix B.2) yields $\omega_{\mathrm{red}}^p \in \Omega^{2d-2}(M/\!\!/G)$ as a differential form. The following is a reformulation of a result of [12].

Theorem 2.6 (Bott-Tolman-Weitsman). *Suppose G is simply connected. Let (M, ω, Φ) be a compact connected q-Hamiltonian G-space, with e a regular value of the moment map. Assume that for* all $p \in \mathrm{Pol}(\mathfrak{g})^G$ *there exists a form $\omega^p \in \Omega_G(M)$ satisfying* (9). *Then the classes $[\omega_{\mathrm{red}}^p]$, together with the image of the Kirwan map, generate the cohomology ring of $M/\!\!/G$.*

Note that $\Phi^*\eta_G^p$ is exact for all $p \in \mathrm{Pol}(\mathfrak{g})^G$, if and only if it is exact for some set of generators of the ring $\mathrm{Pol}(\mathfrak{g})^G$. That is, it is enough to consider forms ω^p for such a set of generators.

Corollary 2.7. *If $G = \mathrm{SU}(2)$, the image of the Kirwan map together with the reduced symplectic form generate the cohomology ring of $M/\!\!/G$.*

Remark 2.8. The condition (9) means that the pair (ω^p, η_G^p) defines a cocycle for the relative equivariant de Rham complex $\Omega_G^\bullet(\Phi) = \Omega_G^{\bullet-1}(M) \oplus \Omega_G^\bullet(G)$. The class $[\omega_{\mathrm{red}}^p]$ depends only on the relative cohomology class $[(\omega^p, \eta_G^p)]$.

In Section 7, we will give an explicit construction of the higher q-Hamiltonian forms ω^p for the moduli space example (4).

3. Linearization and Abelianization

In this Section, we will describe two methods of relating q-Hamiltonian G-spaces to more standard Hamiltonian spaces: Linearization (replacing the target G of the moment map with $\mathfrak{g} \cong \mathfrak{g}^*$), and Abelianization (replacing G by the maximal torus T).

3.1. The linearization of a q-Hamiltonian G-space.

For any invariant polynomial $p \in \mathrm{Pol}^\bullet(\mathfrak{g})^G$ let $\eta_{\mathfrak{g}}^p \in \Omega_G^{2\bullet-1}(\mathfrak{g})$ denote the (exact) equivariant form,

$$\eta_{\mathfrak{g}}^p(\xi) = -\mathrm{d}\langle \cdot, p'(\xi) \rangle.$$

Thus, $\eta_{\mathfrak{g}}^p$ is a linearized version of the form η_G^p. Let

$$\varpi^p = \mathrm{h}(\exp^* \eta_G^p - \eta_{\mathfrak{g}}^p) \in \Omega_G^{2\bullet-2}(\mathfrak{g}), \tag{10}$$

where $\mathrm{h} : \Omega_G^\bullet(\mathfrak{g}) \to \Omega_G^{\bullet-1}(\mathfrak{g})$ is the G-equivariant homotopy operator (Section B.5) for the vector space $\mathfrak{g}$. For $p(\xi) = \frac{1}{2}||\xi||^2$ we omit the superscript p, writing $\eta_{\mathfrak{g}} = -\langle \cdot, \xi \rangle$ and

$$\varpi = \mathrm{h}(\exp^* \eta_G - \eta_{\mathfrak{g}}) \in \Omega_G^2(\mathfrak{g}). \tag{11}$$

Since $(\exp^* \eta_G - \eta_{\mathfrak{g}})^{[1]} = 0$, one has $\varpi^{[0]} = 0$, i.e. the equivariant 2-form ϖ is an ordinary invariant 2-form.

Suppose now that (M, ω, Φ) is a q-Hamiltonian G-space, possibly degenerate. Let $V \subset G$ be an invariant open neighborhood of $e \in G$, given as the diffeomorphic image of an invariant open neighborhood V_0 of $0 \in \mathfrak{g}$ under the exponential map. Let $\log : V_0 \to V$ be the inverse map. Replacing M with $\Phi^{-1}(V)$ if necessary, assume that Φ takes values in V, and let

$$\Phi_0 = \log(\Phi), \quad \omega_0 = \omega - \Phi^* \varpi.$$

Clearly, $\mathrm{d}_G \omega = \Phi^* \eta_G$ implies $\mathrm{d}_G \omega_0 = \Phi_0^* \eta_{\mathfrak{g}}$, which is the usual moment map condition for a Hamiltonian G-space. We will refer to the Hamiltonian G-space (M, ω_0, Φ_0) as the *linearization* of the q-Hamiltonian space (M, ω, Φ). Given a higher q-Hamiltonian form ω^p on M, the form

$$\omega_0^p = \omega^p - \Phi_0^* \varpi^p$$

has the property $\mathrm{d}_G \omega_0^p = \Phi_0^* \eta_{\mathfrak{g}}^p$.

Remarks 3.1. (a) If ω satisfies the minimal degeneracy condition, then the 2-form ω_0 is symplectic [1, Proposition 3.4]. Let $\Gamma_0 = (\exp \omega_0)^{[\dim M]}$ denote the symplectic (Liouville) volume form on M_0, and Γ the volume form on M. Then $\Gamma = \Phi_0^* J^{1/2}\, \Gamma_0$, where $J^{1/2} \in C^\infty(\mathfrak{g})$ is the unique smooth square root with $J^{1/2}(0) = 1$. See [4, Section 3.6].

(b) As a typical application of the linearization construction, the symplectic reduction theorem for q-Hamiltonian G-spaces follows directly from the usual Hamiltonian setting, together with the fact that ϖ vanishes at $\mu = 0$.

3.2. **The Abelianization of a q-Hamiltonian G-space.** The theory of G-valued moment maps is substantially different from the theory of ordinary $\mathfrak{g}^*$-valued moment maps only if the group G is non-Abelian. In this Section we introduce an Abelianization procedure for q-Hamiltonian G-spaces, replacing the group G by its maximal torus T. For any G-manifold M, we denote by $\kappa_T : \Omega_G(M) \to \Omega_T(M)$ the natural map restricting the action.

Recall that $\mathfrak{p} = \mathfrak{t}^\perp$, and consider the two maps

$$\begin{tikzcd} & T \times \mathfrak{p} \arrow[dl, "\pi_T"'] \arrow[dr, "F"] & \\ T & & G \end{tikzcd}$$

where $\pi_T(t, \mu) = t$ and $F(t, \mu) = t \exp(\mu)$. For any $p \in \mathrm{Pol}^\bullet(\mathfrak{g})^G$, define a T-equivariant form

$$\gamma^p = \mathrm{h}\left(F^* \kappa_T(\eta_G^p)\right), \quad \in \Omega_T^{2\bullet-2}(T \times \mathfrak{p}) \tag{12}$$

where h is the T-equivariant homotopy operator (Section B.5) for $T \times \mathfrak{p} \to T$.

Proposition 3.2. *The pull-back of the forms γ^p to $T \times \{0\} \subset T \times \mathfrak{p}$ vanishes. One has,*

$$\mathrm{d}_T \gamma^p = F^* \kappa_T(\eta_G^p) - \pi_T^* \eta_T^p, \quad \xi \in \mathfrak{t}.$$

Proof. Let $\iota_T : T \to T \times \mathfrak{p}$ denote the inclusion. The equation for $\mathrm{d}_T\gamma^p$ follows from the property $\mathrm{d}_T \mathrm{h} + \mathrm{h}\, \mathrm{d}_T = \mathrm{id} - \pi_T^* \iota_T^*$ of the homotopy operator, since $\iota_T^*(F^*\kappa_T(\eta_G^p)) = \eta_T^p$. Note that one can also write $\gamma^p = \mathrm{h}\left(F^*\kappa_T(\eta_G^p) - \pi_T^*\eta_T^p\right)$ since $\mathrm{h} \circ \pi_T^* = 0$. The pull-back of $F^*\kappa_T(\eta_G^p) - \eta_T^p$ to T is zero, hence the same is true for γ^p. ☐

Consider in particular the equivariant 2-form $\gamma \in \Omega_T^2(T \times \mathfrak{p})$ corresponding to the quadratic polynomial $p(\xi) = \frac{1}{2}||\xi||^2$. It turns out that $\gamma^{[0]} = 0$ so that γ is in ordinary 2-form:

Proposition 3.3. *The equivariant 2-form γ is a T-invariant 2-form, given by the formula,*

$$\gamma = \varpi_{\mathfrak{p}} + \frac{1}{2}\theta_T \cdot \exp_{\mathfrak{p}}^* \theta^R \in \Omega^2(T \times \mathfrak{p}),$$

where $\varpi_{\mathfrak{p}}$ is the pull-back of the form $\varpi \in \Omega^2(\mathfrak{g})^G$ to $\mathfrak{p}$, and $\exp_{\mathfrak{p}}$ is the restriction of the exponential map. The pull-back of γ to T vanishes.

Proof. We use the following formula for the pull-back of the form η_G under group multiplication:

$$\mathrm{Mult}_G^* \eta_G = \pi_T^*\eta_G + \mathrm{pr}_2^* \eta_G + \frac{1}{2} d_G(\pi_T^*\theta^L \cdot \mathrm{pr}_2^* \theta^R).$$

Here pr_i denote the projections from $G \times G$ to the two factors. The map F can be written as a composition of the map $T \times \mathfrak{p} \to G \times G$, $(t, \mu) \mapsto (t, \exp_{\mathfrak{p}}(\mu)$ followed by group multiplication. Hence,

$$F^*\kappa_T(\eta_G) = \eta_T + \exp_{\mathfrak{p}}^* \kappa_T(\eta_G) + \frac{1}{2}\mathrm{d}_T(\theta_T \cdot \exp_{\mathfrak{p}}^* \theta^R).$$

Now apply the homotopy operator h for $T \times \mathfrak{p}$. Clearly, $\mathrm{h}\, \eta_T = 0$. Furthermore,

$$\mathrm{h} \exp_{\mathfrak{p}}^* \kappa_T(\eta_G) = \varpi_{\mathfrak{p}}$$

since pull-back from $\mathfrak{g}$ to $\mathfrak{p}$ intertwines the homotopy operators for the two vector spaces. Finally,

$$\mathrm{h}\, \mathrm{d}_T(\theta_T \cdot \exp_{\mathfrak{p}}^* \theta^R) = \theta_T \cdot \exp_{\mathfrak{p}}^* \theta^R$$

since $\theta_T \cdot \exp_{\mathfrak{p}}^* \theta^R$ pulls back to 0 on $T \subset T \times \mathfrak{p}$, and since

$$\mathrm{h}(\theta_T \cdot \exp_{\mathfrak{p}}^* \theta^R) = -\theta^T \cdot \mu = 0.$$

☐

We may use the form γ to turn any q-Hamiltonian G-space (M, ω, Φ) into a (degenerate) q-Hamiltonian T-space, at least on a neighborhood of $\Phi^{-1}(T)$.

For $\epsilon > 0$ let $B_\epsilon(0) \subset \mathfrak{p}$ denote the open ball of radius ϵ. Choose ϵ sufficiently small, so that the map F restricts to a a diffeomorphism $T \times B_\epsilon(0)$ onto an open subset $U \subset G$. Let $\pi : U \to T$ be the projection corresponding to $\pi_T : T \times \mathfrak{p} \to T$, and use F to view γ^p as forms on U. Thus,

$$\mathrm{d}_T\gamma^p = \kappa_T(\eta_G^p)|_U - \pi^*\eta_T^p.$$

The pre-image

$$N := \Phi^{-1}(U)$$

is a T-invariant open neighborhood of $\Phi^{-1}(T)$ in M. Define an invariant 2-form $\omega_N \in \Omega^2(N)$ and an equivariant map $\Phi_N \in C^\infty(N, T)$ by

$$\omega_N = \omega|_N - \Phi^*\gamma, \quad \Phi_N = \pi \circ \Phi|_N.$$

More generally, if ω^p is a higher q-Hamiltonian form for an invariant polynomial p, i.e. ω^p is a primitive for $\Phi^*\eta_G^p$, we define

$$\omega_N^p = \kappa_T(\omega^p)|_N - \Phi^*\gamma^p.$$

Proposition 3.4. *The triple (N, ω_N, Φ_N) is a (degenerate) q-Hamiltonian T-space. If $p \in \mathrm{Pol}(\mathfrak{g})^G$ is an invariant polynomial and ω^p a corresponding higher q-Hamiltonian form for (M, ω, Φ), then ω_N^p is a higher q-Hamiltonian form for (N, ω_N, Φ_N).*

Proof. The equation $\mathrm{d}_T\omega_N^p = \Phi^*\eta_T^p$ follows directly from $\mathrm{d}_G\omega^p = \Phi^*\eta_G^p$ and the property of the form γ^p. $\square$

Notice that the 2-form ω_N is closed, but is usually not symplectic even if ω is minimally degenerate.

4. DH-distributions for Hamiltonian G-spaces

For any manifold M, let $\mathcal{C}^\bullet(M)$ denote the complex of de Rham *currents* on M. Thus $\mathcal{C}^k(M)$ is the topological dual space of $\Omega^{n-k}(M)_{\mathrm{comp}}$, and the differential is defined by duality. If M is oriented, the natural inclusion $\Omega^\bullet(M) \to \mathcal{C}^\bullet(M)$ is a quasi-isomorphism. For any vector field X on M, the operators of Lie derivative and contraction L_X, ι_X extend to the complex $\mathcal{C}^\bullet(M)$. Given a G-action on M, the space $\mathcal{C}^\bullet(M)$ is a G-differential space in the sense of [28], and therefore a differential space

$$\mathcal{C}_G^\bullet(M) = (\mathrm{Pol}(\mathfrak{g}) \otimes \mathcal{C}^\bullet(M))^G$$

of equivariant currents is defined.

In the present Section, we will avoid identifications of the Lie algebra with its dual. Suppose (M, ω, Φ) is a Hamiltonian G-space, i.e. with an ordinary moment map $\Phi : M \to \mathfrak{g}^*$. For any equivariant cocycle $\beta \in \Omega_G(M)$, the expression $\beta e^{2\pi i \omega} \in \Omega_G(M)$ is a cocycle for the twisted differential, $\mathrm{d}_G - 2\pi i \Phi^*\eta_{\mathfrak{g}}$. (Note that the equivariant 3-form $\eta_{\mathfrak{g}}(\xi) = -\mathrm{d}\langle \cdot, \xi\rangle$ is invariantly defined as a form on $\mathfrak{g}^*$.) Hence, its push-forward under the moment map is an equivariant current on $\mathfrak{g}^*$, closed under the differential $\mathrm{d}_G' = \mathrm{d}_G - 2\pi i \eta_{\mathfrak{g}}$. We will show in this Section that the cohomology space of this differential is naturally identified with invariant distributions. The distributions associated with the current $\Phi_*(\beta e^{2\pi i \omega})$ will be called Duistermaat-Heckman distributions.

4.1. **Equivariant currents on $\mathfrak{g}^*$.** Even though $\eta_\mathfrak{g}$ is equivariantly exact, it still defines a non-trivial twisting of the cohomology, as the following result shows.

Proposition 4.1. *Let $\mathrm{d}'_G = \mathrm{d}_G - 2\pi i \eta_\mathfrak{g}$. For any G-invariant open subset $V \subset \mathfrak{g}^*$, the map*

$$\mathcal{C}_G(V) \to \mathcal{D}'(V)^G, \ \phi \mapsto \mathfrak{n} = \sum_I \big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\big)^I \phi_I^{[\dim \mathfrak{g}]}. \tag{13}$$

vanishes on d'_G-coboundaries, and descends to an isomorphism,

$$H(\mathcal{C}_G(V), \mathrm{d}'_G) \to \mathcal{D}'(V)^G.$$

If $\phi \in \mathcal{C}_G(V)$ is compactly supported, the associated distribution $\mathfrak{n}$ is given in terms of its Fourier transform by

$$\langle \mathfrak{n}, e^{-2\pi i \langle\cdot,\xi\rangle}\rangle = \langle \phi(\xi),\, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle. \tag{14}$$

The map (13) restricts to a map for smooth equivariant forms, $\Omega_G(V) \to \Omega^{\dim\mathfrak{g}}(V)^G$, which descends to to an isomorphism $H(\Omega_G(V), \mathrm{d}'_G) \to (\Omega^{\dim\mathfrak{g}}(V))^G$.

Proof. The formula (14) for compactly supported ϕ follows from

$$\langle \mathfrak{n},\, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle = \big\langle \sum_I \big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\big)^I \phi_I,\, e^{-2\pi i\langle\cdot,\xi\rangle}\big\rangle = \sum_I \xi^I \langle \phi_I, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle.$$

Note that the right hand side of (14) may also be viewed as the integral (push-forward to a point) of the current $\phi(\xi)e^{-2\pi i\langle\cdot,\xi\rangle}$. We next show that the distribution $\mathfrak{n}$ corresponding to a coboundary $\mathrm{d}'_G\psi$ is zero. Using a partition of unity, it suffices to prove this if ψ has compact support. But in this case, the integral of $(\mathrm{d}'_G\psi)(\xi)e^{-2\pi i\langle\cdot,\xi\rangle} = \mathrm{d}_\xi(\psi(\xi)e^{-2\pi i\langle\cdot,\xi\rangle})$ vanishes by Stokes' theorem. It remains to show that the induced map in d'_G-cohomology is an isomorphism. To this end, we view the space $\mathcal{C}(V)$ of currents as a G-differential space, with the standard G-action and the standard differential, but with the contraction operators

$$\iota'_\xi = \iota(\xi_{\mathfrak{g}^*}) + 2\pi i \eta_\mathfrak{g}(\xi).$$

¿From this perspective, the d'_G-cohomology of $\mathcal{C}_G(V)$ is just the equivariant cohomology of the G-differential space $\mathcal{C}_G(V)$. Let $P \in C^\infty(\mathfrak{g}^*, \wedge^2\mathfrak{g}^*)$ be the Kirillov-Poisson bivector field on $\mathfrak{g}^*$. If f_{ab}^c are the structure constants in a given basis e_a of $\mathfrak{g}$, with dual basis e^a, and μ_a the associated coordinates,

$$P = \frac{1}{2}\sum_{abc} f_{ab}^c \mu_c \frac{\partial}{\partial\mu_a}\frac{\partial}{\partial\mu_b}.$$

Let $\iota(P) : \mathcal{C}^\bullet(V) \to \mathcal{C}^{\bullet-2}(V)$ denote the operator of contraction by P. Then conjugation by $\exp(\frac{1}{2\pi i}\iota(P))$ is a G-equivariant automorphism of $\mathcal{C}(V)$ which simplifies the contraction operators: Indeed,

$$[\iota(P),\, \iota(\xi_{\mathfrak{g}^*})] = 0, \quad [\iota(P),\, \eta_\mathfrak{g}(\xi)] = -\iota(\xi_{\mathfrak{g}^*}),$$

and therefore

$$\iota''_\xi := \mathrm{Ad}\,\Big(\exp(\frac{1}{2\pi i}\iota(P))\Big)\iota'_\xi = 2\pi i\eta_\mathfrak{g}(\xi) = 2\pi i \mathrm{d}\langle\cdot,\xi\rangle.$$

Clearly, the horizontal subspace of $\mathcal{C}(V)$ with respect to ι''_ξ is the space of currents of top degree, that is $\mathcal{D}'(V)$, and so the basic subspace is $\mathcal{D}'(V)^G$. It is easy to see that the conjugated differential $\mathrm{d}'' = \mathrm{Ad}\big(\exp(\frac{1}{2\pi i}\iota(P))\big)\mathrm{d}$ vanishes on $\mathcal{D}'(V)^G$: One one hand, it preserves the basic subspace, on the other hand, it changes parity while everything in the basic subspace has fixed parity given by $\dim \mathfrak{g}$. This shows that there is an isomorphism

$$\mathcal{D}'(V)^G \to \mathcal{C}(V)_{\mathrm{basic}}, \quad \lambda \mapsto \exp(\frac{1}{2\pi i}\iota(P))\lambda$$

taking values in cocycles. Viewing $\mathcal{C}(V)_{\mathrm{basic}}$ as a subspace of $\mathcal{C}_G(V)$, it is obvious that its composition with the map $\mathcal{C}_G(V) \to \mathcal{D}'(V)$ is the identity. To complete the proof, it suffices to show that the inclusion of the basic subcomplex $\mathcal{C}(V)_{\mathrm{basic}} \to \mathcal{C}_G(V)$ induces an isomorphism in cohomology. By a generalized version of Cartan's theorem, due to Guillemin-Sternberg [20], this will be the case if there exists a G-equivariant linear map $\theta : \mathfrak{g}^* \to \mathrm{End}^{\mathrm{odd}}(\mathcal{C}(V))$ such that $[\iota'_\xi, \theta(\mu)] = \langle \mu, \xi\rangle \,\mathrm{Id}$ and such that the operators $\theta(\mu)$ and $[\mathrm{d}, \theta(\mu)]$ generate a super-commutative subalgebra of $\mathrm{End}(\mathcal{C}(V))$. [1]. Indeed, such a map is given by contraction with the constant vector fields defined by elements of $\mathfrak{g}^*$:

$$\theta(e^a) = \frac{1}{2\pi i}\iota(\frac{\partial}{\partial \mu_a}).$$

Clearly, all of the above goes through for $\mathcal{C}_G(V)$ replaced with $\Omega_G(V)$. ☐

Remark 4.2. The map (13) restricts to a map for smooth equivariant forms, $\Omega_G(V) \to \Omega^{\dim \mathfrak{g}}(V)^G$, and the proof of Proposition 4.1 shows that, again, this map descends to to an isomorphism $H(\Omega_G(V), \mathrm{d}'_G) \cong \Omega^{\dim\mathfrak{g}}(V)^G$.

Later we will need the following consequence of the proof of Proposition 4.1:

Corollary 4.3. *Suppose $\phi \in \mathcal{C}_G(V)$ is a cocycle for $\mathrm{d}'_G = \mathrm{d}_G - 2\pi i \eta_{\mathfrak{g}}$, and $\mathcal{O} \subset V$ is a coadjoint orbit. Then there exists a d'_G-cocycle $\tilde\phi$ with compact support in any given neighborhood of $\mathcal{O}$, such that $\tilde\phi = \phi$ on a smaller neighborhood of $\mathcal{O}$. If ϕ is smooth near $\mathcal{O}$, one can take $\tilde\phi$ to be smooth.*

Proof. Let $\mathfrak{n} \in \mathcal{D}'(V)^G$ be the distribution defined by ϕ. Let V_1, V_2 be open neighborhoods of $\mathcal{O}$, with $\overline{V_1} \subset V$ and $\overline{V_2} \subset V_1$. Let $\chi \in C^\infty(V)^G_{\mathrm{comp}}$ with $\chi|_{V_1} = 1$, and $\chi_1 \in C^\infty(V_1)^G_{\mathrm{comp}}$ with $\chi_1|_{V_2} = 1$. Let $\mathfrak{n}_1 = \chi\,\mathfrak{n}$, and let $\phi_1 = \exp(\frac{1}{2\pi i}\iota(P))\mathfrak{n}_1$ as in the proof of Proposition 4.1. Since $\mathfrak{n}_1 = \mathfrak{n}$ on V_1, it follows that the restrictions of ϕ, ϕ_1 to V_1 are d'_G-cohomologous: $\phi|_{V_1} = \phi_1|_{V_1} + \mathrm{d}'_G(\psi)$. Define $\tilde\phi$ by $\phi_1|_{V_1} + \mathrm{d}'_G(\chi_1\psi)$ on V_1, and equal to ϕ_1 outside V_1. ☐

4.2. **DH-distributions.** Suppose (M, ω, Φ) is an oriented Hamiltonian G-space, with a possibly degenerate 2-form ω. We assume that the moment map $\Phi : M \to \mathfrak{g}^*$

[1] These conditions are a translation of the notion of a W^*-module, as introduced in [20]. As shown in [2], one may in fact drop the super-commutativity condition.

takes values in an invariant open subset $V \subset \mathfrak{g}^*$, and is proper as a map into V. For any equivariant cocycle $\beta \in \Omega_G(M)$, the push-forward

$$\Phi_*(\beta e^{2\pi i \omega}) \in \mathcal{C}_G(V)$$

is a well-defined d'_G-cocycle. We define the Duistermaat-Heckman distribution $\mathfrak{n}^\beta \in \mathcal{D}'(V)^G$ to be its image under the map (13), that is,

$$\mathfrak{n}^\beta = \sum_I \big(\frac{1}{2\pi i}\frac{\partial}{\partial \mu}\big)^I \,\Phi_*(\beta e^{2\pi i\omega})_I^{[\dim M]}.$$

For $\beta = 1$, this simplifies to $\mathfrak{n} = \Phi_*(e^{2\pi i\omega})^{[\dim M]}$, which (up to $2\pi i$ factors) is the original definition of the Duistermaat-Heckman measure in [17], as a push-forward of the Liouville measure. The more general DH-distributions $\mathfrak{n}^\beta$ were introduced by Jeffrey-Kirwan [25], in terms of their Fourier coefficients (see Proposition 4.4(d) below).

We list some of the basic properties of DH-distributions. Given β, let us call $\mu \in \mathfrak{g}^*$ a *β-regular value* of Φ if the differential $\mathrm{d}_x\Phi$ has maximal rank for all $x \in \Phi^{-1}(\mu) \cap \mathrm{supp}(\beta)$. This includes of course the set of regular values of Φ.

Proposition 4.4 (Properties of Hamiltonian DH-distributions). *Let (M, ω, Φ) be an oriented Hamiltonian G-space (with possibly degenerate 2-form ω), with proper moment map $\Phi: M \to V \subset \mathfrak{g}^*$.*

(a) *The definition of $\mathfrak{n}^\beta$ is local, in the sense that the restriction of $\mathfrak{n}^\beta$ to an invariant open subset $V' \subset V$ is the DH-distribution associated to $\beta|_{\Phi^{-1}(V')}$. Its support and singular support satisfy*

$$\mathrm{supp}(\mathfrak{n}^\beta) \subset \Phi(\mathrm{supp}(\beta)), \quad \mathrm{singsupp}(\mathfrak{n}^\beta) \subset \Phi(\mathrm{supp}(\beta) \cap (M \backslash M_*)),$$

where $M_* \subset M$ is the set of $x \in M$ such that $\mathrm{d}_x\Phi$ has maximal rank.

(b) *The map $\Omega_G(M) \to \mathcal{D}'(V)^G$, $\beta \mapsto \mathfrak{n}^\beta$ vanishes on d_G-coboundaries. In particular, it descends to cohomology.*

(c) *Suppose $p \in \mathrm{Pol}(\mathfrak{g})^G \subset \Omega_G(M)$ is an invariant polynomial, viewed as an equivariant cocycle on M. Then*

$$\mathfrak{n}^{p\beta} = p\big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\big)\,\mathfrak{n}^\beta.$$

(d) *If β has compact support, then so does $\mathfrak{n}^\beta$ and its Fourier transform is given by*

$$\langle \mathfrak{n}^\beta, e^{-2\pi i\langle \cdot, \xi\rangle}\rangle = \int_M \beta(\xi) e^{2\pi i(\omega - \langle \Phi, \xi\rangle)}, \quad \xi \in \mathfrak{g}.$$

(e) *The distribution $\mathfrak{n}^\beta$ is smooth near β-regular values of Φ. If $\mathrm{d}_G\beta = 0$, and $\mu \in \mathfrak{t}^*_+ \subset \mathfrak{g}^*$ is a β-regular value,*

$$\mathfrak{n}^\beta(\mu) = (2\pi i)^{\dim G_\mu/2}\frac{\mathrm{vol}_G}{\mathrm{Vol}(G.\mu)}\sum_j \frac{1}{k_j}\int_{\Phi^{-1}(\mu)_j/G_\mu} \beta_{\mathrm{red}} e^{2\pi i\omega_{\mathrm{red}}}$$

Here the sum is over connected components $\Phi^{-1}(\mu)_j$ of the level set and k_j is the number of elements in a principal stabilizer for the G_μ-action on

jth component. Furthermore, β_{red} is the image of β under the chain map $\Omega_G(M) \to \Omega(\Phi^{-1}(\mu)/G_\mu)$, given as pullback to the level set followed by the Cartan map for a principal G_μ-connection on $\Phi^{-1}(\mu)$, and similarly for ω_{red}.

Proof. Properties (a), (b) and (c) are clear from the definition or from the corresponding properties of d'_G-closed currents on V. The integral formula (d) follows from

$$\langle \mathfrak{n}^\beta, e^{-2\pi i \langle \cdot, \xi\rangle}\rangle = \langle \Phi_*(\beta(\xi)e^{2\pi i\omega}), e^{-2\pi i\langle\cdot,\xi\rangle}\rangle = \int_M \beta(\xi) e^{2\pi i(\omega - \langle \Phi, \xi\rangle)}.$$

Proofs of part (e) may be found in [25, 40, 15]. □

Remark 4.5. A well-known result of Kirwan [27] says that if M is compact and connected, and the 2-form ω is symplectic, then all level sets $\Phi^{-1}(\mu)$ are connected. Hence, in this case (e) directly relates pairings on $\Phi^{-1}(\mu)/G_\mu$. More generally, this is the case if each component of the level set $\Phi^{-1}(\mu)$ intersects the principal stratum for the G-action on M.

Since the integrand in (d) is closed under $\mathrm{d}_\xi = \mathrm{d} - \iota_{\xi_M}$, the Fourier transform of $\mathfrak{n}^\beta$ for a compactly supported cocycle β may be calculated by localization (Theorem B.4), as a sum over fixed point manifolds for the vector field ξ_M:

$$\langle \mathfrak{n}^\beta, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle = \sum_{F \in \mathcal{F}(\xi)} \int_F \frac{\beta(\xi) e^{2\pi i(\omega - \langle\Phi,\xi\rangle)}}{\mathrm{Eul}(\nu_F, \xi)}, \quad \xi \in \mathfrak{g}.$$

Combining this with the interpretation (e) of $\mathfrak{n}^\beta$ leads to formulas for intersection pairings on reduced spaces in terms of fixed point data, such as the Jeffrey-Kirwan theorem [25].

4.3. **The isomorphism $R_\mathfrak{g}$.** Choose orientations on $\mathfrak{g}$ and $\mathfrak{t}$, compatible with the orientation of $\mathfrak{p}$ given by the set $\mathfrak{R}_+$ of positive roots. Let $\mathrm{d\,vol}_{\mathfrak{g}^*}$ denote the volume form on $\mathfrak{g}^*$, corresponding to the given invariant inner product on $\mathfrak{g}$, and similarly $\mathrm{d\,vol}_{\mathfrak{t}^*}$ the volume form for the restriction of the inner product.

Let $\mathcal{D}'(\mathfrak{t}^*)^{W-\mathrm{alt}}$ denote the subspace of distributions on $\mathfrak{t}^*$ which *alternate* under the action of the Weyl group W, that is, $w_*\mathfrak{m} = (-1)^{l(w)}\mathfrak{m}$ where $l(w)$ is the length of $w \in W$. More generally, for an invariant open subset $V \subset \mathfrak{g}^*$ we may consider the space $\mathcal{D}'(V \cap \mathfrak{t}^*)^{W-\mathrm{alt}}$.

Lemma 4.6. *There is a unique linear isomorphism*

$$R_\mathfrak{g} : \mathcal{D}'(V)^G \to \mathcal{D}'(V \cap \mathfrak{t}^*)^{W-\mathrm{alt}},$$

with the following two properties:

(a) **(C^∞-linearity)** $R_\mathfrak{g}(f\mathfrak{n}) = f|_{\mathfrak{t}^*} R_\mathfrak{g}(\mathfrak{n})$ *for all* $f \in C^\infty(V)^G$ *and* $\mathfrak{n} \in \mathcal{D}'(V)^G$.

(b) **(Fourier coefficients)** *For compactly supported* $\mathfrak{n}$*, the Fourier transform of* $R_\mathfrak{g}\mathfrak{n}$ *is given by*

$$\langle R_\mathfrak{g}\mathfrak{n},\, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle = (-1)^{n_+} \prod_{\alpha \in \mathfrak{R}_+} 2\pi i \langle \alpha, \xi\rangle \; \langle \mathfrak{n},\, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle, \quad \xi \in \mathfrak{t}.$$

The image of the volume form is given by

$$R_{\mathfrak{g}}(\mathrm{d}\,\mathrm{vol}_{\mathfrak{g}^*}) = (\prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha)\,\mathrm{d}\,\mathrm{vol}_{\mathfrak{t}^*};$$

here $\alpha \in \mathfrak{R}_+ \subset \mathfrak{t}^*$ is viewed a linear form on $\mathfrak{t}^*$, using the inner product.

Proof. The map on test functions

$$C^\infty(V)^G_{\mathrm{comp}} \to C^\infty(V\cap\mathfrak{t}^*)^{W-\mathrm{alt}}_{\mathrm{comp}},\ f \mapsto \frac{\mathrm{vol}_{G/T}}{\#W}(\prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha) f|_{V\cap\mathfrak{t}^*} \tag{15}$$

is a continuous linear isomorphism, independent of the choice of inner product. Let $R_{\mathfrak{g}} : \mathcal{D}'(V)^G \to \mathcal{D}'(V\cap\mathfrak{t}^*)^{W-\mathrm{alt}}$ be defined as the inverse of the dual map. Clearly, $R_{\mathfrak{g}}$ satisfies (a). To establish (b), it suffices to consider the case that $\mathfrak{n}$ is the Liouville measure of a coadjoint orbit $G.\mu$ for a regular element $\mu \in \mathfrak{t}^*_+$, since linear combinations of such measures are dense in $\mathcal{D}'(\mathfrak{g}^*)^G$. That is, we assume that $\mathfrak{n}$ is the unique invariant measure supported on $G.\mu$, with total integral equal to the symplectic volume $\mathrm{Vol}(G.\mu) = (\prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha\cdot\mu)\,\mathrm{vol}_{G/T}$. In this case,

$$R_{\mathfrak{g}}(\mathfrak{n}) = \sum_{w\in W} (-1)^{l(w)}\delta_{w\mu}, \tag{16}$$

as one verifies on test functions. Now (b) follows by the formula for Fourier transforms of coadjoint orbits [8, Chapter 7]. Finally, the formula for $R_{\mathfrak{g}}(\mathrm{d}\,\mathrm{vol}_{\mathfrak{g}^*})$ follows from the Weyl integration formula,

$$\int_{\mathfrak{g}^*} f\,\mathrm{d}\,\mathrm{vol}_{\mathfrak{g}^*} = \frac{\mathrm{vol}_{G/T}}{\#W}\int_{\mathfrak{t}^*} f|_{\mathfrak{t}^*}\,(\prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha)^2\mathrm{d}\,\mathrm{vol}_{\mathfrak{t}^*}\,.$$

□

Example 4.7 (Duistermaat-Heckman measures of coadjoint orbits). Let $\delta_0 \in \mathcal{D}'(\mathfrak{g}^*)^G$ be the delta-measure supported at $0 \in \mathfrak{g}^*$. Then $R_{\mathfrak{g}}(\delta_0)$ is expressed in terms of derivatives of the delta-measure on $\mathfrak{t}^*$:

$$R_{\mathfrak{g}}(\delta_0) = (-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha, \frac{\partial}{\partial\mu}\rangle\ \delta_0.$$

This follows from (b), by taking Fourier transforms. Using the characterization of $R_{\mathfrak{g}}$ as the inverse dual map to (15), applied to $f = 1$, one obtains the identity,

$$\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha, \frac{\partial}{\partial\mu}\rangle\ \prod_{\alpha\in\mathfrak{R}_+} 2\pi\langle\alpha,\cdot\rangle = \frac{\#W}{\mathrm{vol}_{G/T}}. \tag{17}$$

Proposition 4.8. *Assume $V \subset \mathfrak{g}^*$ is an invariant open subset containing the origin. Suppose $\mathfrak{n} = f\,\mathrm{d}\,\mathrm{vol}_{\mathfrak{g}^*}$ with $f \in C^\infty(V)^G$, maps to $R_{\mathfrak{g}}(\mathfrak{n}) = F\,\mathrm{d}\,\mathrm{vol}_{\mathfrak{t}^*}$ with $F \in C^\infty(V\cap\mathfrak{t}^*)^{W-\mathrm{alt}}$. Then*

$$f(0) = \frac{\mathrm{vol}_{G/T}}{\#W}\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha, \frac{\partial}{\partial\mu}\rangle\Big|_{\mu=0} F.$$

Proof. This follows from (17) since $F = \prod_{\alpha\in\mathfrak{R}_+} 2\pi\langle\alpha,\cdot\rangle\ f|_{\mathfrak{t}^*}$. $\square$

By Proposition 4.1, the space $\mathcal{D}'(V)^G$ can be thought of as the cohomology for the 'twisted' equivariant differential on the space $\mathcal{C}_G(V)$ of equivariant currents while $\mathcal{D}'(V\cap\mathfrak{t}^*)$ is the cohomology of the space of equivariant currents on $V\cap\mathfrak{t}^*$ (where T acts trivially). Hence, $R_\mathfrak{g}$ may be viewed as a map of cohomology spaces, $H(\mathcal{C}_G(V),\mathrm{d}'_G)\to H(\mathcal{C}_T(V\cap\mathfrak{t}^*),\mathrm{d}'_T)$. For our applications to Duistermaat-Heckman theory, it will be important to realize this map on the 'chain level'. Let $\pi:\mathfrak{g}^*\to\mathfrak{t}^*$ denote projection to the first factor in $\mathfrak{g}^*=\mathfrak{t}^*\oplus\mathfrak{p}^*$, and choose a representative of the T-equivariant Thom form τ of the vector bundle $\pi^{-1}(V\cap\mathfrak{t}^*)\to V\cap\mathfrak{t}^*$, with fiberwise compact support in the intersection $V\cap\pi^{-1}(V\cap\mathfrak{t}^*)$. Recall that the pull-back of τ to $V\cap\mathfrak{t}^*$ represents the equivariant Euler class, hence it is T-equivariantly cohomologous to $(-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha,\cdot\rangle$. Define a map

$$\mathcal{C}_G(V)\to\mathcal{C}_T(V\cap\mathfrak{t}^*)^{W-\mathrm{alt}},\ \phi\mapsto\psi=(2\pi i)^{n_+}\pi_*(\tau\,\kappa_T(\phi)). \tag{18}$$

Clearly, (18) intertwines d'_G with d'_T.

Proposition 4.9. *The induced map in cohomology*

$$H(\mathcal{C}_G(V),\mathrm{d}'_G)=\mathcal{D}'(V)^G\to H(\mathcal{C}_T(V\cap\mathfrak{t}^*),\mathrm{d}'_T)=\mathcal{D}'(V\cap\mathfrak{t}^*)$$

takes values in $\mathcal{D}'(V\cap\mathfrak{t}^*)^{W-\mathrm{alt}}$, and coincides with the map $R_\mathfrak{g}$.

Proof. Suppose the induced map in cohomology takes $\mathfrak{n}\in\mathcal{D}'(V)^G$ to $\mathfrak{m}\in\mathcal{D}'(V\cap\mathfrak{t}^*)$. We will show $\mathfrak{m}=R_\mathfrak{g}(\mathfrak{n})$. Using a partition of unity, we may assume that $\mathfrak{n}$ has compact support. As shown above, $\mathfrak{n}$ has a unique representative $\phi\in\mathcal{C}(V)_{\mathrm{basic}}$, given as $\phi=\exp(\frac{1}{2\pi i}\iota(P))\mathfrak{n}$. Since$\mathfrak{n}$ has compact support, so does ϕ. By Fourier transform, for $\xi\in\mathfrak{t}$,

$$\langle\mathfrak{m},e^{-2\pi i\langle\cdot,\xi\rangle}\rangle=(2\pi i)^{n_+}\int_{\mathfrak{t}^*}\pi_*(\tau(\xi)\,\phi)e^{-2\pi i\langle\cdot,\xi\rangle}=(2\pi i)^{n_+}\int_{\mathfrak{g}^*}\tau(\xi)\,\phi\,e^{-2\pi i\langle\cdot,\xi\rangle}. \tag{19}$$

Since ϕ has compact support, we can replace the equivariant Thom form by the cohomologous form (with non-compact support) $(-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha,\cdot\rangle\in\mathrm{Pol}(\mathfrak{t})\subset\Omega_T(\mathfrak{g}^*)$. Hence (19) equals

$$(-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}2\pi i\langle\alpha,\xi\rangle\ \langle\mathfrak{n},e^{-2\pi i\langle\cdot,\xi\rangle}\rangle=\langle R_\mathfrak{g}\mathfrak{n},\ e^{-2\pi i\langle\cdot,\xi\rangle}\rangle.$$

$\square$

4.4. **The distributions $\mathfrak{m}^\beta$.** We return to the setting of Section 4.2: (M,ω,Φ) is an oriented Hamiltonian G-space, with proper moment map $\Phi:M\to V\subset\mathfrak{g}^*$. Let $\mathfrak{n}^\beta\in\mathcal{D}'(V)$ be the invariant distribution associated to an equivariant cocycle $\beta\in\Omega_G(M)$. We may then define a distribution $\mathfrak{m}^\beta$ on $V\cap\mathfrak{t}^*$ as simply the image of $\mathfrak{n}^\beta$ under the isomorphism $R_\mathfrak{g}$:

$$\mathfrak{m}^\beta=R_\mathfrak{g}(\mathfrak{n}^\beta)\in D'(V\cap\mathfrak{t}^*)^{W-\mathrm{alt}}.$$

A more geometric construction of $\mathfrak{m}^\beta$ is obtained as follows. Consider the Hamiltonian T-space (M,ω,Φ_T) where the T-moment map Φ_T is Φ followed by projection

$\mathfrak{g}^* \to \mathfrak{t}^*$. Then $\mathfrak{m}^\beta$ is the DH-distribution for this Hamiltonian T-space, corresponding to the T-equivariant cocycle $(2\pi i)^{n_+}\tau_M\kappa_T(\beta)$ where $\tau_M = \Phi^*\tau$:

$$\mathfrak{m}^\beta = (2\pi i)^{n_+}\sum_I \big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\big)^I (\Phi_T)_*\big(\tau_M\kappa_T(\beta)e^{2\pi i\omega}\big)_I^{[\dim M]}. \tag{20}$$

If β is compactly supported, this can be written in terms of Fourier transform:

$$\langle \mathfrak{m}^\beta, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle = (2\pi i)^{n_+}\int_M \beta(\xi)\tau_M(\xi)e^{2\pi i(\omega-\langle\Phi_T,\xi\rangle)},\quad \xi\in\mathfrak{t}.$$

Due to the factor τ_M, the integral is localized to an arbitrary small neighborhood of $X=\Phi^{-1}(V\cap\mathfrak{t}^*)$. That is, it depends only on the restriction of β to an arbitrarily small neighborhood of X. If the moment map Φ is transverse to $\mathfrak{t}^*$, so that X is a smooth submanifold, this becomes more concrete:

Proposition 4.10. *Suppose the moment map Φ is transversal to $\mathfrak{t}^*$, and let $X=\Phi^{-1}(V\cap\mathfrak{t}^*)$ with moment map $\Phi_X=\iota^*\Phi$ and 2-form $\omega_X=\iota^*\omega$ given by pull-back under the inclusion $\iota: X\hookrightarrow M$. Then the distribution $\mathfrak{m}^\beta$ for a cocycle $\beta\in\Omega_G(M)$ is just the DH-distribution for the Hamiltonian T-space (X,ω_X,Φ_X), associated to the cocycle $\iota_X^*\beta$.*

Proof. In the transverse case, a tubular neighborhood of X in M may be identified in a T-equivariant way with $X\times\mathfrak{p}$, in such a way that the diagram

$$\begin{tikzcd} X\times\mathfrak{p} \arrow[r] \arrow[d] & M \arrow[d, "\Phi"'] \\ (V\cap T)\times\mathfrak{p} \arrow[r] & V \end{tikzcd}$$

commutes. Then $\tau_M=\Phi^*\tau$ represents the Thom class of the vector bundle $\pi_X: X\times\mathfrak{p}\to X$. The distribution $\mathfrak{m}^\beta$ does not change if $\kappa_T(\beta)e^{2\pi i\omega}$ is replaced by $\pi_X^*(\beta_X e^{2\pi i\omega_X})$, since the difference is d_T-exact. But then, using $\Phi_T=\Phi_X\circ\pi_X$,

$$(\Phi_T)_*(\tau_M\kappa_T(\beta)e^{2\pi i\omega}) = (\Phi_X)_*(\pi_X)_*(\tau_M\pi_X^*(\beta_X e^{2\pi i\omega_X}) = (\Phi_X)_*(\beta_X e^{2\pi i\omega_X}).$$

□

We refer to Woodward [42] for interesting examples of Hamiltonian G-spaces with Φ transverse to $\mathfrak{t}^*$. In the case of a regular coadjoint orbit, Proposition 4.10 is illustrated by Equation (16).

5. DH-distributions for q-Hamiltonian G-spaces

Throughout this Section, we assume that ρ (the half-sum of positive roots) is a weight of G. For instance, G may be a product of a simply connected group and a torus, or $G=\mathrm{U}(n)$ with n odd. We identify $\mathfrak{g}\cong\mathfrak{g}^*$ by means of an invariant inner product on $\mathfrak{g}$. Similar to the Hamiltonian setting, any equivariant cocycle $\beta\in\Omega_G(M)$ on a q-Hamiltonian G-space (M,ω,Φ) defines an equivariant current $\Phi_*(\beta e^{2\pi i\omega})$ on G, closed under $\mathrm{d}_G'=\mathrm{d}_G-2\pi i\eta_G$. Our first goal is, therefore, to associate to any d_G'-cocycle on G an invariant distribution.

5.1. **The isomorphism R_G.** The group analogue to the product of positive roots $\mu \mapsto \prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha\cdot\mu$ is the function $t \mapsto i^{-n_+}A(t)$, where

$$A = \sum_{w\in W} (-1)^{l(w)} \epsilon_{w\rho}$$

is the Weyl denominator. Indeed,

$$i^{-n_+}A(\exp\mu) = \prod_{\alpha\in\mathfrak{R}_+} 2\sin(\pi\alpha\cdot\mu).$$

Pick orientations on G, T, compatible with the given orientation on $\mathfrak{p}$, and let $\mathrm{d}\,\mathrm{vol}_G$ and $\mathrm{d}\,\mathrm{vol}_T$ be the volume forms defined by the orientation and the inner product. Let $V \subset G$ be an invariant open subset. Similar to Lemma 4.6 we have:

Lemma 5.1. *There is a unique linear isomorphism $R_G : \mathcal{D}'(V)^G \to \mathcal{D}'(V\cap T)^{W-\mathrm{alt}}$, with the following two properties:*

(a) **(C^∞-linearity)** $R_G(f\mathfrak{n}) = f|_{V\cap T}R_G(\mathfrak{n})$ *for all* $f \in C^\infty(V)^G$ *and* $\mathfrak{n} \in \mathcal{D}'(V)^G$.

(b) **(Fourier coefficients)** *For compactly supported $\mathfrak{n}$, the Fourier transform of $R_\mathfrak{g}\mathfrak{n}$ is given by*

$$\langle R_G(\mathfrak{n}), \overline{\epsilon_{\lambda+\rho}}\rangle = \frac{i^{-n_+}}{\mathrm{vol}_{G/T}} \langle \mathfrak{n}, \overline{\chi_\lambda}\rangle,\ \ \lambda\in\Lambda^*_+.$$

For the volume form on G one finds,

$$R_G(\mathrm{d}\,\mathrm{vol}_G) = i^{-n_+}A\ \mathrm{d}\,\mathrm{vol}_T$$

Proof. Define R_G as the inverse dual map to the isomorphism,

$$C^\infty(V)^G_{\mathrm{comp}} \to C^\infty(V\cap T)^{W-\mathrm{alt}}_{\mathrm{comp}},\ \ f \mapsto \frac{\mathrm{vol}_{G/T}}{\#W}\ i^{-n_+}A\, f|_{V\cap T}$$

Then R_G clearly satisfies (a), while Equation (b) follows from the Weyl character formula:

$$\begin{aligned}
\langle\mathfrak{n}, \overline{\chi_\lambda}\rangle &= \frac{\mathrm{vol}_{G/T}}{\#W}\ i^{n_+}\langle R_G\mathfrak{n},\ \overline{A\,\chi_\lambda|_T}\rangle \\
&= \frac{\mathrm{vol}_{G/T}}{\#W}\ i^{n_+}\sum_{w\in W}(-1)^{l(w)}\langle R_G\mathfrak{n}, \overline{\epsilon_{w(\lambda+\rho)}}\rangle \\
&= \mathrm{vol}_{G/T}\ i^{n_+}\langle R_G\mathfrak{n}, \overline{\epsilon_{\lambda+\rho}}\rangle.
\end{aligned}$$

Similarly, the formula for $R_G(\mathrm{d}\,\mathrm{vol}_G)$ is a consequence of the Weyl integration formula. □

The maps $R_\mathfrak{g}$ and R_G are related as follows:

Lemma 5.2. *The Duflo map* $\mathrm{Duf}_G : \mathcal{D}'(\mathfrak{g})^G_{\mathrm{comp}} \to \mathcal{D}'(G)^G,\ \mathfrak{n}\mapsto \exp_* J^{1/2}\mathfrak{n}$ *intertwines the two maps $R_\mathfrak{g}$ and R_G. That is,*

$$R_G \circ \mathrm{Duf}_G = \exp_* \circ R_\mathfrak{g}$$

Proof. The identity $i^{-n_+}A(\exp\mu) = J^{1/2}(\mu)\ \prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha\cdot\mu$ for $\mu\in\mathfrak{t}$ shows that for any test function $f\in C^\infty(G)^G$,

$$\exp^*(i^{-n_+}A\,f|_T) = \Big(\prod_{\alpha\in\mathfrak{R}_+} 2\pi\alpha\Big)\ J^{1/2}\exp^* f.$$

Using the definition of $R_\mathfrak{g}, R_G$ as dual maps to maps on test functions, this yields $(R_G)^{-1}\circ\exp_* = \mathrm{Duf}_G\circ(R_\mathfrak{g})^{-1}$. □

We will now use R_G to construct a map $\mathcal{C}_G(V)\to\mathcal{D}'(V)^G$, vanishing on cocycles for $\mathrm{d}'_G = \mathrm{d}_G - 2\pi i\eta_G$. This map will be given in terms of a commutative diagram

$$\begin{tikzcd} \mathcal{C}_G(V) \arrow[r] \arrow[d] & \mathcal{D}'(V)^G \arrow[d, "R_G"', "\cong"] \\ \mathcal{C}_T(V\cap T)^{W-\mathrm{alt}} \arrow[r] & \mathcal{D}'(V\cap T)^{W-\mathrm{alt}} \end{tikzcd} \tag{21}$$

Here the W-action on $\mathcal{C}_T(V\cap T)$ is $(w.\phi)(\xi) = w_*(\phi(w^{-1}.\xi))$. The lower horizontal map in the diagram is given by

$$\mathcal{C}_T(V\cap T)\to\mathcal{D}'(V\cap T),\quad \psi\mapsto\mathfrak{m} = \sum_I\Big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\Big)^I\psi_I^{[\dim\mathfrak{t}]} \tag{22}$$

where we use $(\frac{1}{2\pi i}\frac{\partial}{\partial\mu})^I$ to denote both a constant coefficient differential operator on $\mathfrak{t}$, and also the induced differential operator on T. Note that this map is W-equivariant and vanishes on $\mathrm{d}'_T = \mathrm{d} - 2\pi i\eta_T$-cocycles. If ψ has compact support, this map can be characterized in terms of Fourier coefficients by

$$\langle\mathfrak{m},\overline{\epsilon_\lambda}\rangle = \langle\psi(\lambda),\overline{\epsilon_\lambda}\rangle,\quad \lambda\in\Lambda^*. \tag{23}$$

To construct the left vertical map, consider the T-equivariant tubular neighborhood $\pi: U\to T$ of $T\subset G$, described in Section 3.2. Recall the 2-form $\gamma\in\Omega^2(U)^T$, satisfying $\mathrm{d}_T\gamma = \kappa_T(\eta_G) - \pi^*\eta_T$. Restrict the bundle $\pi: U\to T$ to the intersection $V\cap T$, and let $\tau\in\Omega_T^{2n_+}(\pi^{-1}(V\cap T))$ be a representative of the T-equivariant Thom class, supported in the intersection $\pi^{-1}(V\cap T)\cap V$. Then

$$\mathcal{C}_G(V)\to\mathcal{C}_T(V\cap T),\quad \phi\mapsto\psi = (2\pi i)^{n_+}\pi_*(\tau\, e^{2\pi i\gamma}\,\kappa_T(\phi)) \tag{24}$$

is a well-defined map intertwining $\mathrm{d}'_G = \mathrm{d}_G - 2\pi i\eta_G$ with $\mathrm{d}'_T = \mathrm{d}_T - 2\pi i\eta_T$. Since the action of $w\in W$ on $\mathfrak{p}$ changes the orientation by $(-1)^{l(w)}$, we may choose τ in such a way that for $g\in N_G(T)$, $g^*\tau(\xi) = (-1)^{l(w)}\tau(w^{-1}.\xi)$. With this choice, the map (24) takes values in $\mathcal{C}_T(V\cap T)^{W-\mathrm{alt}}$. We define the map $\mathcal{C}_G(V)\to\mathcal{D}'(V)^G,\ \phi\mapsto\mathfrak{n}$ in the unique way making (21) commute. That is,

$$\mathfrak{n} = (2\pi i)^{n_+}R_G^{-1}\Big(\sum_I\Big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\Big)^I\pi_*(\tau\,e^{2\pi i\gamma}\,\kappa_T(\phi))_I^{[\dim\mathfrak{g}]}\Big). \tag{25}$$

By construction, (25) vanishes on d'_G-coboundaries, and the induced map in d'_G-cohomology is independent of the choice of τ.

5.2. **DH-distributions for q-Hamiltonian G-spaces.** Let (M,ω,Φ) be an oriented q-Hamiltonian G-space, possibly degenerate, with proper moment map $\Phi: M \to V \subset G$. (Recall that if ω is minimally degenerate, there exists a distinguished orientation, see Lemma 2.1). Let $\beta \in \Omega_G(M)$ be a closed equivariant differential form. Then the current

$$\phi = \Phi_*(\beta e^{2\pi i\omega}) \in C_G(V)$$

is $\mathrm{d}_G' = \mathrm{d}_G - 2\pi i \eta_G$-closed. Let $\mathfrak{n}^\beta \in \mathcal{D}'(V)^G$ be defined by (25). The distribution $\mathfrak{m}^\beta = R_G(\mathfrak{n}^\beta) \in \mathcal{D}'(V \cap T)^{W-\mathrm{alt}}$ admits an interpretation similar to the Hamiltonian case, Section 4.4. Indeed, let (N,Ω_N,Φ_N) be the q-Hamiltonian T-space given as the 'Abelianization' of (M,ω,Φ). Then $\mathfrak{m}^\beta$ is the DH-distribution corresponding to the cocycle $(2\pi i)^{n_+}\tau_N \kappa_T(\beta) \in \Omega_T(N)$ where $\tau_N = \Phi^*\tau$:

$$\mathfrak{m}^\beta = (2\pi i)^{n_+} \sum_I \big(\frac{1}{2\pi i}\frac{\partial}{\partial \mu}\big)^I (\Phi_N)_* \Big(\tau_N \kappa_T(\beta) e^{2\pi i \omega_N}\Big)_I^{[\dim M]}. \tag{26}$$

If Φ is transverse to the maximal torus, the same argument as for Proposition 4.10 shows that $\mathfrak{m}^\beta$ is a DH-distribution for the q-Hamiltonian T-space (X,ω_X,Φ_X) where $X = \Phi^{-1}(T)$ and ω_X, Φ_X are pull-backs of ω, Φ.

Theorem 5.3 (Properties of the q-Hamiltonian DH-distributions)**.** *Let $\beta \in \Omega_G(M)$ be an equivariant cocycle, and $\mathfrak{n}^\beta$ the associated distribution.*

(a) *$\mathfrak{n}^\beta$ depends only on the cohomology class of β.*
(b) *If $V' \subset V$ is an invariant open subset, then $\mathfrak{n}^\beta|_{V'}$ is the DH-distribution corresponding to $\beta|_{\Phi^{-1}(V')}$.*
(c) *If $g \in G$ is a β-regular value of Φ, the distribution $\mathfrak{n}^\beta$ is smooth at g.*
(d) *If β has compact support, then the distribution $\mathfrak{n}^\beta$ has compact support contained in $\Phi(\mathrm{supp}(\beta))$. Its Fourier coefficients $\langle \mathfrak{n}^\beta, \overline{\chi_\lambda}\rangle$ are given by the integral*

$$\langle \mathfrak{n}^\beta, \overline{\chi_\lambda}\rangle = (-2\pi)^{n_+} \mathrm{vol}_{G/T} \int_N \tau_N(\lambda+\rho)\beta(\lambda+\rho) e^{2\pi i \omega_N} \Phi_N^* \overline{\epsilon_{\lambda+\rho}}, \quad \lambda \in \Lambda_+^*$$

which localizes to the fixed point set of $(\lambda+\rho)_M$:

$$\langle \mathfrak{n}^\beta, \overline{\chi_\lambda}\rangle = \dim V_\lambda \sum_{F \in \mathcal{F}(\lambda+\rho)} \int_F \frac{\iota_F^*\beta(\lambda+\rho)\, e^{2\pi i \omega_F}\, \Phi_F^* \overline{\epsilon_{\lambda+\rho}}}{\mathrm{Eul}(\nu_F, \lambda+\rho)}.$$

Here $\omega_F \in \Omega^2(F)$ and $\Phi_F: F \to T$ are defined as pull-backs of ω and Φ to the fixed point manifold F.

Proof. (a), (b) and (c) follow from the properties of the map $\mathcal{C}_G(V) \to \mathcal{D}'(V)^G$. In particular, if $g \in G$ is a β-regular value, the current $\Phi_*(\beta e^{2\pi i \omega})$ is smooth near g, hence so is the distribution $\mathfrak{n}^\beta$.

Suppose next that β is compactly supported. Using $(\frac{1}{2\pi i}\frac{\partial}{\partial \mu})^I \epsilon_\lambda = \lambda^I \epsilon_\lambda$, the Fourier coefficients of the distribution $\mathfrak{m}^\beta$ may be written,

$$\langle \mathfrak{m}^\beta, \overline{\epsilon_{\lambda+\rho}}\rangle = (2\pi i)^{n_+} \int_N \tau_N(\lambda+\rho)\beta(\lambda+\rho) e^{2\pi i \omega_N} \Phi_N^* \overline{\epsilon_{\lambda+\rho}}, \quad \lambda \in \Lambda_+^*.$$

This implies the first Formula in (d), by Lemma 5.1(b). Since the integrand is $\mathrm{d}_{\lambda+\rho}$-closed, the integral may be computed by localization, see Theorem B.2. The fixed point components F are all contained in $\Phi^{-1}(T)$, by equivariance of the moment map. The second Formula in (d) follows since $\iota_F^*\Phi_N = \iota_F^*\Phi$ and $\iota_F^*(\omega_N) = \iota_F^*\omega$, and since $\iota_T^*\tau$ is cohomologous to

$$(-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}\alpha\cdot(\lambda+\rho) = (-2\pi)^{-n_+}\frac{\dim V_\lambda}{\mathrm{vol}_{G/T}},$$

where we used the Weyl dimension formula. □

5.3. **Relation to intersection pairings.** Our goal in this Section is to prove:

Theorem 5.4. *Let $V\subset G$ be an invariant open subset containing the group unit e, and (M,ω,Φ) a q-Hamiltonian G-space, with proper moment map $\Phi: M\to V\subset G$. Suppose $\beta\in\Omega_G(M)$ is an equivariant cocycle, and that $e\in V$ is a β-regular value of Φ. Then the distribution $\mathfrak{n}^\beta$ is smooth near e, and*

$$\mathfrak{n}^\beta(e) = (2\pi i)^{\dim G}\,\mathrm{vol}_G\sum_j\frac{1}{k_j}\int_{(M/\!\!/G)_j}\beta_{\mathrm{red}}e^{2\pi i\omega_{\mathrm{red}}}.$$

Here the sum is over connected components of $M/\!\!/G$, and k_j is the cardinality of a generic stabilizer for the G-action on the jth component of $\Phi^{-1}(e)$.

The idea of proof is to compare with the DH distribution for the Hamiltonian G-space described in 3.1. We first discuss a similar problem for equivariant currents. Recall that in (11), we defined a 2-form $\varpi\in\Omega^2(\mathfrak{g})^G$ with $\mathrm{d}_G\varpi = \exp^*\eta_G-\eta_{\mathfrak{g}}$.

Lemma 5.5. *Suppose $V_0\subset\mathfrak{g}$ is an invariant open subset of $\mathfrak{g}$, such that* exp *restricts to a diffeomorphism onto the image $V=\exp(V_0)$. Let $\phi\in\Omega_G(V)\subset\mathcal{C}_G(V)$ be a smooth equivariant current, and $\phi_0 := e^{-2\pi i\varpi}\exp^*\phi\in\Omega_G(V_0)$. Then ϕ_0 is closed under $\mathrm{d}_G-2\pi i\eta_{\mathfrak{g}}$ if and only if ϕ is closed under $\mathrm{d}_G-2\pi i\eta_G$. The corresponding (smooth) measures $\mathfrak{n}_0$ on V_0 and $\mathfrak{n}$ on V are related by the Duflo map:*

$$\mathfrak{n} = \exp_*(J^{1/2}\mathfrak{n}_0). \tag{27}$$

Proof. The first claim is clear since conjugation by $e^{2\pi i\varpi}$ intertwines $\mathrm{d}_G-2\pi i\exp^*\eta_G$ with $\mathrm{d}_G-2\pi i\eta_{\mathfrak{g}}$. For the second claim, it suffices, by Corollary 4.3, to consider the case that ϕ,ϕ_0 have compact support. Let $\mathfrak{m}=R_G(\mathfrak{n})$ and $\mathfrak{m}_0=R_{\mathfrak{g}}(\mathfrak{n}_0)$. By Lemma 5.2 we have to show

$$\mathfrak{m} = \exp_*\mathfrak{m}_0.$$

Since $\mathfrak{m}_0,\mathfrak{m}$ are compactly supported and W-alternating, it is enough to show that

$$\langle\mathfrak{m},\overline{\epsilon_{\lambda+\rho}}\rangle = \langle\mathfrak{m}_0, e^{-2\pi i\langle\cdot,\lambda+\rho\rangle}\rangle,\quad \lambda\in\Lambda_+^*. \tag{28}$$

By definition

$$\langle\mathfrak{m},\overline{\epsilon_{\lambda+\rho}}\rangle = (2\pi i)^{n_+}\int_U\tau(\lambda+\rho)e^{2\pi i\gamma}\phi(\lambda+\rho)\pi^*\overline{\epsilon_{\lambda+\rho}}.$$

where the integrand is closed for the differential $\mathrm{d}_{\lambda+\rho}$. Integrating over the fibers of $\pi : U \to T$, and using $\iota_T^* \gamma = 0$, this gives

$$\langle \mathfrak{m}, \overline{\epsilon_{\lambda+\rho}} \rangle = (2\pi i)^{n_+} \int_T \iota_T^* \phi'(\lambda+\rho) \overline{\epsilon_{\lambda+\rho}}.$$

A similar argument for ϕ_0 shows that

$$\langle \mathfrak{m}_0, e^{-2\pi i \langle \cdot, \xi \rangle} \rangle = (2\pi i)^{n_+} \int_{\mathfrak{t}} \iota_{\mathfrak{t}}^* \phi_0(\xi) e^{-2\pi i \langle \cdot, \xi \rangle}, \quad \xi \in \mathrm{int}(\mathfrak{t}_+).$$

But $\exp_T^* \iota_T^* \phi = \iota_{\mathfrak{t}}^* \phi_0$, since $\iota_{\mathfrak{t}}^* \varpi = 0$. Hence we have proved (28) and therefore the Lemma. ☐

Proof of Theorem 5.4. Replacing V with a smaller neighborhood of e if necessary (and M with the corresponding pre-image under Φ), we may assume that all points of V are β-regular values of Φ. Furthermore, we may assume that V is the diffeomorphic image under exp of an invariant open neighborhood $V_0 \subset \mathfrak{g}$. Let (M, ω_0, Φ_0) be given by linearization (cf. Section 3.1). By construction, the currents $\phi = \Phi_*(\beta e^{2\pi i \omega})$ on V and $\phi_0 = (\Phi_0)_*(\beta e^{2\pi i \omega_0})$ are smooth, and $\phi = \exp_*(e^{2\pi i \varpi} \phi_0)$. Hence, Lemma 5.5 shows that the measures $\mathfrak{n}_0^\beta$ and $\mathfrak{n}^\beta$ are related by the Duflo map $\mathrm{Duf}_G : \mathcal{D}'(V_0)^G \to \mathcal{D}'(V)^G$. Since $J^{1/2}(0) = 1$, it follows in particular that

$$\mathfrak{n}^\beta(e) = \mathfrak{n}_0^\beta(0)$$

where we use the Riemannian measures on $\mathfrak{g}$ and on G to identify smooth measures with functions. Since $\omega_{\mathrm{red}} = (\omega_0)_{\mathrm{red}}$, the Theorem follows from the interpretation of DH-distributions for Hamiltonian spaces, Proposition 4.4(e). ☐

Remark 5.6. More generally, the value of $\mathfrak{n}^\beta$ at any β-regular value g of Φ is given by an integral over components of $\Phi^{-1}(g)/G_g$, similar to 4.4(e), with the symplectic volume of a (co-)adjoint orbit $G.\mu$ replaced by the q-Hamiltonian volume of a conjugacy class (7). For $g = \exp \mu$ with μ sufficiently small, this follows by the same argument as for $g = e$, using that $\mathrm{Vol}(G.\exp \mu) = J^{1/2}(\mu)\,\mathrm{Vol}(G.\mu)$. For general g, the result can be obtained using cross-sections [1].

5.4. **Fixed point formula for intersection pairings.** Suppose (M, ω, Φ) is an oriented q-Hamiltonian G-space, with proper moment map $\Phi : M \to G$. (In particular, M is compact.) Assume that $e \in G$ is a regular value of Φ.

Given an equivariant cocycle $\beta \in \Omega_G(M)$ consider the Fourier expansion of $\mathfrak{n}^\beta$:

$$\mathfrak{n}^\beta = \frac{1}{\mathrm{vol}_G} \sum_{\lambda \in \Lambda_+^*} \langle \mathfrak{n}^\beta, \overline{\chi_\lambda} \rangle\, \chi_\lambda$$

where $\mathrm{d}\,\mathrm{vol}_G$ is used to identify distributional measures on G with distributional functions. Combining Theorem 5.4 with 5.3(d), and using $\chi_\lambda(e) = \dim V_\lambda$, one recovers the following result from [3]:

$$\sum_j \frac{1}{k_j}\int_{(M/\!\!/G)_j}\beta_{\mathrm{red}}e^{2\pi i\omega_{\mathrm{red}}}$$
$$= (2\pi i)^{-\dim G}\sum_{\lambda\in\Lambda^*_+}(\frac{\dim V_\lambda}{\mathrm{vol}_G})^2\sum_{F\in\mathcal{F}(\lambda+\rho)}\int_F\frac{\iota_F^*\beta(\lambda+\rho)\,e^{2\pi i\omega_F}}{\mathrm{Eul}(\nu_F,\lambda+\rho)}\Phi_F^*\overline{\epsilon_{\lambda+\rho}}$$

Of course, the formal substitution $g = e$ is only valid if the Fourier coefficients decay sufficiently fast, to ensure convergence of this series. In the general case, one can introduce a convergence factor $\exp(-\epsilon||\lambda+\rho||^2)$ in the sum, and obtains an equality for $\epsilon \to 0$. See [29] and [4] for more detailed discussion.

Remark 5.7. If G is simply connected, and M a compact, connected q-Hamiltonian G-space with a minimally degenerate 2-form ω, then the fibers of the moment map Φ are connected [1]. In this case, the formula directly gives intersection pairings on $M/\!\!/G$ in terms of fixed point contributions.

6. DH-distributions for higher q-Hamiltonian forms

6.1. **Currents associated to higher q-Hamiltonian forms.** Suppose (M,ω,Φ) is an oriented q-Hamiltonian G-space, with proper moment map $\Phi : M \to V \subset G$, and let $\beta \in \Omega_G(M)$ be an equivariant cocycle. Given an invariant polynomial $p \in \mathrm{Pol}(\mathfrak{g})^G$, suppose $\omega^p \in \Omega_G(M)$ is a higher q-Hamiltonian form, i.e. satisfying $\mathrm{d}_G\omega^p = \Phi^*\eta^p_G$. Then

$$\phi = \Phi_*(\beta e^{2\pi i\omega^p}) \tag{29}$$

is closed with respect to the differential $\mathrm{d}_G - 2\pi i\eta^p_G$. Note however that $\beta(\xi)e^{2\pi i\omega^p(\xi)}$ does not lie in $\Omega_G(M)$, in general, since the exponential does not depend polynomially on $\xi \in \mathfrak{g}$. On the other hand, we have to insist on polynomial dependence in order for formulas such as (25) to make sense.

To get around this difficulty, we take p to be an invariant polynomial of the form

$$p(\xi) = p_1(\xi) + \sum_{j=2}^{l}\delta_j p_j(\xi) \tag{30}$$

where $p_1(\xi) = \frac{1}{2}||\xi||^2$, the p_j for $j \ge 2$ are invariant polynomials, and $\delta = (\delta_2,\ldots,\delta_l)$ are formal parameters. That is, instead of a general invariant polynomial p we consider only deformations of the quadratic polynomial. It will be convenient to introduce a notation for the perturbation term,

$$q(\xi) = \sum_{j=2}^{l}\delta_j p_j(\xi).$$

We also assume that the leading term of ω^p is the given 2-form ω, so that

$$\omega^p = \omega + \omega^q = \omega + \sum_{j=2}^{l}\delta_j\omega^{p_j} \tag{31}$$

where $\omega^{p_j} \in \Omega_G(M)$ are higher q-Hamiltonian forms corresponding to p_j. Then $\beta e^{2\pi i \omega^p}$ is defined as an element of $\Omega_G(M)[[\delta]]$ where $\mathbb{C}[[\delta]] = \mathbb{C}[[\delta_2, \ldots, \delta_l]]$ denotes the ring of formal power series. Similarly, Equation (29) defines an element of $\mathcal{C}_G(V)[[\delta]]$, closed under the differential $\mathrm{d}_G - 2\pi i \eta^p_G$.

6.2. **Witten's change of variables.** Suppose $\phi \in \mathcal{C}_G(V)[[\delta]]$ is closed under the differential $\mathrm{d}_G - 2\pi i \eta^p_G$. We would like to associate to ϕ an invariant distribution $\mathfrak{n} \in \mathcal{D}'(V)^G[[\delta]]$.

As a first step, we define a current on $V \cap T$, by a formula similar to (24)

$$\psi = (2\pi i)^{n_+} \pi_* (\tau e^{-2\pi i \gamma^p} \kappa_T(\phi)) \in \mathcal{C}_T(V \cap T)[[\delta]], \tag{32}$$

where γ^p is given by (12). Note that ψ is well-defined since $e^{-2\pi i \gamma^p} = e^{-2\pi i \gamma} e^{-2\pi i \gamma^q}$ lies in $\Omega_T(U)[[\delta]]$, i.e. the coefficients depend *polynomially* on the Lie algebra variables. Furthermore, ψ is closed for the differential $\mathrm{d} - 2\pi i \eta^p_T$.

The second step associates to any $\mathrm{d} - 2\pi i \eta^p_T$-cohomology class a distribution on $V \cap T$. For this, we cannot directly use the map (22), since this map does not vanish on $\mathrm{d} - 2\pi i \eta^p_T$-coboundaries unless $p = \frac{1}{2}||\xi||^2$. The underlying problem is that the current $\psi(\lambda)\overline{\epsilon_\lambda}$ (for weights $\lambda \in \Lambda^*$) is not d-closed, in general. Instead, using $(\mathrm{d} + 2\pi i \lambda \cdot \theta_T)\overline{\epsilon_\lambda} = 0$ and $(\mathrm{d} - 2\pi i p'(\xi) \cdot \theta_T)\psi(\xi) = 0$, we have:

Lemma 6.1. *For $\xi \in \mathfrak{t}$ and $\lambda \in \Lambda^*$, the current $\psi(\xi)\overline{\epsilon_\lambda} \in \mathcal{C}(V \cap T)$ is* d*-closed if and only if $p'(\xi) = \lambda$.*

This observation motivates a 'change of variables' $\xi = (p')^{-1}(\lambda)$, as in Witten's paper [41, Equation (5.9)]. Let $\mathrm{Pol}(\mathfrak{g}, \mathfrak{g}) \subset C^\infty(\mathfrak{g}, \mathfrak{g})$ be the algebra of polynomial maps from $\mathfrak{g}$ to itself. The transformation $\xi \mapsto p'(\xi)$ is a well-defined invertible element of the algebra $\mathrm{Pol}(\mathfrak{g}, \mathfrak{g})[[\delta]]$, with leading term the identity map. (See Appendix A for more on the change of variables). We will need the following fact regarding the Jacobian p'' of the change of variables. View p'' as an element of $\mathrm{Pol}(\mathfrak{g}, \mathrm{End}(\mathfrak{g}))[[\delta]]$, given in terms of an orthonormal basis e_a of $\mathfrak{g}$ and the associated coordinates ξ^a by the matrix $\frac{\partial^2 p}{\partial \xi^a \partial \xi^b}$.

Lemma 6.2. *For any $\xi \in \mathfrak{t}$, the linear map $p''(\xi) \in \mathrm{End}(\mathfrak{g})[[\delta]]$ preserves the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Furthermore, if ξ is a regular element of $\mathfrak{t}$ the determinant of $p''(\xi)|_{\mathfrak{p}}$ is given by*

$$(\det p''(\xi)|_{\mathfrak{p}})^{1/2} = \prod_{\alpha \in \mathfrak{R}_+} \frac{\alpha \cdot p'(\xi)}{\alpha \cdot \xi}.$$

Proof. Let us treat the δ_i as real variables rather than as formal parameters. For sufficiently small δ_i, the derivative $p' : \mathfrak{g} \to \mathfrak{g}$ is a well-defined diffeomorphism of neighborhoods of 0 containing ξ resp. $p'(\xi)$. By invariance, the derivatives of p at ξ vanish in $\mathfrak{p}$-directions, which implies that $p''(\xi)$ has block form

$$p''(\xi) = \begin{pmatrix} p''(\xi)|_{\mathfrak{t}} & 0 \\ 0 & p''(\xi)|_{\mathfrak{p}} \end{pmatrix}.$$

Suppose now that ξ is a regular element. The matrix $p''(\xi)|_{\mathfrak{p}}$ is the Jacobian for the transformation

$$T_\xi(G\cdot\xi)\cong\mathfrak{p}\to T_{p'(\xi)}(G\cdot p'(\xi))\cong\mathfrak{p}$$

induced by p'. Hence its determinant is the ratio of the Riemannian volumes of the orbits through ξ resp. $p'(\xi)$, with the Riemannian metric induced from $\mathfrak{g}$. The Lemma follows, since the Riemannian volume of an adjoint orbit $G\cdot\zeta\subset\mathfrak{g}$ through a regular element $\zeta\in\mathfrak{t}$ is equal to $|\det_{\mathfrak{p}}(\mathrm{ad}_\zeta)|\ \mathrm{Vol}_{G/T}$ where

$$|\det_{\mathfrak{p}}(\mathrm{ad}_\zeta)|=(\prod_{\alpha\in\mathfrak{R}_+}2\pi\alpha\cdot\zeta)^2.$$

□

Let $\mathfrak{A}_{\mathfrak{g}}$ denote the algebra automorphism of $\mathrm{Pol}(\mathfrak{g})[[\delta]]$ given by

$$(\mathfrak{A}_{\mathfrak{g}}F)(p'(\xi))=F(\xi).$$

Since the change of variables operator $\mathfrak{A}_{\mathfrak{g}}$ commutes with the adjoint action, it induces algebra automorphisms of the Cartan complexes $\Omega_G(M)[[\delta]]$ and $\mathcal{C}_G(M)[[\delta]]$, for any G-manifold M. Let $\mathfrak{A}_{\mathfrak{t}}$ be defined similarly, using the restriction $p|_{\mathfrak{t}}$.

Lemma 6.3. *The automorphism $\mathfrak{A}_{\mathfrak{t}}$ of $\mathcal{C}_T(V\cap T)[[\delta]]$ intertwines the differential $\mathrm{d}-2\pi i\eta_T^p$ with $\mathrm{d}-2\pi i\eta_T$. In particular, if $\psi\in\mathcal{C}_T(V\cap T)[[\delta]]$ is closed under $\mathrm{d}-2\pi i\eta_T^p$, then $\mathfrak{A}_{\mathfrak{t}}\psi$ is closed under $\mathrm{d}-2\pi i\eta_T$.*

Proof. It is obvious that $\mathfrak{A}_{\mathfrak{t}}$ commutes with d and intertwines $\eta_T^p(\xi)=p'(\xi)\cdot\theta_T$ with $\eta_T(\xi)=\xi\cdot\theta_T$. □

Hence, by composing $\mathfrak{A}_{\mathfrak{t}}$ with the map (22), we obtain a map $\mathcal{C}_T(V\cap T)[[\delta]]\to\mathcal{D}'(V\cap T)[[\delta]]$, vanishing on $\mathrm{d}-2\pi i\eta_T^p$-coboundaries. As it turns out, it will be convenient to modify this map by the factor $S=(\det p''|_{\mathfrak{p}})^{1/2}$, and to define a W-equivariant map $\mathcal{C}_T(V\cap T)[[\delta]]\to\mathcal{D}'(V\cap T)[[\delta]]$ by

$$\psi\mapsto\mathfrak{m}=\sum_I(\frac{1}{2\pi i}\frac{\partial}{\partial\mu})^I(\mathfrak{A}_{\mathfrak{t}}(S\psi))_I^{[\dim\mathfrak{t}]}. \tag{33}$$

Hence there is a unique map $\mathcal{C}_G(V)[[\delta]]\to\mathcal{D}'(V)^G[[\delta]]$ vanishing on $\mathrm{d}_G-2\pi i\eta_G^p$-coboundaries, for which the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}_G(V)[[\delta]] \arrow[r] \arrow[d] & \mathcal{D}'(V)^G[[\delta]] \arrow[d, "R_G"', "\cong"] \\ \mathcal{C}_T(V\cap T)^{W-\mathrm{alt}}[[\delta]] \arrow[r] & \mathcal{D}'(V\cap T)^{W-\mathrm{alt}}[[\delta]] \end{tikzcd} \tag{34}$$

Here the lower horizontal map is (33) and the left vertical map is (32).

Lemma 6.4. *Suppose $\phi\in\mathcal{C}_G(V)[[\delta]]$ is closed under $\mathrm{d}_G-2\pi i\eta_G^p$, and let $\mathfrak{n}\in\mathcal{D}'(V)^G[[\delta]]$ be its image under the above map. If ϕ is compactly supported, the*

Fourier coefficients of $\mathfrak{n}$ *are given by the formula,*

$$\langle \mathfrak{n}, \overline{\chi_\lambda} \rangle = \frac{\dim V_\lambda}{(-1)^{n_+} \prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot\xi} \int_U \tau(\xi) e^{-2\pi i \gamma^p(\xi)}\, \phi(\xi) \pi^* \overline{\epsilon_{\lambda+\rho}}$$

where ξ *is the solution of* $p'(\xi) = \lambda + \rho$.

Proof. The Lemma follows from $\langle \mathfrak{n}, \overline{\chi_\lambda}\rangle = i^{n_+} \operatorname{vol}_{G/T} \langle \mathfrak{m}, \overline{\epsilon_{\lambda+\rho}}\rangle$ and the calculation,

$$\begin{aligned} \langle \mathfrak{m}, \overline{\epsilon_{\lambda+\rho}}\rangle &= S(\xi)\, \langle \psi(\xi), \overline{\epsilon_{\lambda+\rho}}\rangle \\ &= \prod_{\alpha\in\mathfrak{R}_+} \frac{\alpha\cdot(\lambda+\rho)}{\alpha\cdot\xi} (2\pi i)^{n_+} \langle \tau(\xi)\, e^{-2\pi i\gamma^p(\xi)}\, \phi(\xi), \pi^*\overline{\epsilon_{\lambda+\rho}}\rangle. \end{aligned}$$

using $\prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot(\lambda+\rho) = (2\pi)^{-n_+} \frac{\dim V_\lambda}{\operatorname{vol}_{G/T}}$. □

Returning to the case of a q-Hamiltonian G-space, consider ω^p of the form (31), and an equivariant cocycle $\beta \in \Omega_G(M)$. We define a generalized DH-distribution $\mathfrak{n}^\beta \in \mathcal{D}'(V)^G[[\delta]]$ as the image of the current $\phi = \Phi_*(\beta e^{2\pi i \omega^p}) \in \mathcal{C}_G(V)[[\delta]]$ under the map (34).

If β is compactly supported (in particular if M is compact), the Fourier coefficients of $\mathfrak{n}^\beta$ can be expressed in terms of the Abelianization (N, ω_N, Φ_N). Recall that by Proposition 3.4, the form $\omega_N^p = \omega^p - \Phi^*\gamma^p$ satisfies $\mathrm{d}_T \omega_N^p = \Phi_N^* \eta_T^p$. We obtain:

$$\langle \mathfrak{n}^\beta, \overline{\chi_\lambda}\rangle = \frac{\dim V_\lambda}{(-1)^{n_+} \prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot\xi} \int_U \tau(\xi)\beta(\xi) e^{2\pi i \omega_N^p(\xi)}\, \pi^*\overline{\epsilon_{\lambda+\rho}}. \tag{35}$$

Using localization, we find:

Proposition 6.5. *Suppose* $\beta \in \Omega_G(M)$ *is a compactly supported cocycle. Then*

$$\langle \mathfrak{n}^\beta, \overline{\chi_\lambda}\rangle = \dim V_\lambda \sum_{F\in\mathcal{F}(\lambda+\rho)} \int_F \frac{\beta(\xi) e^{2\pi i\omega^p(\xi)} \Phi_F^* \overline{\epsilon_{\lambda+\rho}}}{\operatorname{Eul}(\nu_F, \xi)}, \quad \lambda \in \Lambda_+^*$$

where ξ *is the solution of* $p'(\xi) = \lambda+\rho$.

Proof. We view the δ_j as real variables rather than just formal parameters. For δ_j sufficiently small, the inverse $\xi = (p')^{-1}(\lambda+\rho)$ is a well-defined smooth function of δ_j. Moreover, $\xi \to \lambda+\rho$ as $\delta_j \to 0$. This implies that $\xi_M^{-1}(0) \subset (\lambda+\rho)_M^{-1}(0)$, for δ_j sufficiently small. Since the integrand in the definition of $\mathfrak{m}^\beta_{\lambda+\rho}$ is d_ξ-closed, Proposition 6.5 follows from the localization formula (Theorem B.2), using that the pull-back of $\tau_M(\xi)$ to any F is T-equivariantly cohomologous to $(-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha,\xi\rangle$. □

6.3. **Interpretation.** By construction, the distribution $\mathfrak{n}^\beta \in \mathcal{D}'(V)[[\delta]]$ is smooth at β-regular values of Φ. Its restriction to the set of β-regular values $g \in V$ encodes intersection pairings on symplectic quotients $\Phi^{-1}(g)/G_g$. We will need the precise relationship only for $g = e$. Generalizing Theorem 5.4 we will prove:

Theorem 6.6. *Let (M, ω, Φ) be a q-Hamiltonian G-space, with proper moment map $\Phi : M \to V \subset G$, and $\beta \in \Omega_G(M)$ is an equivariant cocycle. Let ω^p be a higher q-Hamiltonian form as in (31), and $\mathfrak{n}^\beta \in \mathcal{D}'(V)[[\delta]]$ the DH-distribution defined by these data. Suppose V contains the group unit e, and is a β-regular value of Φ. Then $\mathfrak{n}^\beta$ is smooth near e, and using $\mathrm{d}\,\mathrm{vol}_G$ to identify measures and functions,*

$$\mathfrak{n}^\beta(e) = (2\pi i)^{\dim G} \operatorname{vol}_G \sum_j \frac{1}{k_j} \int_{(M/\!\!/G)_j} (\det(p'')\,\beta)_{\mathrm{red}} e^{2\pi i \omega^p_{\mathrm{red}}}.$$

A combination of Theorem 6.6 and Proposition 6.5 (applied to $\mathfrak{n}^{\beta_1}$ where $\beta_1 = \frac{\beta}{\det(p'')}$) gives a localization formula for intersection pairings:

Theorem 6.7 (Localization formula for higher q-Hamiltonian forms)**.** *Let (M, ω, Φ) be a compact, connected q-Hamiltonian G-space, with group unit e a regular value of the moment map. and $\beta \in \Omega_G(M)$ an equivariant cocycle. Let $p(\xi) = \frac{1}{2}||\xi||^2 + \sum_{j=2}^{l} \delta_j p_j(\xi)$ be an invariant polynomial and $\omega^p = \omega + \sum \delta_j \omega^{p_j} \in \Omega_G(M)$ a corresponding higher q-Hamiltonian form. Then*

$$\begin{aligned} &\sum_j \frac{1}{k_j} \int_{(M/\!\!/G)_j} \beta_{\mathrm{red}} e^{2\pi i \omega^p_{\mathrm{red}}} \\ &= \frac{(2\pi i)^{-\dim G}}{\operatorname{vol}_G^2} \sum_{\lambda \in \Lambda^*_+} \frac{(\dim V_\lambda)^2}{\det p''(\xi)} \sum_{F \in \mathcal{F}(\lambda+\rho)} \int_F \frac{\iota_F^* \beta(\xi)\, e^{2\pi i \omega^p_F(\xi)}}{\mathrm{Eul}(\nu_F, \xi)} \Phi_F^* \overline{\epsilon_{\lambda+\rho}} \end{aligned}$$

where in each summand, $\xi \in \mathfrak{g}[[\delta]]$ is given as the solution of $p'(\xi) = \lambda + \rho$.

6.4. **Proof of Theorem 6.6.** The idea of proof is to relate the distribution $\mathfrak{n}^\beta$ to a similar distribution for ordinary Hamiltonian spaces. Since we are only considering a neighborhood of e, we may assume that V is the diffeomorphic image of an invariant neighborhood $V_0 \subset \mathfrak{g}$ of 0 under the exponential map. Furthermore, we may assume that V consists of β-regular values of Φ so that the current ϕ is in fact smooth.

We begin by considering arbitrary $\mathrm{d}_G - 2\pi i \eta^p_G$-closed currents $\phi \in \mathcal{C}_G(V)[[\delta]]$, and the corresponding distributions $\mathfrak{n} \in \mathcal{D}'(V)[[\delta]]$ and $\mathfrak{m} = R_G(\mathfrak{n})$.

Let $\varpi^p \in \Omega_G(\mathfrak{g})[[\delta]]$ be defined as in (10). Conjugation by $e^{2\pi i \varpi^p}$ is an automorphism of $\mathcal{C}_G(\mathfrak{g})[[\delta]]$, intertwining $\mathrm{d}_G - 2\pi i \exp^* \eta^p_G$ with $\mathrm{d}_G - 2\pi i \eta^p_{\mathfrak{g}}$. Hence the form

$$\phi_0 = e^{-2\pi i \varpi^p} \exp^* \phi \in \mathcal{C}_G(V_0)[[\delta]] \tag{36}$$

is closed for the differential $\mathrm{d}_G - 2\pi i \eta^p_{\mathfrak{g}}$. The change of variables operator $\mathfrak{A}_{\mathfrak{g}}$ intertwines $\eta^p_{\mathfrak{g}}(\xi) = -\mathrm{d}\langle \cdot, p'(\xi)\rangle$ with $\eta_{\mathfrak{g}}(\xi) = -\mathrm{d}\langle \cdot, \xi\rangle$. Therefore $\mathfrak{A}_{\mathfrak{g}}\phi_0$ is closed for $\mathrm{d}_G - 2\pi i \eta_{\mathfrak{g}}$, and we have the associated distribution,

$$\mathfrak{n}_0 = \sum_I \Big(\frac{1}{2\pi i}\frac{\partial}{\partial \mu}\Big)^I (\mathfrak{A}_{\mathfrak{g}}\phi_0)_I^{[\dim \mathfrak{g}]}.$$

Similar to ψ, define a $\mathrm{d}_T - 2\pi i \eta^p_{\mathfrak{t}}$-cocycle

$$\psi_0 = (2\pi i)^{n_+} (\pi_0)_* (\tau_0\, \kappa_T(\phi_0)) \in \mathcal{C}_T(V_0 \cap \mathfrak{t})[[\delta]], \tag{37}$$

and the corresponding distribution,

$$\mathfrak{m}_0 = \sum_I \big(\frac{1}{2\pi i}\frac{\partial}{\partial\mu}\big)^I (\mathfrak{A}_{\mathfrak{t}} S\, \psi_0)_I^{[\dim \mathfrak{t}]}.$$

Lemma 6.8. *The distributions $\mathfrak{n}_0$ and $\mathfrak{m}_0$ are related by $\mathfrak{m}_0 = R_{\mathfrak{g}}(\mathfrak{n}_0)$.*

Proof. It suffices to prove this for compactly supported ϕ_0. In this case, the Fourier transforms of $\mathfrak{n}_0$ is $\langle \mathfrak{n}_0, e^{-2\pi i\langle\cdot, p'(\xi)\rangle}\rangle = \langle \phi_0(\xi), e^{-2\pi i \langle \cdot, p'(\xi)\rangle}\rangle$. For the Fourier transform of $\mathfrak{m}_0$ we compute,

$$\begin{aligned} \langle \mathfrak{m}_0, e^{-2\pi i\langle\cdot, p'(\xi)\rangle}\rangle &= S(\xi)\langle \psi_0(\xi), e^{-2\pi i\langle\cdot,\xi\rangle}\rangle \\ &= (2\pi i)^{n_+}\, S(\xi)\langle \tau_0(\xi)\phi_0(\xi), e^{-2\pi i\langle\cdot,\xi\rangle}\rangle. \end{aligned}$$

Since ϕ_0 is compactly supported, $\tau_0(\xi)$ may be replaced by the cohomologous form of non-compact support $(-1)^{n_+}\prod_{\alpha\in\mathfrak{R}_+}\langle\alpha,\xi\rangle$. This factor combines with the denominator of S, and we obtain

$$\langle \mathfrak{m}_0, e^{-2\pi i\langle\cdot, p'(\xi)\rangle}\rangle = (-1)^{n_+} \prod_{\alpha\in\mathfrak{R}_+} 2\pi i\langle \alpha, p'(\xi)\rangle\ \langle \mathfrak{n}_0, e^{-2\pi i\langle\cdot, p'(\xi)\rangle}\rangle.$$

Now use the characterization of $R_{\mathfrak{g}}$ by Fourier transforms, Lemma 4.6(b). ☐

Lemma 6.9. *Suppose $\phi \in \Omega_G(V)[[\delta]]$ is a smooth equivariant current, closed under $\mathrm{d}_G - 2\pi i\eta^p_G$. Let $\mathfrak{n}$ and $\phi_0, \mathfrak{n}_0$ be as above. Then $\mathfrak{n}, \mathfrak{n}_0$ are related by the Duflo map:*

$$\mathfrak{n} = \exp_*(J^{1/2}\mathfrak{n}_0).$$

Proof. Again, it is enough to prove this for compactly supported ϕ. We have to show that $\mathfrak{m} = R_G(\mathfrak{n})$ and $\mathfrak{m}_0 = R_{\mathfrak{g}}(\mathfrak{n}_0)$ are related by $\mathfrak{m} = (\exp_T)_*\mathfrak{m}_0$. As in the proof of Lemma 5.5, this is verified by comparing the Fourier coefficients of the two measures: First, defining ξ by $p'(\xi) = \lambda + \rho$, the localization formula (or integration over the fiber, using the properties of the Thom form) gives

$$\langle \mathfrak{m}, \overline{\epsilon_{\lambda+\rho}}\rangle = (2\pi i)^{n_+} S(\xi) \int_U \tau(\xi)\phi(\xi) e^{-2\pi i\gamma^p(\xi)}\pi^*\overline{\epsilon_{\lambda+\rho}} = S(\xi)\int_T \iota_T^*\phi(\xi)\overline{\epsilon_{\lambda+\rho}}$$

where we have used that the integrand is d_ξ-closed. The Fourier coefficient $\langle \mathfrak{m}_0', e^{-2\pi i\langle\cdot,\lambda+\rho\rangle}\rangle$ is computed similarly, and agrees with $\langle \mathfrak{m}', \overline{\epsilon_{\lambda+\rho}}\rangle$. ☐

Above, we have seen that $\mathfrak{A}\phi_0$ is a $\mathrm{d}_G - 2\pi i\eta_{\mathfrak{g}}$-closed current. Another way of making ϕ_0 closed under $\mathrm{d}_G - 2\pi i \eta_{\mathfrak{g}}$ is to define

$$\tilde\phi_0(\xi) = e^{2\pi i\langle\cdot, q'(\xi)\rangle}\phi_0(\xi) \in \mathcal{C}_G(V_0)[[\delta]].$$

Let $\tilde{\mathfrak{n}}_0 \in \mathcal{D}'(V_0)[[\delta]]$ be the corresponding distribution. As it turns out, we will find it much easier to interpret $\tilde{\mathfrak{n}}_0$ rather than $\mathfrak{n}_0$, in the q-Hamiltonian setting.

Lemma 6.10. *The distributions* $\mathfrak{n}_0$ *and* $\tilde{\mathfrak{n}}_0$ *are related as follows,*

$$\mathfrak{n}_0(\mu) = e^{2\pi i \langle \mu, q'(\frac{1}{2\pi i}\frac{\partial}{\partial\nu})\rangle} \det(p''(\frac{1}{2\pi i}\frac{\partial}{\partial\nu}))\, \tilde{\mathfrak{n}}_0(\nu)\Big|_{\nu=\mu}.$$

In particular,

$$\mathfrak{n}_0(0) = \det(p'')(\frac{1}{2\pi i}\frac{\partial}{\partial\nu})\, \tilde{\mathfrak{n}}_0(\nu)\Big|_{\nu=0}$$

Proof. It suffices to prove this for the case that ϕ_0 is compactly supported. The calculation

$$\langle \mathfrak{n}_0, e^{-2\pi i\langle\cdot,p'(\xi)\rangle}\rangle = \langle \phi_0(\xi), e^{-2\pi i\langle\cdot,p'(\xi)\rangle}\rangle = \langle \tilde{\phi}_0(\xi), e^{-2\pi i\langle\cdot,\xi\rangle}\rangle = \langle \tilde{\mathfrak{n}}_0, e^{-2\pi i\langle\cdot,\xi\rangle}\rangle.$$

shows that the Fourier coefficients of $\mathfrak{n}_0$ and $\tilde{\mathfrak{n}}_0$ are related by a change of variables. Hence, the Lemma boils down to a description of the Fourier transform of the change-of-variables operator $\mathfrak{A}_\mathfrak{g}$, which is accomplished in Appendix A. □

Proof of Theorem 6.6. With our preparations, the proof has now become a fairly straightforward extension of the proof of Theorem 5.4. We may assume that V consists of β-regular values of Φ and is the diffeomorphic image under exp of an invariant open subset $V_0 \subset \mathfrak{g}$. Let (M, ω_0, Φ_0) be as in the proof of Theorem 5.4. We use the currents $\phi = \Phi_*(\beta e^{2\pi i\omega^p})$ and $\phi_0 = (\Phi_0)_*(\beta e^{2\pi i\omega_0^p})$ to define $\mathfrak{n}^\beta \in \mathcal{D}'(V)^G[[\delta]]$ and $\mathfrak{n}_0^\beta \in \mathcal{D}'(V_0)^G[[\delta]]$. ¿From Lemma 6.9, we see that $\mathfrak{n}^\beta(e) = \mathfrak{n}_0^\beta(0)$. To identify $\mathfrak{n}_0^\beta(0)$ it is better to work with $\tilde{\mathfrak{n}}_0^\beta$, defined by the $\mathrm{d}_G - 2\pi i\eta_\mathfrak{g}$-closed form

$$\tilde{\phi}_0(\xi) = (\Phi_0)_*(\beta e^{2\pi i(\omega_0^p - \langle\Phi_0, q'(\cdot)\rangle)}) = \Phi_0^*(\beta e^{2\pi i(\omega_0^q - \langle\Phi_0,q'(\cdot)\rangle)} e^{2\pi i\omega_0}) \in \mathcal{C}_G(V_0)[[\delta]].$$

Indeed, $\tilde{\mathfrak{n}}_0^\beta$ is just the DH-distribution for the d_G-closed equivariant form $\beta e^{2\pi i(\omega_0^q - \langle\Phi_0,q'(\cdot)\rangle)}$, hence its interpretation in terms of intersection pairings is given by Proposition 4.4(e). According to Lemma 6.10, the value of $\mathfrak{n}_0^\beta$ at 0 is equal to the value of $(\det p'')(\frac{1}{2\pi i}\frac{\partial}{\partial\mu})\ \tilde{\mathfrak{n}}_0^\beta$ at 0. By Proposition 4.4(c),

$$(\det p'')(\frac{1}{2\pi i}\frac{\partial}{\partial\mu})\ \tilde{\mathfrak{n}}_0^\beta = \tilde{\mathfrak{n}}_0^{\det(p'')\beta},$$

so we find that $\mathfrak{n}^\beta(e) = \mathfrak{n}_0^\beta(0) = \tilde{\mathfrak{n}}_0^{\det(p'')\beta}(0)$ is given by

$$(2\pi i)^{\dim G}\, \mathrm{vol}_G \sum_j \frac{1}{k_j}\int_{(M_0/\!\!/G)_j} \big(\det(p'')\,\beta e^{2\pi i(\omega_0^q - \langle\Phi_0,q'(\cdot)\rangle)}\big)_{\mathrm{red}} e^{2\pi i(\omega_0)_{\mathrm{red}}}.$$

Finally, observe that under the natural identification $M/\!\!/G = M_0/\!\!/G$,

$$(\omega_0^q - \langle\Phi_0, q'(\cdot)\rangle)_{\mathrm{red}} = \omega_{\mathrm{red}}^q,\ \ (\omega_0)_{\mathrm{red}} = \omega_{\mathrm{red}}.$$

□

7. Proof of the Witten formulas

In this Section we will apply Theorem 6.7 to our main examples. In each case, we first describe the Fourier coefficients of the distribution $\mathfrak{n}^\beta \in \mathcal{D}'(G)[[\delta]]$, by working out the fixed point contributions from Proposition 6.5. Throughout this Section we assume that G is simply connected and simple. Thus in particular, ρ is a weight. It is easy to generalize the discussion to general semi-simple compact groups, by passing to covers.

7.1. **Conjugacy classes.** Any conjugacy class $\mathcal{C} \subset G$ has a unique structure of a q-Hamiltonian G-space, in such a way that the moment map is the inclusion $\Phi : \mathcal{C} \hookrightarrow G$. The 2-form ω given by the formula [1]

$$\omega(\xi_\mathcal{C}, \xi'_\mathcal{C})|_g = \frac{1}{2}\xi \cdot (\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi'.$$

More generally, if $p \in \mathrm{Pol}(\mathfrak{g})^G$ is an invariant polynomial, the pull-back $\Phi^*\eta^p_G$ is exact since the equivariant cohomology of a homogeneous space G/K lives in even degree. In fact, evaluation at the base point $eK \in G/K$ is a homotopy equivalence,

$$\Omega^\bullet_G(\mathcal{C}) \to \Omega^\bullet_K(\mathrm{pt}) = \mathrm{Pol}^{2\bullet}(\mathfrak{k})^K, \tag{38}$$

with homotopy inverse $\Omega_K(\mathrm{pt}) \to \Omega_G(\mathcal{C})$ given by induction. Hence, up to equivariant coboundaries we can fix a primitive $\omega^p \in \Omega_G(\mathcal{C})$ of $\Phi^*\eta^p_G$ by requiring that it lies in the kernel of (38). Now let $p \in \mathrm{Pol}(\mathfrak{g})[[\delta]]$ have the form (30), and take $\omega^p \in \Omega_G(\mathcal{C})[[\delta]]$ as in (31), normalized to lie in the kernel of (38).

Theorem 7.1. *Let $\beta \in \Omega_G(\mathcal{C})$ be the equivariant cocycle defined by induction from an invariant polynomial $Q \in \mathrm{Pol}(\mathfrak{k})^K$. Then the Fourier coefficients of the corresponding distribution $\mathfrak{n}^\beta \in \mathcal{D}'(G)[[\delta]]$ are given by,*

$$\langle \mathfrak{n}^\beta, \overline{\chi_\lambda} \rangle = (2\pi i)^{\dim(\mathcal{C})/2} Q(\xi) \Big(\frac{\det p''(\xi)}{\det p''_\mathfrak{k}(\xi)} \Big)^{1/2} \overline{\chi_\lambda(\mathcal{C})}\ \mathrm{Vol}(\mathcal{C}).$$

with $p'(\xi) = \lambda + \rho$. Here $\chi_\lambda(\mathcal{C})$ denotes the value of the character χ_λ on the conjugacy class, and $\mathrm{Vol}(\mathcal{C})$ *is given by* (7).

Proof. Let $\mu \in \mathfrak{A}$ be the unique point in alcove with $\exp(\mu) \in \mathcal{C}$, as in Example 2.3. The centralizer $K = G_{\exp\mu}$ is a connected subgroup containing the maximal torus T. We denote by $W_K \subset W$ the Weyl group of K, by $\mathfrak{R}_{K,+} \subset \mathfrak{R}_+$ the set of positive roots, and by ρ_K its half-sum. ρ_K need not be a weight of K, in general. On the other hand, $2\mu \cdot \rho_K$ is an integer: This follows since $2\rho_K$ is in the weight lattice for the adjoint group $K/Z(K)$, while μ is in the integral lattice for this group (since $\exp_K(\mu) \in Z(K)$). The orientation on $\mathcal{C}$ differs from the orientation on the homogeneous space G/K by a sign, $(-1)^{2\mu\cdot\rho_K}$.

The set of fixed points for the vector field $(\lambda+\rho)_\mathcal{C}$ is the Weyl group orbit $W.\exp\mu = \mathcal{C} \cap T$. It is thus parametrized by W/W_K. Consider the fixed point $F = \{\exp\mu\}$. The space $\nu_F = T_{\exp\mu}\mathcal{C}$ is isomorphic to $\mathfrak{k}^\perp \subset \mathfrak{g}$ as a K-module;

hence the equivariant Euler form is

$$\mathrm{Eul}(\nu_F,\xi)=(-1)^{2\mu\cdot\rho_K}\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}(-\alpha)\cdot\xi,\quad \xi\in\mathfrak{t}$$

By equivariance, $\mathrm{Eul}(\nu_{w.F},\xi)=\mathrm{Eul}(\nu_F,w\xi)$. We therefore obtain,

$$\begin{aligned}\frac{\langle\mathfrak{n}^\beta,\overline{\chi_\lambda}\rangle}{\dim V_\lambda} &= (-1)^{2\mu\cdot\rho_K}Q(\xi)\sum_{w\in W/W_K}\frac{\overline{\epsilon_{w.(\lambda+\rho)}}(g)}{\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}(-\alpha)\cdot w\xi}\\ &= (-1)^{2\mu\cdot\rho_K}Q(\xi)\Big(\frac{\det p''(\xi)}{\det p''_{\mathfrak{k}}(\xi)}\Big)^{1/2}\sum_{w\in W/W_K}\frac{\overline{\epsilon_{w.(\lambda+\rho)}}(g)}{\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}(-\alpha)\cdot w(\lambda+\rho)}\end{aligned}$$

where we have used Lemma 6.2 to write

$$\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}\frac{\alpha\cdot p'(\xi)}{\alpha\cdot\xi}=\Big(\frac{\det p''(\xi)}{\det p''_{\mathfrak{k}}(\xi)}\Big)^{1/2}$$

The proof is complete by the following Lemma. □

Lemma 7.2. *Let K be the centralizer of $g=\exp\mu$, with $\mu\in\mathfrak{A}$. Let $\mathcal{C}=G.\exp\mu$ be the corresponding conjugacy class. The following formula holds for all $\lambda\in\Lambda^*_+$:*

$$\frac{\chi_\lambda(\mathcal{C})}{\dim V_\lambda}=\frac{(-1)^{2\mu\cdot\rho_K}}{\mathrm{Vol}(\mathcal{C})}(2\pi i)^{-\dim(\mathcal{C})/2}\sum_{w\in W/W_K}\frac{\epsilon_{w.(\lambda+\rho)}(g)}{\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}\alpha\cdot w(\lambda+\rho)},$$

with $\mathrm{Vol}(\mathcal{C})$ *as given in* (7).

Proof. Let $\tilde K$ denotes any connected finite cover of K for which ρ_K is in the weight lattice, and $\chi_\nu^{\tilde K}\in C^\infty(K)$ denotes the irreducible character corresponding to a weight $\nu\in\Lambda^*_{\tilde K,+}$. By two applications of the Weyl character formula,

$$\begin{aligned}\chi_\lambda(\mathcal{C}) &= \sum_{w\in W/W_K}(-1)^{l(w)}\frac{\chi^{\tilde K}_{w(\lambda+\rho)-\rho_K}(\exp_{\tilde K}\mu)}{\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}2i\sin(\pi\alpha\cdot\mu)}\\ &= \frac{\mathrm{vol}_{G/K}}{\mathrm{Vol}(\mathcal{C})}i^{-\dim(\mathcal{C})/2}\sum_{w\in W/W_K}(-1)^{l(w)}\chi^{\tilde K}_{w(\lambda+\rho)-\rho_K}(\exp_{\tilde K}\mu).\end{aligned}$$

Since $\exp_{\tilde K}\mu$ is in the center of $\tilde K$,

$$\chi_\nu^{\tilde K}(\exp_{\tilde K}\mu)=\dim V_\nu^{\tilde K}\,e^{2\pi i\nu\cdot\mu}$$

for all $\nu\in\Lambda^*_{\tilde K,+}$. In our case $\nu=w(\lambda+\rho)-\rho_K$, and therefore $e^{2\pi i\nu\cdot\mu}=(-1)^{2\rho_K\cdot\mu}e^{2\pi i w(\lambda+\rho)\cdot\mu}$, while the dimension of $V_\nu^{\tilde K}$ can be re-written by means of the Weyl dimension formula:

$$\begin{aligned}\dim V^{\tilde K}_{w(\lambda+\rho)-\rho_K} &= \mathrm{vol}_{K/T}\prod_{\alpha\in\mathfrak{R}_{K,+}}2\pi\alpha\cdot w(\lambda+\rho)\\ &= \frac{(-1)^{l(w)}\dim V_\lambda}{\mathrm{vol}_{G/K}\prod_{\alpha\in\mathfrak{R}_+\backslash\mathfrak{R}_{K,+}}2\pi\alpha\cdot w(\lambda+\rho)}.\end{aligned}$$

Putting all this together, we obtain the expression for $\chi_\lambda(\mathcal{C})$ as given in the Lemma. $\square$

7.2. **Fusion.** Suppose (M, ω, Φ) is a q-Hamiltonian G^2-space. That is, $\Phi = (\Phi_0, \Phi_1) : M \to G^2$ is a G^2-equivariant map, and the 2-form ω satisfies the condition $\mathrm{d}_{G^2}\omega = \Phi_0^*\eta_G + \Phi_1^*\eta_G$. Then, as explained in [1] the space M with diagonal G-action becomes a q-Hamiltonian G-space $(M, \omega_{\mathrm{fus}}, \Phi_{\mathrm{fus}})$ with moment map $\Phi_{\mathrm{fus}} = \Phi_0\Phi_1$ (pointwise product) and 2-form

$$\omega_{\mathrm{fus}} = \omega + \Phi^*\varphi,$$

where $\varphi = \frac{1}{2}\operatorname{pr}_1^*\theta^L \cdot \operatorname{pr}_2^*\theta^R \in \Omega^2(G \times G)$. In the appendix, we explain how the Bott-Shulman machinery defines higher analogues $\varphi^p \in \Omega_G^{2\bullet-2}(G \times G)$ of this form, for any $p \in \mathrm{Pol}^\bullet(\mathfrak{g})^G$, with $\varphi^p = \varphi$ for $p(\xi) = \frac{1}{2}||\xi||^2$. Some basic properties of these forms are:

(a) The form degree 0 part $(\varphi^p)^{[0]}$ vanishes.
(b) Letting $\mathrm{Mult} : G \times G \to G$ denote the group multiplication,

$$\mathrm{Mult}^*\,\eta_G^p = \operatorname{pr}_1^*\,\eta_G^p + \operatorname{pr}_2^*\,\eta_G^p + \mathrm{d}_G\varphi^p.$$

(c) The equivariant forms on $G \times G \times G$,

$$(\operatorname{pr}_1 \times \operatorname{pr}_2)^*\varphi^p + (\mathrm{Mult} \circ (\operatorname{pr}_1 \times \operatorname{pr}_2) \times \operatorname{pr}_3)^*\varphi^p$$

and

$$(\operatorname{pr}_2 \times \operatorname{pr}_3)^*\varphi^p + (\operatorname{pr}_1 \times (\mathrm{Mult} \circ (\operatorname{pr}_2 \times \operatorname{pr}_3))^*\varphi^p$$

differ by a d_G-coboundary. For $p(\xi) = \frac{1}{2}||\xi||^2$ they are in fact equal.

Hence, given an equivariant form $\omega^p \in \Omega_{G^2}(M)$ with $\mathrm{d}_{G^2}\omega^p = \Phi_0^*\eta_G^p + \Phi_1^*\eta_G^p$, the form

$$\omega_{\mathrm{fus}}^p = \omega^p + \Phi^*\varphi^p$$

has the property $\mathrm{d}_G\omega_{\mathrm{fus}}^p = \Phi_{\mathrm{fus}}^*\eta_G^p$.

As an application, suppose $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are conjugacy classes, with moment maps Φ_j the inclusion and with given higher q-Hamiltonian forms ω_j^p, normalized as in 7.1. Then the product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ becomes a q-Hamiltonian G-space, with moment map the product $\Phi_1 \cdots \Phi_r$, and with a higher q-Hamiltonian form ω^p obtained by iterated fusion. Note that ω^p is not canonically defined since the fusion procedure is not associative (except for $p(\xi) = \frac{1}{2}||\xi||^2$). However, (c) above shows that any two forms obtained in this way differ by a d_G-coboundary.

The fixed point set for the action of $(\lambda + \rho)_M$ on $M = \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ are simply products of the fixed point sets for each $\mathcal{C}_j$. Since they are 0-dimensional, the terms $\Phi^*\varphi^p$ vanish if pulled back to the fixed point sets, using (a). Hence, the fixed point contribution for the product is just the product of fixed point contributions from each factor.

7.3. **The double.** The *double* is the q-Hamiltonian G^2-space $(D(G), \omega, (\Phi_0, \Phi_1))$ where $D(G) = G \times G$, with group action

$$(g_0, g_1).(h, k) = (g_0 h g_1^{-1},\ g_1 k g_1^{-1})$$

and moment map components $\Phi_0(h, k) = k$, $\Phi_1(h, k) = \mathrm{Ad}_h(k^{-1})$. The 2-form ω satisfying $\mathrm{d}_{G^2}\omega = \Phi_0^*\eta_G + \Phi_1^*\eta_G$ arises naturally in the Bott-Shulman construction (cf. Appendix B.9), and equals the form denote λ in (46). The explicit form of ω is not important for what follows, except for the fact that its pull-back to $D(T) = T \times T \subset G \times G$ is the standard symplectic form on $D(T) = T \times T$:

$$\iota_{D(T)}^*\omega = h^*\theta_T \cdot k^*\theta_T.$$

More generally, for any invariant polynomial $p \in \mathrm{Pol}(\mathfrak{g})^G$, the Bott-Shulman construction defines forms $\omega^p \in \Omega_{G^2}(D(G))$ (see Appendix B.9, where these forms are denoted λ^p), with the property $\mathrm{d}_{G^2}\omega^p = \Phi_0^*\eta_G^p + \Phi_1^*\eta_G^p$. The pull-back of these forms to $D(T)$ is given by

$$\iota_{T\times T}^*\omega^p(\xi_0, \xi_1) = \sum_{jk} A_{rs}(\xi_0, \xi_1) h^*\theta_T^r k^*\theta_T^s$$

with $A_{rs}(\xi_0, \xi_1) = \int_0^1 \mathrm{d}t\, p_\mathfrak{t}''(\xi)_{rs}((1-t)\xi_0 + t\xi_1)$. Here $p_\mathfrak{t}$ is the restriction of p to $\mathfrak{t}$, $p_\mathfrak{t}''(\xi)_{jk}$ is the matrix of second derivatives in a given orthonormal basis of $\mathfrak{t}$, and θ_T^j are the corresponding components of the Maurer-Cartan form.

The fused double $\tilde{D}(G) = D(G)_{\mathrm{fus}}$ is a q-Hamiltonian G-space with moment map $\Phi_{\mathrm{fus}} = \Phi_0\Phi_1$ given by the group commutator $(h, k) \mapsto [k, h] = khk^{-1}h^{-1}$. As explained above, the forms ω^p gives rise to higher q-Hamiltonian forms ω_{fus}^p on $\tilde{D}(G)$. On $\tilde{D}(T) \subset \tilde{D}(G)$, the 'fusion term' $\Phi^*\varphi^p$ vanishes, as one can see from Lemma B.4. Hence, the pull-back of ω_{fus}^p to $\tilde{D}(T)$ is simply

$$\iota_{T\times T}^*\omega_{\mathrm{fus}}^p(\xi) = \sum_{jk} p_\mathfrak{t}''(\xi)_{rs} h^*\theta_T^r k^*\theta_T^s$$

Dropping the subscript 'fus', consider now the fixed point contributions for $(\tilde{D}(G), \Phi, \omega)$, for p be as in (30), and ω^p obtained by fusion as explained above. Since G acts by conjugation on each factor in $\tilde{D}(G) = G \times G$, the fixed point set for $(\lambda + \rho)_M$ is just $F = \tilde{D}(T)$. On F the moment map becomes trivial, and the normal bundle is T-equivariantly isomorphic to $\mathfrak{p} \oplus \mathfrak{p}$. The equivariant Euler form is (cohomologous to)

$$\mathrm{Eul}(\nu_F, \xi) = (-1)^{n_+} \Big(\prod_{\alpha \in \mathfrak{R}_+} \alpha \cdot \xi \Big)^2$$

In the simplest case $\beta = 1$, the fixed point contributions read,

$$\begin{aligned}
\frac{\langle \mathfrak{n}, \overline{\chi_\lambda}\rangle}{\dim V_\lambda} &= (-1)^{n_+} \frac{\int_F e^{2\pi i \omega^p(\xi)}}{(\prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot\xi)^2}\\
&= (-1)^{n_+}(2\pi i)^{\dim T}\,(\mathrm{vol}_T)^2\,\frac{\det p_{\mathfrak{t}}''(\xi)}{(\prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot\xi)^2}\\
&= (-1)^{n_+}(2\pi i)^{\dim T}\,(\mathrm{vol}_T)^2\,\frac{\det p''(\xi)}{(\prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot(\lambda+\rho))^2}\\
&= (-1)^{n_+}(2\pi i)^{\dim T}\,\frac{(\mathrm{vol}_T)^2}{(\dim V_\lambda)^2}\,\frac{\det p''(\xi)}{(\prod_{\alpha\in\mathfrak{R}_+} \alpha\cdot\rho)^2}\\
&= (2\pi i)^{\dim G}\frac{(\mathrm{vol}_G)^2\det p''(\xi)}{(\dim V_\lambda)^2}.
\end{aligned}$$

We now consider more general β. The equivariant cohomology algebra of $G\times G$ is a tensor product (as modules over $H_G(\mathrm{pt})$) of the cohomology algebras of the two factors

$$H_G(G\times G) = H_G(G)\otimes_{H_G(\mathrm{pt})} H_G(G).$$

Let $p_1,\ldots,p_N$ be a set of generators for the ring of invariant polynomials, and let $\eta_G^i = \eta_G^{p_i}$ be the corresponding generators of $H_G(G)$. Following [41] and [3, Section 6], set

$$\beta(\xi) = \exp\Big(\sum_{j=1}^N \sigma_j p_j(\xi) + \sum_{i=1}^N \epsilon_i^{(1)} h^*\eta_G^{p_i}(\xi) + \epsilon_i^{(2)} k^*\eta_G^{p_i}(\xi)\Big).$$

where σ_i are formal even parameters, and $\epsilon_i^{(1)},\epsilon_i^{(2)}$ are *odd* parameters (anti-commuting with each other but also with odd differential forms). To simplify notation, denote

$$P^{(1)}(\xi) = \sum_{i=1}^N \epsilon_i^{(1)} p_i(\xi),\quad P^{(2)}(\xi) = \sum_{i=1}^N \epsilon_i^{(2)} p_i(\xi),\quad R(\xi) = \sum_{i=1}^N \sigma_i p_i(\xi)$$

Thus

$$\beta = \exp(R + h^*\eta_G^{P^{(1)}} + k^*\eta_G^{P^{(2)}}).$$

The integral over $F = T^2$ now becomes

$$\int_{T^2}\beta(\xi)e^{2\pi i\omega^p(\xi)} = e^{R(\xi)}\int_{T^2} e^{2\pi i(p_{\mathfrak{t}}''(\xi)h^*\theta^T)\cdot k^*\theta^T) + (P^{(1)})'(\xi)\cdot h^*\theta^T + (P^{(2)})'(\xi)\cdot k^*\theta^T}.$$

This integral is solved by completion of the square, writing the exponent as

$$\big(2\pi i\, p_{\mathfrak{t}}''(\xi)h^*\theta^T + Q'(\xi)\big)\cdot\big(k^*\theta^T - \frac{1}{2\pi i}p_{\mathfrak{t}}''(\xi)^{-1}P'(\xi)\big) - \frac{1}{2\pi i}p_{\mathfrak{t}}''(\xi)^{-1}(P^{(1)})'(\xi)\cdot(P^{(2)})'(\xi).$$

This yields,

$$\langle \mathfrak{n}^\beta, \overline{\chi_\lambda}\rangle = (2\pi i)^{\dim G}\frac{(\mathrm{vol}_G)^2\det p''(\xi)}{(\dim V_\lambda)^2} e^{R(\xi) - \frac{1}{2\pi i}p_{\mathfrak{t}}''(\xi)^{-1}(P^{(1)})'(\xi)\cdot(P^{(2)})'(\xi)}$$

with $p'(\xi) = \lambda + \rho$.

7.4. **Moduli spaces of flat bundles.** By fusion of s copies of the double G^2 and r conjugacy classes $\mathcal{C}_1,\ldots,\mathcal{C}_r$, the space $M = G^{2s}\times\mathcal{C}_1\times\cdots\times\mathcal{C}_r$ considered in (4) acquires the structure of a q-Hamiltonian G-space, with moment map (5). For any $p\in\mathrm{Pol}(\mathfrak{g})^G$, this space carries a higher q-Hamiltonian form ω^p, which is canonically defined up to a d_G-coboundary.

The equivariant cohomology ring of M is simply the tensor product (over $H_G(\mathrm{pt})$) of $2s$ factors of $H_G(G)$ with the cohomology rings of the conjugacy classes $\mathcal{C}_l = G/K_l$, i.e. $\mathrm{Pol}(\mathfrak{k}_l)^{K_l}$. Let $R=\sum\sigma_i p_i$ be as above, and write

$$P_j^{(1)} = \sum_{i=1}^N \epsilon_{ij}^{(1)} p_i,\quad P_j^{(2)} = \sum_{i=1}^N \epsilon_{ij}^{(2)} p_i$$

where $\epsilon_{ij}^{(1)},\epsilon_{ij}^{(2)}$ are odd parameters, and the index j stands for the G^2-factor. Let $Q_l\in\mathrm{Pol}(\mathfrak{k}_l)^{K_l}$ be invariant polynomials, with corresponding cocycles $\beta_l\in\Omega_G(\mathcal{C}_l)$, and let

$$\beta = \exp\Big(R + \sum_{j=1}^s (h_j^*\eta_G^{P_j^{(1)}} + k_j^*\eta_G^{P_j^{(2)}})\Big)\prod_{l=1}^r \beta_l$$

Consider now the fixed point contributions. The fixed point sets F for $\lambda+\rho$ is just the product of the fixed point sets for the factors G^2 resp. $\mathcal{C}_l$. In particular, each of the $r+s$ moment maps is *constant* on F. This implies that the 'fusion terms', i.e. terms involving φ^p, all vanish if pulled back to F. Hence, the fixed point contribution from any such F is simply the product of the fixed point contributions from the factors. We therefore obtain the following result for the Fourier coefficients,

$$\langle \mathfrak{n}^\beta,\overline{\chi_\lambda}\rangle = (2\pi i)^{\frac{\dim M}{2}}(\mathrm{vol}_G)^{2s}\Big(\frac{\det^{1/2}p''(\xi)}{\dim V_\lambda}\Big)^{2s+r} e^{\tilde{R}(\xi)}\prod_{l=1}^r \frac{Q_l(\xi)\,\mathrm{Vol}(\mathcal{C}_l)\overline{\chi_\lambda(\mathcal{C}_l)}}{\det^{1/2}p''_{\mathfrak{k}_l}(\xi)}$$

where

$$\tilde{R}(\xi) = R(\xi) - \frac{1}{2\pi i}\sum_j p''_{\mathfrak{t}}(\xi)^{-1}(P_j^{(1)})'(\xi)\cdot(P_j^{(2)})'(\xi)$$

We finally state the resulting Witten formula for intersection pairings. Since we assume G is simple and simply connected, the level set $\Phi^{-1}(e)\subset M$ is connected, and so is the moduli space $\mathcal{M} = M/\!/G$. If e is a regular value, the action of G on $\Phi^{-1}(e)$ is locally free. We assume that the generic stabilizer for the G-action on $\Phi^{-1}(e)$ is equal to the center $Z(G)$. This is automatic for $s\geq 2$, or for $2s+r\geq 3$ and sufficiently 'generic' conjugacy classes. (See [4] for discussion.) Write $\mathfrak{f}^p = \omega^p_{\mathrm{red}}$ since these generalize the Atiyah-Bott classes of 'type $\mathfrak{f}$', and let β be given as above. Then

$$\begin{aligned}&\frac{1}{(2\pi i)^{\dim\mathcal{M}/2}}\int_{\mathcal{M}}\beta_{\mathrm{red}}e^{2\pi i\mathfrak{f}^p}\\ =\;& \#Z(G)(\mathrm{vol}_G)^{2s-2}\sum_{\lambda\in\Lambda^*_+} e^{\tilde{R}(\xi)}\Big(\frac{\det^{1/2}p''(\xi)}{\dim V_\lambda}\Big)^{2s+r-2}\prod_{l=1}^r\Big(\frac{Q_l(\xi)\,\mathrm{Vol}(\mathcal{C}_l)\overline{\chi_\lambda(\mathcal{C}_l)}}{\det^{1/2}p''_{\mathfrak{k}_l}(\xi)}\Big)\end{aligned}$$

Appendix A. Formal change of variables

A *formal diffeomorphism* of a manifold X, is an invertible elements of the algebra $C^\infty(X,X)[[\delta]]$ where δ_j are given formal parameters. If $X = V$ is a vector space, we can consider the smaller group of *polynomial* formal diffeomorphisms, consisting of invertible elements $P = \sum_I \delta^I P_I$ (using multi-index notation) of the algebra $\mathrm{Pol}(V,V)[[\delta]]$, where $\mathrm{Pol}(V,V)$ are V-valued polynomials on V.

Suppose V carries a scalar product $\cdot$, and let $\mathcal{F}_V^{-1} : \mathcal{D}'(V)_{\mathrm{comp}} \to C^\infty(V)$ denote the inverse Fourier transform, defined by

$$(\mathcal{F}_V^{-1}\mathfrak{n})(\xi) = \langle \mathfrak{n}, e^{-2\pi i \langle \cdot, \xi\rangle}\rangle, \quad \xi \in V.$$

Proposition A.1. *Suppose $P = \sum_I \delta^I P_I$ is a formal diffeomorphism of V, such that all P_I are polynomials and $P_\emptyset = \mathrm{Id}_V$. Let $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{D}'(V)_{\mathrm{comp}}[[\delta]]$, with*

$$(\mathcal{F}_V^{-1}\mathfrak{n}_1)(\xi) = (\mathcal{F}_V^{-1}\mathfrak{n}_2)(P^{-1}(\xi))$$

as elements of $C^\infty(V)[[\delta]]$. Then

$$\mathfrak{n}_1(\mu) = e^{2\pi i \mu \cdot Q(\frac{1}{2\pi i}\frac{\partial}{\partial\nu})} \det\big(P'(\frac{1}{2\pi i}\frac{\partial}{\partial \nu})\big)\, \mathfrak{n}_2(\nu)\Big|_{\nu=\mu}$$

where $Q(\xi) = P(\xi) - \xi$.

Proof. Results of this type are well-known from the theory of Fourier integral operators – see e.g. [16]. It is enough to consider the case $\mathfrak{n}_i$ smooth, so that $f_i(\xi) = (\mathcal{F}_V^{-1}\mathfrak{n}_i)(\xi)$ are rapidly decreasing functions of ξ. The desired identity follows from the calculation,

$$\begin{aligned}
\mathfrak{n}_1(\mu) &= \int_V f_2(P^{-1}(\xi)) e^{2\pi i \mu\cdot\xi} \mathrm{d}\xi \\
&= \int_V f_2(\zeta)\, e^{2\pi i \mu \cdot P(\zeta)} (\det P'(\zeta)) \mathrm{d}\zeta \\
&= \int_V e^{2\pi i\mu\cdot Q(\zeta)} (\det P'(\zeta))\, f_2(\zeta)\, e^{2\pi i \mu\cdot\zeta}\mathrm{d}\zeta.
\end{aligned}$$

(To justify the second equality, use Borel summation to temporarily replace P by a genuine function $V \to V$, depending on δ_j as parameters.) ☐

Appendix B. Equivariant de Rham theory

B.1. **The Cartan model of equivariant cohomology.** Let G be a compact Lie group acting smoothly on a manifold M. That is, we are given a group homomorphism $G \to \mathrm{Diff}(M)$, $g \mapsto \mathcal{A}_g$, such that the action map $G \times M \to M$, $(g,x) \mapsto g.x = \mathcal{A}_g(x)$ is smooth. For any Lie algebra element $\xi \in \mathfrak{g}$, the corresponding generating vector field ξ_M is the derivation of $C^\infty(M)$ given by $\xi_M(f) = \frac{d}{dt}|_{t=0}\mathcal{A}^*_{\exp(-t\xi)}f$. For each $\xi \in \mathfrak{g}$, define an odd derivation of $\Omega(M)$ by

$$\mathrm{d}_\xi = \mathrm{d} - \iota(\xi_M).$$

The Cartan complex of equivariant differential forms is the graded algebra $\Omega^\bullet_G(M) = (\mathrm{Pol}(\mathfrak{g}) \otimes \Omega(M))^G$ of G-equivariant polynomial maps $\beta : \mathfrak{g} \to \Omega(M)$, equipped with the equivariant differential

$$(\mathrm{d}_G\beta)(\xi) = \mathrm{d}_\xi\beta(\xi),$$

and with grading given by the differential form degree plus twice the polynomial degree. Its cohomology algebra coincides with the equivariant cohomology $H^\bullet_G(M) = H^\bullet(EG \times_G M)$.

B.2. **The Cartan map.** Let G be a compact Lie group and $\pi : P \to B$ a principal G-bundle. Then the pull-back map $\pi^* : \Omega(B) \to \Omega(P)_{\mathrm{basic}} \subset \Omega_G(P)$ is a quasi-isomorphism. If $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ is a principal connection, one has an explicit homotopy inverse

$$\mathrm{Car}^\theta : \ \Omega_G(P) \to \Omega(P)_{G-\mathrm{basic}} \cong \Omega(B),$$

known as the *Cartan map*. The definition of this map is as follows: Let $\mathrm{Hor}^\theta : \Omega^\bullet(P) \to \Omega^\bullet(P)$ denote the horizontal projection, and let $F^\theta = \mathrm{d}\theta + \frac{1}{2}[\theta,\theta] \in \Omega^2(P) \otimes \mathfrak{g}$ be the curvature. There is a unique algebra homomorphism

$$\mathrm{Pol}^\bullet(\mathfrak{g}) \to \Omega^{2\bullet}(P), \ p \mapsto p(F^\theta)$$

given on linear polynomials $\mu \in \mathfrak{g}^*$ by $\mu \mapsto \langle \mu, F^\theta\rangle$. Tensoring with $\Omega(P)$, this yields an algebra homomorphism

$$\Omega_G(P) = (\mathrm{Pol}(\mathfrak{g}) \otimes \Omega(P))^G \to \Omega(P)^G, \ \beta \mapsto \beta(F^\theta).$$

The Cartan map is defined by

$$\mathrm{Car}^\theta(\beta) = \mathrm{Hor}^\theta(\beta(F^\theta)).$$

It was proved by Cartan that Car^θ is a chain map, inducing the inverse map $(\pi^*)^{-1}$ in cohomology. For a nice proof of Cartan's theorem, showing in particular that Car^θ and π^* are homotopy inverses, see Nicolaescu [34]. The proof carries over to the case that the G-action on P is not free but only *locally* free, i.e. has finite stabilizers, and it also generalizes to the case that $P \to B$ is an L-equivariant principal G-bundle with a L-invariant connection, where L is a second Lie group.

One application of the Cartan map is *induction.* Suppose G is a compact connected Lie group, and K a maximal rank subgroup. Let the principal K-bundle $G \to G/K$ be equipped with the unique G-invariant connection. Given a K-manifold Y, let the principal K-bundle $G \times Y \to G \times_K Y$ carry the pull-back connection. The *induction map*

$$\mathrm{Ind}_K^G : \Omega_K(Y) \to \Omega_G(G \times_K Y),$$

is a homotopy equivalence, given as the pull-back map $\Omega_K(Y) \to \Omega_{G\times K}(G \times Y)$ followed by the Cartan map. A homotopy inverse $\Omega_G(G \times_K Y) \to \Omega_K(Y)$ is given as pull-back to $Y \subset G \times_K Y$.

Example B.1. In the special case $Y = \mathrm{pt}$, we obtain homotopy inverses

$$\Omega_K(\mathrm{pt}) \to \Omega_G(G/K), \quad \Omega_G(G/K) \to \Omega_K(\mathrm{pt})$$

where the first map is the induction map and the second map is pull-back to the base point $eK \in G/K$.

B.3. **Equivariant Thom form, equivariant Euler form.** Let $\pi : E \to B$ be a G-equivariant real vector bundle of even rank, with a fiberwise orientation. An equivariant Thom form for E is an equivariant form $\tau_E \in \Omega_G(E)$, compactly supported in fiber directions, with fiber integral $\pi_*\tau_E = 1$. Any two Thom forms differ by the equivariant coboundary of a form with fiberwise compact support. Assume the base B is oriented, and give E the product orientation. Then

$$\int_E \tau_E(\xi)\beta = \int_B \beta, \tag{39}$$

for any differential form $\beta \in \Omega(E)$ such that $\tau(\xi)\beta$ has compact support and $\mathrm{d}_\xi\beta = 0$. The proof boils down to the fact that if one is working with equivariant currents, the Thom class is represented by $\iota_*(1) \in \mathcal{C}_G(E)$, where $\iota : B \to E$ is the inclusion of the base.

The pull-back $\mathrm{Eul}(E) = \iota^*\tau_E \in \Omega_G(B)$ is called the equivariant Euler form. Given an invariant Riemannian metric and compatible connection on E, the Mathai-Quillen construction [30] gives explicit representatives for τ_E, and therefore of $\mathrm{Eul}(E)$.

An important special case is: $G = T$, $B = \mathrm{pt}$, and $E = V$ a complex vector space. Let $a_1, \ldots, a_n \in \Lambda^*$ be the (real) weights for the action on V. Then the equivariant Euler form is simply a polynomial on t: $\mathrm{Eul}(V, \xi) = (-1)^n \prod_{j=1}^n \langle a_j, \xi\rangle$.

B.4. **The Berline-Vergne localization formula.** Let G be a compact Lie group and M an oriented G-manifold. For any $\xi \in \mathfrak{g}$, consider the derivation $\mathrm{d}_\xi = \mathrm{d} - \iota(\xi_M)$. Let M^ξ be the set of zeroes of ξ_M, or equivalently the fixed point set of the 1-parameter subgroup generated by ξ.

Theorem B.2 (Berline-Vergne)**.** [9] *Suppose $\alpha \in \Omega(M)_{\mathrm{comp}}$ is a compactly supported differential form with $\mathrm{d}_\xi\alpha = 0$. Let $S \subset M$ be an embedded G_ξ-invariant oriented submanifold of even codimension, containing the fixed point set M^ξ. (We allow for S to consist of several components of varying dimension.) Let $\mathrm{Eul}(\nu_S, \cdot) \in \Omega_{G_\xi}(S)$ be the G_ξ-equivariant Euler form of the normal bundle of S, for some choice of invariant Euclidean metric and connection. Then*

$$\int_M \alpha = \int_S \frac{\iota_S^*\alpha}{\mathrm{Eul}(\nu_S, \xi)}$$

Remarks B.3. (a) Suppose $\beta = \alpha(\xi)$ where $\alpha \in \Omega_G(M)$ is a G-equivariant cocycle. Then $\mathrm{d}_\xi\beta = 0$, and the localization formula is a version of the localization formula in equivariant cohomology (see Atiyah-Bott [6]). However, not all d_ξ-cocycles arise in this way.

(b) The theorem is usually stated for the special case $S = M^\xi$. In fact, the general case may be deduced from this special case, by further localizing the integral over S to $S' = M^\xi \subset S$. If $\zeta \in \mathfrak{g}$ is sufficiently close to ξ and commutes with ξ, then also $S = M^\zeta$ satisfies the conditions of the theorem.

B.5. **Equivariant homotopy operators.** We will need the following facts about homotopy operators. For any G-equivariant vector bundle $\pi : E \to B$, let $\mathrm{h} : \Omega^\bullet(E) \to \Omega^{\bullet-1}(E)$ denote the standard homotopy operator. That is, up to a sign h is defined as pull-back under the map $I \times E \to E$ given by scalar multiplication on the fibers, followed by the push-forward map $(\mathrm{pr}_2)_* : \Omega^\bullet(I \to E) \to \Omega^{\bullet-1}(E)$. Since the projection π is G-equivariant, the homotopy operator defines a degree -1 operator on $\Omega_G(E)$, denoted by the same symbol.

Letting $\iota : B \to E$ be the inclusion of the zero section, the homotopy operator satisfies $\iota^* \circ \mathrm{h} = 0$, $\mathrm{h} \circ \pi^* = 0$ and

$$\mathrm{d}_G \mathrm{h} + \mathrm{h}\, \mathrm{d}_G = \mathrm{id} - \pi^* \circ \iota^*.$$

Another simple fact regarding h is that if a form on E is zero along $B \subset E$, then so is its image under h.

B.6. **Equivariant simplicial differential forms.** Recall the definition of a simplicial manifold [18, 36, 32]. For each positive integer n let $[n]$ denote the ordered sequence $\{0, \ldots, n\}$. A map $f : [m] \to [n]$ is called *increasing* if $f(i) \geq f(j)$ for $i > j$. Of particular interest are the *face maps* $\partial^i : [n-1] \to [n]$ for $i = 0, \ldots, n$, defined as the unique strictly increasing map whose image does not contain i.

A simplicial manifold is a contravariant functor from the category of ordered sequences (with increasing maps as morphisms) into the category of manifolds. That is, a simplicial manifold $X_\bullet$ is a sequence of manifolds $(X_n)_{n=0}^\infty$, together with a map $X(f) : X_n \to X_m$ for each increasing map $f : [m] \to [n]$, such that $X(\mathrm{id}) = \mathrm{id}$ and $X(f \circ g) = X(g) \circ X(f)$. The maps $\partial_i = X(\partial^i) : X_n \to X_{n-1}$ are again referred to as *face maps.* A (smooth) simplicial map between simplicial manifolds $F_\bullet : X_\bullet \to X'_\bullet$ is a collection of smooth maps $F_n : X_n \to X'_n$ intertwining the maps $X(f)$, $X'(f)$. Any manifold M can be viewed as a simplicial manifold $M_\bullet$, where all $M_n = M$ and all $X(f)$ are the identity map. Another example is the simplicial manifold $E_\bullet M$, where $E_n M = M^{n+1}$ and $X(f)(x_0, \ldots, x_n) = (x_{f(0)}, \ldots, x_{f(n)})$.

Let $\Delta^n \subset \mathbb{R}^{n+1}$ denote the standard n-simplex, defined as the intersection of the positive orthant with the affine hyperplane $\sum_{i=0}^n t_i = 1$. Any increasing map $f : [m] \to [n]$ defines a linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ sending the basis vector e_i to $e_{f(i)}$. It induces a map $\Delta(f) : \Delta^m \to \Delta^n$. The geometric realization [31, 36] of a simplicial manifold $|X|$ is the quotient $|X| = \coprod_n (\Delta^n \times X_n)/\sim$ where one divides by the equivalence relation generated by $(\Delta(f)(t), x) \sim (t, X(f)(x))$ for all increasing maps f.

Following Dupont [18], one defines a simplicial r-form on $X_\bullet$ to be a collection of r-forms $\alpha_n \in \Omega^r(\Delta^n \times X_n)$ satisfying relations

$$(\Delta(f) \times \mathrm{id})^* \alpha_n = (\mathrm{id} \times X(f))^* \alpha_m \tag{40}$$

for any increasing map $f : [m] \to [n]$. Under certain technical hypothesis (which holds in our examples), the complex $(\Omega^r_{\mathrm{simp}}(X_\bullet), \mathrm{d})$ computes the cohomology of the geometric realization with coefficients in $\mathbb{R}$. Consider on the other hand the double complex $\Omega^{k,l}(X_\bullet) := \Omega^l(X_k)$, with commuting differentials d and $\delta = \sum_{i=0}^{k+1} (-1)^i \partial_i^*$, and the corresponding total complex $\Omega^r(X_\bullet) = \bigoplus_{k+l=r} \Omega^l(X_k)$ with differential

$\delta + (-1)^k \mathrm{d}$. The maps $\Omega^r(\Delta^n \times X_n) \to \Omega^{r-n}(X_n)$ (integration over simplices) assemble to a chain map

$$\Omega^r_{\mathrm{simp}}(X_\bullet) \to \Omega^r(X_\bullet),$$

As shown by Dupont [18], this map is a chain homotopy equivalence.

There is a straightforward equivariant extension of these concepts: Suppose $K_\bullet$ is a simplicial Lie group (i.e. all K_n are Lie groups and all face maps and degeneracy maps are group homomorphisms). An action of $K_\bullet$ on $X_\bullet$ is a simplicial map $K_\bullet \times X_\bullet \to X_\bullet$ given by a K_n-action on X_n in each degree n. This means that for any increasing map $f : [m] \to [n]$, the maps $X(f) : X_n \to X_m$ are equivariant with respect to the homomorphisms $K(f) : K_n \to K_m$:

$$X(f)(k.x) = (K(f)(k)).(X(f)(x)).$$

Thus one obtains pull-back maps in equivariant cohomology, $X(f)^* : \Omega^r_{K_m}(X_m) \to \Omega^r_{K_n}(X_n)$. We define a space

$$\Omega^r_{K_\bullet}(X_\bullet) := \bigoplus_{n=0}^{r} \Omega^{r-n}_{K_n}(X_n)$$

of $K_\bullet$-equivariant forms on $X_\bullet$, with equivariant differential $\delta + (-1)^n \mathrm{d}_{K_n}$ on $\Omega^{r-n}_{K_n}(X_n)$, and define a $K_\bullet$-equivariant r-form to be a collection of equivariant forms $\alpha_n = \Omega^r_{K^n}(\Delta^n \times X_n)$ satisfying the compatibility relations (40). As before, integration over simplices defines a chain equivalence between these two complexes.

Suppose now that $P_\bullet \to X_\bullet$ is a simplicial $K_\bullet$-equivariant principal G-bundle. A $K_\bullet$-invariant simplicial connection $\sigma_\bullet$ is given by a family of K_n-invariant connection forms

$$\sigma_n \in \Omega^1(\Delta^n \times P_n)^{K_n} \otimes \mathfrak{g}$$

satisfying Dupont's compatibility relations. Given a G-manifold M, the collection of equivariant Cartan maps for $\sigma_\bullet$, followed by integration over simplices, defines a chain map[2]

$$\Omega^\bullet_G(M) \to \bigoplus_n \Omega^{\bullet-n}_{K_n}(P_n \times_G M). \tag{41}$$

This is the simplicial version of the (equivariant) Cartan map. If M is a point, this is known as the simplicial Chern-Weil map.

B.7. **Equivariant Bott-Shulman forms.** Let G be a compact Lie group, and consider the simplicial group $E_\bullet G$. The diagonal action $g.(g_0, \ldots, g_n) = (g_0 g^{-1}, \ldots, g_n g^{-1})$ of G on $E_n G$ makes $E_\bullet G \to B_\bullet G = E_\bullet G/G$ into a simplicial principal G-bundle. There is a distinguished simplicial connection on $E_\bullet G \to B_\bullet G$, given by

$$\sigma_n = \sum_{i=0}^{n} t_i \, \mathrm{pr}_i^* \, \theta^L \in \Omega^1(\Delta^n \times E_n G) \otimes \mathfrak{g},$$

where $\mathrm{pr}_i : E_n G \to G$ is projection to the ith factor and $\theta^L \in \Omega^1(G) \otimes \mathfrak{g}$ are the left-invariant Maurer-Cartan forms. This connection is invariant under the left-action

[2] See [2] for an alternative construction of such a chain map.

of $E_\bullet G$ on itself. It hence defines, for any G-manifold M and any homomorphism $K_\bullet \to E_\bullet G$ of simplicial groups, a $K_\bullet$-equivariant Cartan map (41). To make this map more explicit, use the projection

$$G^{n+1} \times M \to G^n \times M, \quad (g_0, \ldots, g_n, x) \mapsto (g_0 g_1^{-1}, \ldots, g_{n-1} g_n^{-1}, g_n.x)$$

to identify $E_n G \times_G M \cong G^n \times M$. Under this identification the face maps are

$$\partial_i(h_1, \ldots, h_n, x) \quad = \quad \begin{cases} (h_2, \ldots, h_n, x) & \text{for } i = 0, \\ (h_1, \ldots, h_i h_{i+1}, \ldots, h_n, x) & \text{for } 0 < i < n, \\ (h_1, \ldots, h_{n-1}, h_n.x) & \text{for } i = n, \end{cases}$$

and the action of $E_n G = G^{n+1}$ on $G^n \times M$ reads

$$(g_0, \ldots, g_n).(h_1, \ldots, h_n, x) = (g_0 h_1 g_1^{-1},\ g_1 h_2 g_2^{-1}, \ldots, g_{n-1} h_n g_n^{-1}, g_n.x).$$

We have thus constructed a chain map

$$\phi_{K_\bullet} = \bigoplus_{n \geq 0} \phi_{K_n}^{(n)} : \quad \Omega_G^\bullet(M) \to \bigoplus_{n \geq 0} \Omega_{K_n}^{\bullet - n}(G^n \times M) \tag{42}$$

which we will call the *equivariant Bott-Shulman map*. As observed by Bott-Shulman (in a slightly less general setting), the maps $\phi_{K_n}^{(n)}$ vanishes on $\Omega_G^d(M)$ for all $n > d/2$.

To compute the Bott-Shulman map, it is useful to note that the bundle $E_n G \times M \to E_n G \times_G M$ is trivial: The submanifold defined by $g_0 = e$ is a trivializing section. In terms of the identification $E_n G \times_G M = G^n \times M$, this section is the map

$$\iota_n : G^n \times M \to G^{n+1} \times M, \ (h_1, \ldots, h_n; y) \mapsto (g_0, \ldots, g_n; x)$$

where $g_j = (h_1 \cdots h_j)^{-1}$ and $y = (h_1 \cdots h_n)^{-1}.x$.

B.8. **The case $M = \mathrm{pt}$.** The setting originally considered by Bott [11] and Shulman [37] corresponds to the case where M is a point and $K_\bullet$ is trivial. The equivariant extension to $K_\bullet = G$, viewed as the diagonal subgroup of $E_\bullet G$, is discussed by Jeffrey in [24]. In this case, (42) becomes a map

$$\phi_G = \bigoplus_{n \geq 0} \phi_G^{(n)} : \quad \mathrm{Pol}^\bullet(\mathfrak{g})^G \to \bigoplus_{n \geq 0} \Omega_G^{2\bullet - n}(G^n)$$

where G acts on $B_n G = G^n$ by conjugation. To write down concrete formulas, note that the generating vector field for the diagonal action of G on $E_n G = G^{n+1}$, given by the left action on each factor, is the sum of $n+1$ copies of the right-invariant vector field $-\xi^R$. Hence, the equivariant curvature of the connection σ_n reads

$$F_G(\xi) = \sum_{i=0}^n \mathrm{d}t_i\, g_i^* \theta^L - \frac{1}{2} \sum_{i=0}^n t_i\, g_i^*[\theta^L, \theta^L] - \frac{1}{2} \sum_{i,j=0}^n t_i t_j [g_i^* \theta^L, g_j^* \theta^L] + \sum_{i=0}^n t_i \ \mathrm{Ad}_{g_i^{-1}}(\xi).$$

Then $\phi^{(n)}(p)$ is given explicitly as the integral

$$\phi^{(n)}(p) = \int_{\Delta^n} p(\iota_n^* F_G(\xi)).$$

For $p \in \mathrm{Pol}^d(g)^G$, introduce special notation

$$\begin{aligned} \eta_G^p &= \phi_G^{(1)}(p) \in \Omega_G^{2d-1}(G), \\ \varphi^p &= \phi_G^{(2)}(p) \in \Omega_G^{2d-2}(G^2) \end{aligned}$$

(obviously $\phi_G^{(0)}(p) = 0$ if $d > 0$). If $p(\xi) = \frac{1}{2}||\xi||^2$ we will drop the superscript p. It is not hard to see that the general formula for $\phi^{(n)}(p)$ specializes, for $n = 1$, to the formula (8) for η_G^p. Also, taking $n = 2$ and $p(\xi) = \frac{1}{2}||\xi||^2$ one finds that φ is an ordinary 2-form on $G \times G$:

$$\varphi = \frac{1}{2}\,\mathrm{pr}_1^*\,\theta^L \cdot \mathrm{pr}_2^*\,\theta^R.$$

(The forms $\phi_G^{(n)}(p)$ for $n > 2$ vanish for $p(\xi) = \frac{1}{2}||\xi||^2$.)

The fact that $\phi_G(p)$ is closed under the total differential for the double complex $(\Omega_G^k(G^l), \mathrm{d}_G, \delta)$ gives equations,

$$\begin{aligned} \mathrm{d}_G\eta_G^p &= 0, \\ \mathrm{d}_G\varphi^p &= \mathrm{Mult}^*\,\eta_G^p - \mathrm{pr}_1^*\,\eta_G^p - \mathrm{pr}_2^*\,\eta_G^p \end{aligned} \tag{43}$$

where $\mathrm{Mult} : G \times G \to G$ is group multiplication. Furthermore, the form

$$\begin{aligned} &(\mathrm{pr}_1 \times \mathrm{pr}_2)^*\varphi^p + (\mathrm{Mult} \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \times \mathrm{pr}_3)^*\varphi^p \\ &\quad -(\mathrm{pr}_2 \times \mathrm{pr}_3)^*\varphi^p - (\mathrm{pr}_1 \times (\mathrm{Mult} \circ (\mathrm{pr}_2 \times \mathrm{pr}_3))^*\varphi^p \end{aligned}$$

on $G \times G \times G$ is d_G-exact. Another useful property of the forms η_G^p is that they change sign under the inversion map $\mathrm{Inv} : G \to G,\ g \mapsto g^{-1}$

$$\mathrm{Inv}^*\,\eta_G^p = -\eta_G^p. \tag{44}$$

This follows from Remark 2.1(b), by pulling the identity (43) back under the map $g \mapsto (g, g^{-1})$, and using that the pull-back of η_G^p to the group unit vanishes.

While the formulas for the forms φ^p are rather complicated, one has simple expressions if $G = T$ is a torus.

Lemma B.4. *If $G = T$ is a torus, the form $\varphi^p(\xi) \in \Omega_T(T^2)$ is given by*

$$\varphi^p(\xi) = \frac{1}{2}\sum_{rs} p''_{rs}(\xi)\,\mathrm{pr}_1^*\,\theta_T^r\,\mathrm{pr}_2^*\,\theta_T^s.$$

Proof. In the Abelian case, the formula for the equivariant curvature of σ_2 simplifies dramatically:

$$F_T(\xi) = \sum_{i=0}^{2} \mathrm{d}t_i\, g_i^*\theta_T + \xi.$$

Hence, the pull-back under $\iota_2(h_1, h_2) = (e, h_1^{-1}, (h_1h_2)^{-1})$ reads,

$$\iota_2^* F_T(\xi) = \xi - \mathrm{d}t_1\ h_1^*\theta_T - \mathrm{d}t_2(h_1^*\theta_T + h_2^*\theta_T).$$

Hence,

$$\begin{aligned} p(\iota_2^* F_T(\xi)) &= p(\xi - \mathrm{d}t_1\ h_1^*\theta_T - \mathrm{d}t_2(h_1^*\theta_T + h_2^*\theta_T)) \\ &= -\mathrm{d}t_1\mathrm{d}t_2 \sum_{rs} p''_{rs}(\xi)\, h_1^*\theta_T^r\, h_2^*\theta_T^s + \ldots \end{aligned}$$

where we have only written the coefficient of $\mathrm{d}t_1\mathrm{d}t_2$. The Lemma follows since $\int_{\Delta^2} \mathrm{d}t_1\mathrm{d}t_2 = -\frac{1}{2}$, with our conventions. □

B.9. **The case $M = G$.** Consider next the case $M = G$ with G acting by conjugation, and with $K_\bullet = E_\bullet G = G^{\bullet+1}$. We obtain a chain map

$$\psi = \bigoplus_{n=0}^{\infty} \psi^{(n)} : \quad \Omega_G^\bullet(G) \to \bigoplus_{n=0} \Omega_{G^{n+1}}^{\bullet-n}(G^{n+1})$$

As pointed out above, $\psi^{(0)}$ is just the identity map. Consider the degree $n = 1$ component. The action of G^2 is given by $(g_0, g_1).(h, k) = (g_0 h g_1^{-1}, g_1 k g_1^{-1})$, and the face maps are $\partial_0(h, k) = k$ and $\partial_1(h, k) = \mathrm{Ad}_h(k)$. The form

$$\lambda^p := \psi^{(1)}(\eta_G^p) \in \Omega_{G^2}^{2d-2}(G^2)$$

has the property

$$\mathrm{d}_{G^2}\lambda^p(\xi_0, \xi_1) = \partial_0^*\eta_G^p(\xi_1) - \partial_1^*\eta_G^p(\xi_0). \tag{45}$$

Again we drop the superscript for $p(\xi) = \frac{1}{2}||\xi||^2$, and also the subscript G^2 since λ^p does not depend on the equivariant parameter in this case. Explicit calculation gives:

$$\lambda = -\frac{1}{2} h^*\theta^L \cdot \mathrm{Ad}_k(h^*\theta^L) + h^*\theta^L \cdot k^*\frac{\theta^L + \theta^R}{2} \tag{46}$$

(where we view h, k as maps $G^2 \to G$). For more general p, we only state the result if $G = T$ is a torus:

Lemma B.5. *If $G = T$ is a torus,*

$$\lambda^p(\xi_0, \xi_1) = \sum_{rs} A_{rs}(\xi_0, \xi_1) h^*\theta_T^r\, k^*\theta_T^s$$

where $A_{rs}(\xi_0, \xi_1) = \int_0^1 p''_{rs}(t\xi_0 + (1-t)\xi_1)\ \mathrm{d}t$.

Proof. We denote points in $E_n T \times T = T^{n+1} \times T$ by $(g_0, \ldots, g_n, k)$, and points in $E_n T \times_T G = T^n \times T$ by $(h_1, \ldots, h_n, k)$. The $E_n T = T^{n+1}$-equivariant curvature of σ_n is given by

$$F^\sigma_{T^{n+1}}(\xi_0, \ldots, \xi_n) = \sum_{i=0}^n t_i \xi_i + \sum_{i=0}^n \mathrm{d}t_i\ g_i^*\theta_T$$

Recall that $\eta_T^p(\xi) = -p'(\xi) \cdot \theta_T$. To compute $\lambda^{(p)} = \psi^{(1)}(\eta_T^p)$, we have to consider the form in $\Omega_{T^2}(E_1 T \times T)$,

$$\eta^p(F^\sigma_{T^2}(\xi_0, \xi_1)) = -k^*\theta_T \cdot p'(\sum_{i=0}^1 t_i\xi_i - \sum_{i=0}^1 \mathrm{d}t_i\ g_i^*\theta_T)$$

Pulling back under the map $\iota_1(h,k) = (e, h^{-1}, k)$, and setting $t_1 = t$, $t_0 = (1-t)$ we obtain

$$\begin{aligned} \iota_1^* \eta^p(F^\sigma_{T^2}(\xi_0, \xi_1)) &= -k^*\theta_T \cdot p'((1-t)\xi_0 + t\xi_1 + \mathrm{d}t h^*\theta_T) \\ &= \mathrm{d}t \sum_{rs} p''_{rs}((1-t)\xi_0 + t\xi_1) k^*\theta_T^r h^*\theta_T^s + \dots \end{aligned}$$

where we have only written the coefficient of $\mathrm{d}t$. Integrating over Δ^1, the Lemma follows. □

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