Lie groups and Lie algebras

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1. Terminology and notation

1.1. Lie groups.

Definition 1.1. A Lie group is a group G, equipped with a manifold structure such that the group operations

$$\mathrm{Mult}\colon G\times G\to G,\quad (g_1,g_2)\mapsto g_1g_2$$
$$\mathrm{Inv}\colon G\to G,\quad g\mapsto g^{-1}$$

are smooth. A morphism of Lie groups G, G' is a morphism of groups $\phi\colon G\to G'$ that is smooth.

Remark 1.2. Using the implicit function theorem, one can show that smoothness of Inv is in fact automatic. (Exercise)

The first example of a Lie group is the *general linear group*

$$\mathrm{GL}(n,\mathbb{R})=\{A\in \mathrm{Mat}_n(\mathbb{R})|\ \det(A)\neq 0\}$$

of invertible $n\times n$ matrices. It is an open subset of $\mathrm{Mat}_n(\mathbb{R})$, hence a submanifold, and the smoothness of group multiplication follows since the product map for $\mathrm{Mat}_n(\mathbb{R})$ is obviously smooth.

Our next example is the orthogonal group

$$\mathrm{O}(n)=\{A\in \mathrm{Mat}_n(\mathbb{R})|\ A^TA=I\}.$$

To see that it is a Lie group, it suffices to show that $\mathrm{O}(n)$ is an embedded submanifold of $\mathrm{Mat}_n(\mathbb{R})$. In order to construct submanifold charts, we use the exponential map of matrices

$$\exp\colon\ \mathrm{Mat}_n(\mathbb{R})\to \mathrm{Mat}_n(\mathbb{R}),\quad B\mapsto \exp(B)=\sum_{n=0}^\infty \frac{1}{n!}B^n$$

(an absolutely convergent series). One has $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\exp(tB)=B$, hence the differential of exp at 0 is the identity $\mathrm{id}_{\mathrm{Mat}_n(\mathbb{R})}$. By the inverse function theorem, this means that there is $\epsilon>0$ such that exp restricts to a diffeomorphism from the open neighborhood $U=\{B:\ ||B||<\epsilon\}$ of 0 onto an open neighborhood $\exp(U)$ of I. Let

$$\mathfrak{o}(n)=\{B\in \mathrm{Mat}_n(\mathbb{R})|\ B+B^T=0\}.$$

We claim that

$$\exp(\mathfrak{o}(n) \cap U) = \mathrm{O}(n) \cap \exp(U),$$

so that exp gives a submanifold chart for $\mathrm{O}(n)$ over $\exp(U)$. To prove the claim, let $B \in U$. Then

$$\begin{aligned} \exp(B) \in \mathrm{O}(n) &\Leftrightarrow \exp(B)^T = \exp(B)^{-1} \\ &\Leftrightarrow \exp(B^T) = \exp(-B) \\ &\Leftrightarrow B^T = -B \\ &\Leftrightarrow B \in \mathfrak{o}(n). \end{aligned}$$

For a more general $A \in \mathrm{O}(n)$, we use that the map $\mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ given by left multiplication is a diffeomorphism. Hence, $A \exp(U)$ is an open neighborhood of A, and we have

$$A \exp(U) \cap \mathrm{O}(n) = A(\exp(U) \cap \mathrm{O}(n)) = A \exp(U \cap \mathfrak{o}(n)).$$

Thus, we also get a submanifold chart near A. This proves that $\mathrm{O}(n)$ is a submanifold. Hence its group operations are induced from those of $\mathrm{GL}(n, \mathbb{R})$, they are smooth. Hence $\mathrm{O}(n)$ is a Lie group. Notice that $\mathrm{O}(n)$ is compact (the column vectors of an orthogonal matrix are an orthonomal basis of $\mathbb{R}^n$; hence $\mathrm{O}(n)$ is a subset of $S^{n-1} \times \cdots S^{n-1} \subset \mathbb{R}^n \times \cdots \mathbb{R}^n$).

A similar argument shows that the *special linear group*

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathbb{R}) |\ \det(A) = 1\}$$

is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$, and hence is a Lie group. The submanifold charts are obtained by exponentiating the subspace

$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in \mathrm{Mat}_n(\mathbb{R}) |\ \mathrm{tr}(B) = 0\},$$

using the identity $\det(\exp(B)) = \exp(\mathrm{tr}(B))$.

Actually, we could have saved most of this work with $\mathrm{O}(n)$, $\mathrm{SL}(n, \mathbb{R})$ once we have the following beautiful result of E. Cartan:

> **Fact:** *Every closed subgroup of a Lie group is an embedded submanifold, hence is again a Lie group.*

We will prove this very soon, once we have developed some more basics of Lie group theory. A closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ (for suitable n) is called a *matrix Lie group*. Let us now give a few more examples of Lie groups, without detailed justifications.

Examples 1.3.
(a) Any finite-dimensional vector space V over $\mathbb{R}$ is a Lie group, with product Mult given by addition.
(b) Let $\mathcal{A}$ be a finite-dimensional associative algebra over $\mathbb{R}$, with unit $1_{\mathcal{A}}$. Then the group $\mathcal{A}^\times$ of invertible elements is a Lie group. More generally, if $n \in \mathbb{N}$ we can create the algebra $\mathrm{Mat}_n(\mathcal{A})$ of matrices with entries in $\mathcal{A}$, and the *general linear group*

$$\mathrm{GL}(n, \mathcal{A}) := \mathrm{Mat}_n(\mathcal{A})^\times$$

is a Lie group. If $\mathcal{A}$ is *commutative*, one has a determinant map $\det\colon \mathrm{Mat}_n(\mathcal{A}) \to \mathcal{A}$, and $\mathrm{GL}(n, \mathcal{A})$ is the pre-image of $\mathcal{A}^\times$. One may then define a *special linear group*

$$\mathrm{SL}(n, \mathcal{A}) = \{g \in \mathrm{GL}(n, \mathcal{A}) |\ \det(g) = 1\}.$$

(c) We mostly have in mind the cases $\mathcal{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Here $\mathbb{H}$ is the algebra of *quaternions* (due to Hamilton). Recall that $\mathbb{H} = \mathbb{R}^4$ as a vector space, with elements $(a, b, c, d) \in \mathbb{R}^4$ written as

$$x = a + ib + jc + kd$$

with imaginary units i, j, k. The algebra structure is determined by

$$i^2 = j^2 = k^2 = -1, \; ij = k, \; jk = i, \; ki = j.$$

Note that $\mathbb{H}$ is non-commutative (e.g. $ji = -ij$), hence $\mathrm{SL}(n, \mathbb{H})$ is *not* defined. On the other hand, one can define complex conjugates

$$\overline{x} = a - ib - jc - kd$$

and

$$|x|^2 := x\overline{x} = a^2 + b^2 + c^2 + d^2.$$

defines a norm $x \mapsto |x|$, with $|x_1 x_2| = |x_1||x_2|$ just as for complex or real numbers. The spaces $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ inherit norms, by putting

$$||x||^2 = \sum_{i=1}^{n} |x_i|^2, \quad x = (x_1, \ldots, x_n).$$

The subgroups of $\mathrm{GL}(n, \mathbb{R}),\ \mathrm{GL}(n, \mathbb{C}),\ \mathrm{GL}(n, \mathbb{H})$ preserving this norm (in the sense that $||Ax|| = ||x||$ for all x) are denoted

$$\mathrm{O}(n), \; \mathrm{U}(n), \; \mathrm{Sp}(n)$$

and are called the *orthogonal, unitary, and symplectic group*, respectively. Since the norms of $\mathbb{C}, \mathbb{H}$ coincide with those of $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} = \mathbb{C}^2 \cong \mathbb{R}^4$, we have

$$\mathrm{U}(n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n), \quad \mathrm{Sp}(n) = \mathrm{GL}(n, \mathbb{H}) \cap \mathrm{O}(4n).$$

In particular, all of these groups are compact. One can also define

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}), \quad \mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}),$$

these are called the *special orthogonal* and *special unitary* groups. The groups $\mathrm{SO}(n),\ \mathrm{SU}(n), \mathrm{Sp}(n)$ are often called the *classical groups* (but this term is used a bit loosely).

(d) For any Lie group G, its univeral cover $\widetilde{G}$ is again a Lie group. The universal cover $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is an example of a Lie group that is not isomorphic to a matrix Lie group.

1.2. **Lie algebras.**

Definition 1.4. A Lie algebra is a vector space $\mathfrak{g}$, together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying *anti-symmetry*

$$[\xi, \eta] = -[\eta, \xi] \text{ for all } \xi, \eta \in \mathfrak{g},$$

and the *Jacobi identity*,

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0 \text{ for all } \xi, \eta, \zeta \in \mathfrak{g}.$$

The map $[\cdot, \cdot]$ is called the Lie bracket. A morphism of Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is a linear map $\phi \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ preserving brackets.

The space

$$\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}_n(\mathbb{R})$$

is a Lie algebra, with bracket the commutator of matrices. (The notation indicates that we think of $\mathrm{Mat}_n(\mathbb{R})$ as a Lie algebra, not as an algebra.)

A Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, i.e. a subspace preserved under commutators, is called a *matrix Lie algebra.* For instance,

$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in \mathrm{Mat}_n(\mathbb{R}) \colon\ \mathrm{tr}(B) = 0\}$$

and

$$\mathfrak{o}(n) = \{B \in \mathrm{Mat}_n(\mathbb{R}) \colon\ B^T = -B\}$$

are matrix Lie algebras (as one easily verifies). It turns out that every finite-dimensional real Lie algebra is isomorphic to a matrix Lie algebra (*Ado's theorem*), but the proof is not easy.

The following examples of finite-dimensional Lie algebras correspond to our examples for Lie groups. The origin of this correspondence will soon become clear.

Examples 1.5. (a) Any vector space V is a Lie algeba for the zero bracket.

(b) Any associative algebra $\mathcal{A}$ can be viewed as a Lie algebra under commutator. Replacing $\mathcal{A}$ with matrix algebras over $\mathcal{A}$, it follows that $\mathfrak{gl}(n, \mathcal{A}) = \mathrm{Mat}_n(\mathcal{A})$, is a Lie algebra, with bracket the commutator. If $\mathcal{A}$ is commutative, then the subspace $\mathfrak{sl}(n, \mathcal{A}) \subset \mathfrak{gl}(n, \mathcal{A})$ of matrices of trace 0 is a Lie subalgebra.

(c) We are mainly interested in the cases $\mathcal{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Define an inner product on $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ by putting

$$\langle x, y \rangle = \sum_{i=1}^{n} \overline{x}_i y_i,$$

and define $\mathfrak{o}(n),\ \mathfrak{u}(n),\ \mathfrak{sp}(n)$ as the matrices in $\mathfrak{gl}(n, \mathbb{R}),\ \mathfrak{gl}(n, \mathbb{C}),\ \mathfrak{gl}(n, \mathbb{H})$ satisfying

$$\langle Bx, y \rangle = -\langle x, By \rangle$$

for all x, y. These are all Lie algebras called the (infinitesimal) orthogonal, unitary, and symplectic Lie algebras. For $\mathbb{R}, \mathbb{C}$ one can impose the additional condition $\mathrm{tr}(B) = 0$, thus defining the special orthogonal and special unitary Lie algebras $\mathfrak{so}(n),\ \mathfrak{su}(n)$. Actually,

$$\mathfrak{so}(n) = \mathfrak{o}(n)$$

sunce $B^T = -B$ already implies $\mathrm{tr}(B) = 0$.

Exercise 1.6. Show that $\mathrm{Sp}(n)$ can be characterized as follows. Let $J \in U(2n)$ be the unitary matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Then

$$\mathrm{Sp}(n) = \{A \in \mathrm{U}(2n) |\ \overline{A} = JAJ^{-1}\}.$$

Here $\overline{A}$ is the componentwise complex conjugate of A.

Exercise 1.7. Let $R(\theta)$ denote the 2×2 rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that for all $A \in \mathrm{SO}(2m)$ there exists $O \in \mathrm{SO}(2m)$ such that OAO^{-1} is of the block diagonal form

$$\begin{pmatrix} R(\theta_1) & 0 & 0 & \cdots & 0 \\ 0 & R(\theta_2) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & R(\theta_m) \end{pmatrix}.$$

For $A \in \mathrm{SO}(2m+1)$ one has a similar block diagonal presentation, with m 2×2 blocks $R(\theta_i)$ and an extra 1 in the lower right corner. Conclude that $\mathrm{SO}(n)$ is connected.

Exercise 1.8. Let G be a connected Lie group, and U an open neighborhood of the group unit e. Show that any $g \in G$ can be written as a product $g = g_1 \cdots g_N$ of elements $g_i \in U$.

Exercise 1.9. Let $\phi\colon G \to H$ be a morphism of connected Lie groups, and assume that the differential $\mathrm{d}_e\phi\colon T_eG \to T_eH$ is bijective (resp. surjective). Show that ϕ is a covering (resp. surjective). Hint: Use Exercise 1.8.

2. The covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$

The Lie group $\mathrm{SO}(3)$ consists of rotations in 3-dimensional space. Let $D \subset \mathbb{R}^3$ be the closed ball of radius π. Any element $x \in D$ represents a rotation by an angle $||x||$ in the direction of x. This is a 1-1 correspondence for points in the interior of D, but if $x \in \partial D$ is a boundary point then $x, -x$ represent the same rotation. Letting $\sim$ be the equivalence relation on D, given by antipodal identification on the boundary, we have $D^3/\sim = \mathbb{R}P(3)$. Thus

$$\mathrm{SO}(3) = \mathbb{R}P(3)$$

(at least, topogically). With a little extra effort (which we'll make below) one can make this into a diffeomorphism of manifolds.

By definition

$$\mathrm{SU}(2) = \{A \in \mathrm{Mat}_2(\mathbb{C}) |\ A^\dagger = A^{-1},\ \det(A) = 1\}.$$

Using the formula for the inverse matrix, we see that $\mathrm{SU}(2)$ consists of matrices of the form

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} |\ |w|^2 + |z|^2 = 1 \right\}.$$

That is, $\mathrm{SU}(2) = S^3$ as a manifold. Similarly,

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} it & -\overline{u} \\ u & -it \end{pmatrix} |\ t \in \mathbb{R},\ u \in \mathbb{C} \right\}$$

gives an identification $\mathfrak{su}(2) = \mathbb{R} \oplus \mathbb{C} = \mathbb{R}^3$. Note that for a matrix B of this form, $\det(B) = t^2 + |u|^2$, so that det corresponds to $|| \cdot ||^2$ under this identification.

The group $\mathrm{SU}(2)$ acts linearly on the vector space $\mathfrak{su}(2)$, by matrix conjugation: $B \mapsto ABA^{-1}$. Since the conjugation action preserves det, we obtain a linear action on $\mathbb{R}^3$, preserving the norm. This defines a Lie group morphism from $\mathrm{SU}(2)$ into $\mathrm{O}(3)$. Since $\mathrm{SU}(2)$ is connected, this must take values in the identity component:

$$\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3).$$

The kernel of this map consists of matrices $A \in \mathrm{SU}(2)$ such that $ABA^{-1} = B$ for all $B \in \mathfrak{su}(2)$. Thus, A commutes with all skew-adjoint matrices of trace 0. Since A commutes with multiples of the identity, it then commutes with all skew-adjoint matices. But since $\mathrm{Mat}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ (the decomposition into skew-adjoint and self-adjoint parts), it then follows that A is a multiple of the identity matrix. Thus $\ker(\phi) = \{I, -I\}$ is discrete. Since $\mathrm{d}_e\phi$ is an isomorphism, it follows that the map ϕ is a double covering. This exhibits $\mathrm{SU}(2) = S^3$ as the double cover of $\mathrm{SO}(3)$.

Exercise 2.1. Give an explicit construction of a double covering of $\mathrm{SO}(4)$ by $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Hint: Represent the quaternion algebra $\mathbb{H}$ as an algebra of matrices $\mathbb{H} \subset \mathrm{Mat}_2(\mathbb{C})$, by

$$x = a + ib + jc + kd \mapsto x = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}.$$

Note that $|x|^2 = \det(x)$, and that $\mathrm{SU}(2) = \{x \in \mathbb{H} |\ \det(x) = 1\}$. Use this to define an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{H}$ preserving the norm.

3. The Lie algebra of a Lie group

3.1. **Review: Tangent vectors and vector fields.** We begin with a quick reminder of some manifold theory, partly just to set up our notational conventions.

Let M be a manifold, and $C^\infty(M)$ its algebra of smooth real-valued functions. For $m \in M$, we define the tangent space T_mM to be the space of directional derivatives:

$$T_mM = \{v \in \mathrm{Hom}(C^\infty(M), \mathbb{R}) |\ v(fg) = v(f)g + v(g)f\}.$$

Here $v(f)$ is local, in the sense that $v(f) = v(f')$ if $f' - f$ vanishes on a neighborhood of m.

Example 3.1. If $\gamma\colon J \to M$, $J \subset \mathbb{R}$ is a smooth curve we obtain tangent vectors to the curve,

$$\dot\gamma(t) \in T_{\gamma(t)}M, \quad \dot\gamma(t)(f) = \frac{\partial}{\partial t}|_{t=0} f(\gamma(t)).$$

Example 3.2. We have $T_x\mathbb{R}^n = \mathbb{R}^n$, where the isomorphism takes $a \in \mathbb{R}^n$ to the corresponding velocity vector of the curve $x + ta$. That is,

$$v(f) = \frac{\partial}{\partial t}|_{t=0} f(x+ta) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

A smooth map of manifolds $\phi\colon M \to M'$ defines a *tangent map*:

$$\mathrm{d}_m\phi\colon T_mM \to T_{\phi(m)}M', \quad (\mathrm{d}_m\phi(v))(f) = v(f \circ \phi).$$

The locality property ensures that for an open neighborhood $U \subset M$, the inclusion identifies $T_mU = T_mM$. In particular, a coordinate chart $\phi\colon U \to \phi(U) \subset \mathbb{R}^n$ gives an isomorphism

$$\mathrm{d}_m\phi\colon T_mM = T_mU \to T_{\phi(m)}\phi(U) = T_{\phi(m)}\mathbb{R}^n = \mathbb{R}^n.$$

Hence T_mM is a vector space of dimension $n = \dim M$. The union $TM = \bigcup_{m\in M} T_mM$ is a vector bundle over M, called the tangent bundle. Coordinate charts for M give vector bundle charts for TM. For a smooth map of manifolds $\phi\colon M \to M'$, the entirety of all maps $\mathrm{d}_m\phi$ defines a smooth vector bundle map

$$\mathrm{d}\phi\colon TM \to TM'.$$

A *vector field* on M is a derivation $X\colon C^\infty(M) \to C^\infty(M)$. That is, it is a linear map satisfying

$$X(fg) = X(f)g + fX(g).$$

The space of vector fields is denoted $\mathfrak{X}(M) = \mathrm{Der}(C^\infty(M))$. Vector fields are local, in the sense that for any open subset U there is a well-defined restriction $X|_U \in \mathfrak{X}(U)$ such that $X|_U(f|_U) = (X(f))|_U$. For any vector field, one obtains tangent vectors $X_m \in T_mM$ by $X_m(f) = X(f)|_m$. One can think of a vector field as an assignment of tangent vectors, depending smoothly on m. More precisely, a vector field is a smooth section of the tangent bundle TM. In local coordinates, vector fields are of the form $\sum_i a_i \frac{\partial}{\partial x_i}$ where the a_i are smooth functions.

It is a general fact that the commutator of derivations of an algebra is again a derivation. Thus, $\mathfrak{X}(M)$ is a Lie algebra for the bracket

$$[X, Y] = X \circ Y - Y \circ X.$$

In general, smooth maps $\phi\colon M \to M'$ of manifolds do not induce maps of the Lie algebras of vector fields (unless ϕ is a diffeomorphism). One makes the following definition.

Definition 3.3. Let $\phi\colon M \to N$ be a smooth map. Vector fields X, Y on M, N are called ϕ-related, written $X \sim_\phi Y$, if

$$X(f \circ \phi) = Y(f) \circ \phi$$

for all $f \in C^\infty(M')$.

In short, $X \circ \phi^* = \phi^* \circ Y$ where $\phi^*\colon C^\infty(N) \to C^\infty(M),\ f \mapsto f \circ \phi$.

One has $X \sim_\phi Y$ if and only if $Y_{\phi(m)} = \mathrm{d}_m\phi(X_m)$. From the definitions, one checks

$$X_1 \sim_\phi Y_1,\ X_2 \sim_\phi Y_2 \ \Rightarrow\ [X_1, X_2] \sim_\phi [Y_1, Y_2].$$

Example 3.4. Let $j\colon S \hookrightarrow M$ be an embedded submanifold. We say that a vector field X is *tangent to* S if $X_m \in T_mS \subset T_mM$ for all $m \in S$. We claim that if two vector fields are tangent to S then so is their Lie bracket. That is, the vector fields on M that are tangent to S form a Lie subalgebra.

Indeed, the definition means that there exists a vector field $X_S \in \mathfrak{X}(S)$ such that $X_S \sim_j X$. Hence, if X, Y are tangent to S, then $[X_S, Y_S] \sim_j [X, Y]$, so $[X_S, Y_S]$ is tangent.

Similarly, the vector fields vanishing on S are a Lie subalgebra.

Let $X \in \mathfrak{X}(M)$. A curve $\gamma(t),\ t \in J \subset \mathbb{R}$ is called an *integral curve* of X if for all $t \in J$,

$$\dot\gamma(t) = X_{\gamma(t)}.$$

In local coordinates, this is an ODE $\frac{\mathrm{d}x_i}{\mathrm{d}t} = a_i(x(t))$. The existence and uniqueness theorem for ODE's (applied in coordinate charts, and then patching the local solutions) shows that for any $m \in M$, there is a unique maximal integral curve $\gamma(t),\ t \in J_m$ with $\gamma(0) = m$.

Definition 3.5. A vector field X is *complete* if for all $m \in M$, the maximal integral curve with $\gamma(0) = m$ is defined for all $t \in \mathbb{R}$.

In this case, one obtains a *smooth* map

$$\Phi\colon \mathbb{R} \times M \to M,\ (t, m) \mapsto \Phi_t(m)$$

such that $\gamma(t) = \Phi_{-t}(m)$ is the integral curve through m. The uniqueness property gives

$$\Phi_0 = \mathrm{Id},\quad \Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$$

i.e. $t \mapsto \Phi_t$ is a group homomorphism. Conversely, given such a group homomorphism such that the map Φ is smooth, one obtains a vector field X by setting

$$X = \frac{\partial}{\partial t}|_{t=0}\Phi^*_{-t},$$

as operators on functions. That is, $X(f)(m) = \frac{\partial}{\partial t}|_{t=0} f(\Phi_{-t}(m))$. [1]

The Lie bracket of vector fields measure the non-commutativity of their flows. In particular, if X, Y are complete vector fields, with flows $\Phi^X_t,\ \Phi^Y_s$, then $[X, Y] = 0$ if and only if

$$\Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t.$$

In this case, $X + Y$ is again a complete vector field with flow $\Phi^{X+Y}_t = \Phi^X_t \circ \Phi^Y_t$. (The right hand side defines a flow since the flows of X, Y commute, and the corresponding vector field is identified by taking a derivative at $t = 0$.)

3.2. **The Lie algebra of a Lie group.** Let G be a Lie group, and TG its tangent bundle. For all $a \in G$, the left,right translations

$$L_a\colon G \to G,\ g \mapsto ag$$

$$R_a\colon G \to G,\ g \mapsto ga$$

are smooth maps. Their differentials at e define isomorphisms $\mathrm{d}_g L_a\colon T_gG \to T_{ag}G$, and similarly for R_a. Let

$$\mathfrak{g} = T_eG$$

be the tangent space to the group unit.

A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if

$$X \sim_{L_a} X$$

for all $a \in G$, i.e. if it commutes with L^*_a. The space $\mathfrak{X}^L(G)$ of left-invariant vector fields is thus a Lie subalgebra of $\mathfrak{X}(G)$. Similarly the space of right-invariant vector fields $\mathfrak{X}^R(G)$ is a Lie subalgebra.

[1] The minus sign is convention, but it is motivated as follows. Let $\mathrm{Diff}(M)$ be the infinite-dimensional group of diffeomorphisms of M. It acts on $C^\infty(M)$ by $\Phi.f = f \circ \Phi^{-1} = (\Phi^{-1})^* f$. Here, the inverse is needed so that $\Phi_1.\Phi_2.f = (\Phi_1\Phi_2).f$. We think of vector fields as 'infinitesimal flows', i.e. informally as the tangent space at id to $\mathrm{Diff}(M)$. Hence, given a curve $t \mapsto \Phi_t$ through $\Phi_0 = \mathrm{id}$, smooth in the sense that the map $\mathbb{R} \times M \to M,\ (t, m) \mapsto \Phi_t(m)$ is smooth, we define the corresponding vector field $X = \frac{\partial}{\partial t}|_{t=0}\Phi_t$ in terms of the action on functions: as

$$X.f = \frac{\partial}{\partial t}|_{t=0}\Phi_t.f = \frac{\partial}{\partial t}|_{t=0}(\Phi_t^{-1})^* f.$$

If Φ_t is a flow, we have $\Phi_t^{-1} = \Phi_{-t}$.

Lemma 3.6. *The map*

$$\mathfrak{X}^L(G) \to \mathfrak{g},\ X \mapsto X_e$$

is an isomorphism of vector spaces. (Similarly for $\mathfrak{X}^R(G)$.)

Proof. For a left-invariant vector field, $X_a = (\mathrm{d}_e L_a)X_e$, hence the map is injective. To show that it is surjective, let $\xi \in \mathfrak{g}$, and put $X_a = (\mathrm{d}_e L_a)\xi \in T_aG$. We have to show that the map $G \to TG,\ a \mapsto X_a$ is smooth. It is the composition of the map $G \to G \times \mathfrak{g},\ g \mapsto (g, \xi)$ (which is obviously smooth) with the map $G \times \mathfrak{g} \to TG,\ (g, \xi) \mapsto \mathrm{d}_e L_g(\xi)$. The latter map is the restriction of $\mathrm{d}\,\mathrm{Mult}\colon TG \times TG \to TG$ to $G \times \mathfrak{g} \subset TG \times TG$, and hence is smooth. $\square$

We denote by $\xi^L \in \mathfrak{X}^L(G),\ \xi^R \in \mathfrak{X}^R(G)$ the left,right invariant vector fields defined by $\xi \in \mathfrak{g}$. Thus

$$\xi^L|_e = \xi^R|_e = \xi$$

Definition 3.7. The Lie algebra of a Lie group G is the vector space $\mathfrak{g} = T_eG$, equipped with the unique bracket such that

$$[\xi, \eta]^L = [\xi^L, \eta^L], \quad \xi \in \mathfrak{g}.$$

Remark 3.8. If you use the right-invariant vector fields to define the bracket on $\mathfrak{g}$, we get a minus sign. Indeed, note that $\mathrm{Inv}\colon G \to G$ takes left translations to right translations. Thus, ξ^R is Inv-related to some left invariant vector field. Since $\mathrm{d}_e \mathrm{Inv} = -\mathrm{Id}$, we see $\xi^R \sim_{\mathrm{Inv}} -\xi^L$. Consequently,

$$[\xi^R, \eta^R] \sim_{\mathrm{Inv}} [-\xi^L, -\eta^L] = [\xi, \eta]^L.$$

But also $-[\xi, \eta]^R \sim_{\mathrm{Inv}} [\xi, \eta]^L$, hence we get

$$[\xi^R, \zeta^R] = -[\xi, \zeta]^R.$$

The construction of a Lie algebra is compatible with morphisms. That is, we have a *functor* from Lie groups to finite-dimensional Lie algebras.

Theorem 3.9. *For any morphism of Lie groups $\phi\colon G \to G'$, the tangent map $d_e\phi\colon \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras. For all $\xi \in \mathfrak{g},\ \xi' = d_e\phi(\xi)$ one has*

$$\xi^L \sim_\phi (\xi')^L,\ \xi^R \sim_\phi (\xi')^R.$$

Proof. Suppose $\xi \in \mathfrak{g}$, and let $\xi' = \mathrm{d}_e\phi(\xi) \in \mathfrak{g}'$. The property $\phi(ab) = \phi(a)\phi(b)$ shows that $L_{\phi(a)} \circ \phi = \phi \circ L_a$. Taking the differential at e, and applying to ξ we find $(\mathrm{d}_e L_{\phi(a)})\xi' = (\mathrm{d}_a\phi)(\mathrm{d}_e L_a(\xi))$ hence $(\xi')^L_{\phi(a)} = (\mathrm{d}_a\phi)(\xi^L_a)$. That is $\xi^L \sim_\phi (\xi')^L$. The proof for right-invariant vector fields is similar. Since the Lie brackets of two pairs of ϕ-related vector fields are again ϕ-related, it follows that $\mathrm{d}_e\phi$ is a Lie algebra morphism. $\square$

Remark 3.10. Two special cases are worth pointing out.

(a) Let V be a finite-dimensional (real) vector space. A representation of a Lie group G on V is a Lie group morphism $G \to \mathrm{GL}(V)$. A representation of a Lie algebra $\mathfrak{g}$ on V is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$. The Theorem shows that the differential of any Lie group representation is a representation of its a Lie algebra.

(b) An *automorphism of a Lie group G* is a Lie group morphism $\phi\colon G \to G$ from G to itself, with ϕ a diffeomorphism. An *automorphism of a Lie algebra* is an invertible morphism from $\mathfrak{g}$ to itself. By the Theorem, the differential of any Lie group automorphism is an automorphism of its Lie algebra. As an example, $\mathrm{SU}(n)$ has a Lie group automorphism given by complex conjugation of matrices; its differential is a Lie algebra automorphism of $\mathfrak{su}(n)$ given again by complex conjugation.

Exercise 3.11. Let $\phi\colon G \to G$ be a Lie group automorphism. Show that its fixed point set is a closed subgroup of G, hence a Lie subgroup. Similarly for Lie algebra automorphisms. What is the fixed point set for the complex conjugation automorphism of $\mathrm{SU}(n)$?

4. The exponential map

Theorem 4.1. *The left-invariant vector fields ξ^L are complete, i.e. they define a flow Φ_t^ξ such that*

$$\xi^L = \frac{\partial}{\partial t}|_{t=0}(\Phi^\xi_{-t})^*.$$

Letting $\phi^\xi(t)$ denote the unique integral curve with $\phi^\xi(0) = e$. It has the property

$$\phi^\xi(t_1 + t_2) = \phi^\xi(t_1)\phi^\xi(t_2),$$

and the flow of ξ^L is given by right translations:

$$\Phi_t^\xi(g) = g\phi^\xi(-t).$$

Similarly, the right-invariant vector fields ξ^R are complete. $\phi^\xi(t)$ is an integral curve for ξ^R as well, and the flow of ξ^R is given by left translations, $g \mapsto \phi^\xi(-t)g$.

Proof. If $\gamma(t),\ t \in J \subset \mathbb{R}$ is an integral curve of a left-invariant vector field ξ^L, then its left translates $a\gamma(t)$ are again integral curves. In particular, for $t_0 \in J$ the curve $t \mapsto \gamma(t_0)\gamma(t)$ is again an integral curve. Hence it coincides with $\gamma(t_0 + t)$ for all $t \in J \cap (J - t_0)$. In this way, an integral curve defined for small $|t|$ can be extended to an integral curve for all t, i.e. ξ^L is complete.

Since ξ^L is left-invariant, so is its flow Φ_t^ξ. Hence

$$\Phi_t^\xi(g) = \Phi_t^\xi \circ L_g(e) = L_g \circ \Phi_t^\xi(e) = g\Phi_t^\xi(e) = g\phi^\xi(-t).$$

The property $\Phi^\xi_{t_1+t_2} = \Phi^\xi_{t_1}\Phi^\xi_{t_2}$ shows that $\phi^\xi(t_1+t_2) = \phi^\xi(t_1)\phi^\xi(t_2)$. Finally, since $\xi^L \sim_{\mathrm{Inv}} -\xi^R$, the image

$$\mathrm{Inv}(\phi^\xi(t)) = \phi^\xi(t)^{-1} = \phi^\xi(-t)$$

is an integral curve of $-\xi^R$. Equivalently, $\phi^\xi(t)$ is an integral curve of ξ^R. □

Since left and right translations commute, it follows in particular that

$$[\xi^L, \eta^R] = 0.$$

Definition 4.2. A 1-parameter subgroup of G is a group homomorphism $\phi\colon \mathbb{R} \to G$.

We have seen that every $\xi \in \mathfrak{g}$ defines a 1-parameter group, by taking the integral curve through e of the left-invariant vector field ξ^L. Every 1-parameter group arises in this way:

Proposition 4.3. *If ϕ is a 1-parameter subgroup of G, then $\phi = \phi^\xi$ where $\xi = \dot\phi(0)$. One has*
$$\phi^{s\xi}(t) = \phi^\xi(st).$$
The map
$$\mathbb{R} \times \mathfrak{g} \to G,\ (t, \xi) \mapsto \phi^\xi(t)$$
is smooth.

Proof. Let $\phi(t)$ be a 1-parameter group. Then $\Phi_t(g) := g\phi(-t)$ defines a flow. Since this flow commutes with left translations, it is the flow of a left-invariant vector field, ξ^L. Here ξ is determined by taking the derivative of $\Phi_{-t}(e) = \phi(t)$ at $t = 0$: $\xi = \dot\phi(0)$. This shows $\phi = \phi^\xi$. As an application, since $\psi(t) = \phi^\xi(st)$ is a 1-parameter group with $\dot\psi^\xi(0) = s\dot\phi^\xi(0) = s\xi$, we have $\phi^\xi(st) = \phi^{s\xi}(t)$. Smoothness of the map $(t, \xi) \mapsto \phi^\xi(t)$ follows from the smooth dependence of solutions of ODE's on parameters. $\square$

Definition 4.4. The *exponential map* for the Lie group G is the smooth map defined by
$$\exp\colon \mathfrak{g} \to G,\ \xi \mapsto \phi^\xi(1),$$
where $\phi^\xi(t)$ is the 1-parameter subgroup with $\dot\phi^\xi(0) = \xi$.

Proposition 4.5. *We have*
$$\phi^\xi(t) = \exp(t\xi).$$
If $[\xi, \eta] = 0$ then
$$\exp(\xi + \eta) = \exp(\xi)\exp(\eta).$$

Proof. By the previous Proposition, $\phi^\xi(t) = \phi^{t\xi}(1) = \exp(t\xi)$. For the second claim, note that $[\xi, \eta] = 0$ implies that ξ^L, η^L commute. Hence their flows Φ_t^ξ, Φ_t^η, and $\Phi_t^\xi \circ \Phi_t^\eta$ is the flow of $\xi^L + \eta^L$. Hence it coincides with $\Phi_t^{\xi+\eta}$. Applying to e, we get $\phi^\xi(t)\phi^\eta(t) = \phi^{\xi+\eta}(t)$. Now put $t = 1$. $\square$

In terms of the exponential map, we may now write the flow of ξ^L as $\Phi_t^\xi(g) = g\exp(-t\xi)$, and similarly for the flow of ξ^R. That is,
$$\xi^L = \frac{\partial}{\partial t}|_{t=0} R^*_{\exp(t\xi)},\quad \xi^R = \frac{\partial}{\partial t}|_{t=0} L^*_{\exp(t\xi)}.$$

Proposition 4.6. *The exponential map is natural with respect to Lie group homomorphisms $\phi\colon G \to H$. That is,*
$$\phi(\exp(\xi)) = \exp((d_e\phi)(\xi)),\quad \xi \in \mathfrak{g}.$$

Proof. $t \mapsto \phi(\exp(t\xi))$ is a 1-parameter subgroup of H, with differential at e given by
$$\frac{d}{dt}\Big|_{t=0} \phi(\exp(t\xi)) = \mathrm{d}_e\phi(\xi).$$
Hence $\phi(\exp(t\xi)) = \exp(t\mathrm{d}_e\phi(\xi))$. Now put $t = 1$. $\square$

Proposition 4.7. *Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a matrix Lie group, and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ its Lie algebra. Then $\exp\colon \mathfrak{g} \to G$ is just the exponential map for matrices,*
$$\exp(\xi) = \sum_{n=0}^\infty \frac{1}{n!}\xi^n.$$
Furthermore, the Lie bracket on $\mathfrak{g}$ is just the commutator of matrices.

Proof. By the previous Proposition, applied to the inclusion of G in $\mathrm{GL}(n,\mathbb{R})$, the exponential map for G is just the restriction of that for $\mathrm{GL}(n,\mathbb{R})$. Hence it suffices to prove the claim for $G = \mathrm{GL}(n,\mathbb{R})$. The function $\sum_{n=0}^{\infty} \frac{t^n}{n!}\xi^n$ is a 1-parameter group in $\mathrm{GL}(n,\mathbb{R})$, with derivative at 0 equal to $\xi \in \mathfrak{gl}(n,\mathbb{R})$. Hence it coincides with $\exp(t\xi)$. Now put $t = 1$. □

Proposition 4.8. *For a matrix Lie group $G \subset \mathrm{GL}(n,\mathbb{R})$, the Lie bracket on $\mathfrak{g} = T_I G$ is just the commutator of matrices.*

Proof. It suffices to prove for $G = \mathrm{GL}(n,\mathbb{R})$. Using $\xi^L = \frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(t\xi)}$ we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}(R^*_{\exp(-t\xi)}R^*_{\exp(-s\eta)}R^*_{\exp(t\xi)}R^*_{\exp(s\eta)})\\
&= \frac{\partial}{\partial s}\Big|_{s=0}(R^*_{\exp(-s\eta)}\xi^L R^*_{\exp(s\eta)} - \xi^L)\\
&= \xi^L\eta^L - \eta^L\xi^L\\
&= [\xi,\eta]^L.
\end{aligned}$$

On the other hand, write

$$R^*_{\exp(-t\xi)}R^*_{\exp(-s\eta)}R^*_{\exp(t\xi)}R^*_{\exp(s\eta)} = R^*_{\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta)}.$$

Since the Lie group exponential map for $\mathrm{GL}(n,\mathbb{R})$ coincides with the exponential map for matrices, we may use Taylor's expansion,

$$\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta) = I + st(\xi\eta - \eta\xi) + \ldots = \exp(st(\xi\eta-\eta\xi)) + \ldots$$

where $\ldots$ denotes terms that are cubic or higher in s, t. Hence

$$R^*_{\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta)} = R^*_{\exp(st(\xi\eta-\eta\xi)} + \ldots$$

and consequently

$$\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}R^*_{\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta)} = \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}R^*_{\exp(st(\xi\eta-\eta\xi))} = (\xi\eta-\eta\xi)^L.$$

We conclude that $[\xi,\eta] = \xi\eta - \eta\xi$. □

Remark 4.9. Had we defined the Lie algebra using right-invariant vector fields, we would have obtained *minus* the commutator of matrices. Nonetheless, some authors use that convention.

The exponential map gives local coordinates for the group G on a neighborhood of e:

Proposition 4.10. *The differential of the exponential map at the origin is $d_0 \exp = \mathrm{id}$. As a consequence, there is an open neighborhood U of $0 \in \mathfrak{g}$ such that the exponential map restricts to a diffeomorphism $U \to \exp(U)$.*

Proof. Let $\gamma(t) = t\xi$. Then $\dot\gamma(0) = \xi$ since $\exp(\gamma(t)) = \exp(t\xi)$ is the 1-parameter group, we have

$$(\mathrm{d}_0 \exp)(\xi) = \frac{\partial}{\partial t}|_{t=0}\exp(t\xi) = \xi.$$

□

Exercise 4.11. Show hat the exponential map for $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ $\mathrm{U}(n)$ are surjective. (We will soon see that the exponential map for any compact, connected Lie group is surjective.)

Exercise 4.12. A matrix Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$ is called *unipotent* if for all $A \in G$, the matrix $A - I$ is nilpotent (i.e. $(A-I)^r = 0$ for some r). The prototype of such a group are the upper triangular matrices with 1's down the diagonal. Show that for a connected unipotent matrix Lie group, the exponential map is a diffeomorphism.

Exercise 4.13. Show that $\exp\colon \mathfrak{gl}(2, \mathbb{C}) \to \mathrm{GL}(2, \mathbb{C})$ is surjective. More generally, show that the exponential map for $\mathrm{GL}(n, \mathbb{C})$ is surjective. (Hint: First conjugate the given matrix into Jordan normal form).

Exercise 4.14. Show that $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective, by proving that the matrices $\begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ are not in the image. (Hint: Assuming these matrices are of the form $\exp(B)$, what would the eigenvalues of B have to be?) Show that these two matrices represent *all* conjugacy classes of elements that are not in the image of exp. (Hint: Find a classification of the conjugacy classes of $\mathrm{SL}(2, \mathbb{R})$, e.g. in terms of eigenvalues.)

5. Cartan's theorem on closed subgroups

Using the exponential map, we are now in position to prove Cartan's theorem on closed subgroups.

Theorem 5.1. *Let H be a closed subgroup of a Lie group G. Then H is an embedded submanifold, and hence is a Lie subgroup.*

We first need a Lemma. Let V be a Euclidean vector space, and $S(V)$ its unit sphere. For $v \in V \backslash \{0\}$, let $[v] = \frac{v}{||v||} \in S(V)$.

Lemma 5.2. *Let $v_n, v \in V \backslash \{0\}$ with $\lim_{n\to\infty} v_n = 0$. Then*

$$\lim_{n\to\infty} [v_n] = [v] \Leftrightarrow \exists a_n \in \mathbb{N}\colon \lim_{n\to\infty} a_n v_n = v.$$

Proof. The implication $\Leftarrow$ is obvious. For the opposite direction, suppose $\lim_{n\to\infty}[v_n] = [v]$. Let $a_n \in \mathbb{N}$ be defined by $a_n - 1 < \frac{||v||}{||v_n||} \leq a_n$. Since $v_n \to 0$, we have $\lim_{n\to\infty} a_n \frac{||v_n||}{||v||} = 1$, and

$$a_n v_n = \left(a_n \frac{||v_n||}{||v||} \right) [v_n] \, ||v|| \to [v] \, ||v|| = v. \qquad \square$$

Proof of E. Cartan's theorem. It suffices to construct a submanifold chart near $e \in H$. (By left translation, one then obtains submanifold charts near arbitrary $a \in H$.) Choose an inner product on $\mathfrak{g}$.

We begin with a candidate for the Lie algebra of H. Let $W \subset \mathfrak{g}$ be the subset such that $\xi \in W$ if and only if either $\xi = 0$, or $\xi \neq 0$ and there exists $\xi_n \neq 0$ with

$$\exp(\xi_n) \in H, \quad \xi_n \to 0, \quad [\xi_n] \to [\xi].$$

We will now show the following:

(i) $\exp(W) \subset H$,
(ii) W is a subspace of $\mathfrak{g}$,

(iii) There is an open neighborhood U of 0 and a diffeomorphism $\phi\colon U \to \phi(U) \subset G$ with $\phi(0) = e$ such that

$$\phi(U \cap W) = \phi(U) \cap H.$$

(Thus ϕ defines a submanifold chart near e.)

Step (i). Let $\xi \in W\backslash\{0\}$, with sequence ξ_n as in the definition of W. By the Lemma, there are $a_n \in \mathbb{N}$ with $a_n\xi_n \to \xi$. Since $\exp(a_n\xi_n) = \exp(\xi_n)^{a_n} \in H$, and H is closed, it follows that

$$\exp(\xi) = \lim_{n\to\infty} \exp(a_n\xi_n) \in H.$$

Step (ii). Since the subset W is invariant under scalar multiplication, we just have to show that it is closed under addition. Suppose $\xi, \eta \in W$. To show that $\xi + \eta \in W$, we may assume that $\xi, \eta, \xi + \eta$ are all non-zero. For t sufficiently small, we have

$$\exp(t\xi)\exp(t\eta) = \exp(u(t))$$

for some smooth curve $t \mapsto u(t) \in \mathfrak{g}$ with $u(0) = 0$. Then $\exp(u(t)) \in H$ and

$$\lim_{n\to\infty} n\, u(\frac{1}{n}) = \lim_{h\to 0} \frac{u(h)}{h} = \dot{u}(0) = \xi + \eta.$$

hence $u(\frac{1}{n}) \to 0,\ \exp(u(\frac{1}{n}) \in H,\ [u(\frac{1}{n})] \to [\xi + \eta]$. This shows $[\xi + \eta] \in W$, proving (ii).

Step (iii). Let W' be a complement to W in $\mathfrak{g}$, and define

$$\phi\colon \mathfrak{g} \cong W \oplus W' \to G, \quad \phi(\xi + \xi') = \exp(\xi)\exp(\xi').$$

Since $\mathrm{d}_0\phi$ is the identity, there is an open neighborhood $U \subset \mathfrak{g}$ of 0 such that $\phi\colon U \to \phi(U)$ is a diffeomorphism. It is automatic that $\phi(W \cap U) \subset \phi(W) \cap \phi(U) \subset H \cap \phi(U)$. We want to show that we can take U sufficiently small so that we also have the opposite inclusion

$$H \cap \phi(U) \subset \phi(W \cap U).$$

Suppose not. Then, any neighborhood $U_n \subset \mathfrak{g} = W \oplus W'$ of 0 contains an element (η_n, η_n') such that

$$\phi(\eta_n, \eta_n') = \exp(\eta_n)\exp(\eta_n') \in H$$

(i.e. $\exp(\eta_n') \in H$) but $(\eta_n, \eta_n') \notin W$ (i.e. $\eta_n' \neq 0$). Thus, taking U_n to be a nested sequence of neighborhoods with intersection $\{0\}$, we could construct a sequence $\eta_n' \in W' - \{0\}$ with $\eta_n' \to 0$ and $\exp(\eta_n') \in H$. Passing to a subsequence we may assume that $[\eta_n'] \to [\eta]$ for some $\eta \in W'\backslash\{0\}$. On the other hand, such a convergence would mean $\eta \in W$, by definition of W. Contradiction. $\square$

As remarked earlier, Cartan's theorem is very useful in practice. For a given Lie group G, the term 'closed subgroup' is often used as synonymous to 'embedded Lie subgroup'.

Examples 5.3. (a) The matrix groups $G = \mathrm{O}(n), \mathrm{Sp}(n), \mathrm{SL}(n, \mathbb{R}), \ldots$ are all closed subgroups of some $\mathrm{GL}(N, \mathbb{R})$, and hence are Lie groups.

(b) Suppose that $\phi\colon G \to H$ is a morphism of Lie groups. Then $\ker(\phi) = \phi^{-1}(e) \subset G$ is a closed subgroup. Hence it is an embedded Lie subgroup of G.

(c) The center $Z(G)$ of a Lie group G is the set of all $a \in G$ such that $ag = ga$ for all $a \in G$. It is a closed subgroup, and hence an embedded Lie subgroup.

(d) The group of automorphisms of a Lie algebra $\mathfrak{g}$ is closed in the group $\mathrm{End}(\mathfrak{g})^\times$ of vector space automorphisms, hence it is a Lie group.

6. The adjoint representations

6.1. **The adjoint representation of G.** Recall that an automorphism of a Lie group G is an invertible morphism from G to itself. The automorphisms form a group $\mathrm{Aut}(G)$. Any $a \in G$ defines an 'inner' automorphism $\mathrm{Ad}_a \in \mathrm{Aut}(G)$ by conjugation:

$$\mathrm{Ad}_a(g) = aga^{-1}$$

Indeed, Ad_a is an automorphism since $\mathrm{Ad}_a^{-1} = \mathrm{Ad}_{a^{-1}}$ and

$$\mathrm{Ad}_a(g_1 g_2) = a g_1 g_2 a^{-1} = a g_1 a^{-1} a g_2 a^{-1} = \mathrm{Ad}_a(g_1)\,\mathrm{Ad}_a(g_2).$$

Note also that $\mathrm{Ad}_{a_1 a_2} = \mathrm{Ad}_{a_1} \mathrm{Ad}_{a_2}$, thus Ad defines a group morphism $G \to \mathrm{Aut}(G)$ into the group of automorphisms,

Its differential at the identity is a G-representation $G \to \mathrm{Aut}(\mathfrak{g})$ by automorphisms of the Lie algebra $\mathfrak{g}$. This is the *adjoint representation of* G, and it is common to denote it by the same symbol $\mathrm{Ad}_a := \mathrm{d}_e \mathrm{Ad}_a$:

$$\mathrm{Ad}_a \colon \mathfrak{g} \to \mathfrak{g},\ \xi \mapsto \mathrm{Ad}_a \xi.$$

Since the Ad_a are Lie algebra/group morphisms, they are compatible with the exponential map,

$$\exp(\mathrm{Ad}_a \xi) = \mathrm{Ad}_a \exp(\xi).$$

Remark 6.1. If $G \subset \mathrm{GL}(n, \mathbb{R})$ is a matrix Lie group, then $\mathrm{Ad}_a \in \mathrm{Aut}(\mathfrak{g})$ is the conjugation of matrices

$$\mathrm{Ad}_a(\xi) = a\xi a^{-1}.$$

This follows by taking the derivative of $\mathrm{Ad}_a(\exp(t\xi)) = a \exp(t\xi) a^{-1}$, using that exp is just the exponential series for matrices.

6.2. **The adjoint representation of $\mathfrak{g}$.** A derivation of a Lie algebra $\mathfrak{g}$ is an linear map $D \in \mathrm{End}(\mathfrak{g})$ such that $D[\xi, \zeta] = [D\xi, \eta] + [\xi, D\eta]$. Derivations of $\mathfrak{g}$ form a Lie algebra under commutator. For instance, Lie bracket $[\xi, \cdot]$ with a given element of $\mathfrak{g}$ is a derivation (by Jacobi's identity); derivations of this type are called *inner*.

For any Lie algebra $\mathfrak{g}$, one defines the adjoint representation

$$\mathrm{ad} \colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$$

by

$$\mathrm{ad}_\xi = [\xi, \cdot].$$

The fact that this is a representation is again a consequence of the Jacobi identity.

Suppose now that G is a Lie group, with Lie algebra $\mathfrak{g}$. Recall that the differential of any G-representation is a $\mathfrak{g}$-representation.

Theorem 6.2. *If $\mathfrak{g}$ is the Lie algebra of G, then the adjoint representation* ad *of $\mathfrak{g}$ is the differential of the adjoint representation of G. One has the equality of operators*

$$\exp(\mathrm{ad}_\xi) = \mathrm{Ad}(\exp \xi)$$

for all $\xi \in \mathfrak{g}$.

Proof. We have $\exp(s \operatorname{Ad}_{\exp(t\xi)} \eta) = \operatorname{Ad}_{\exp(t\xi)} \exp(s\eta) = \exp(t\xi)\exp(s\eta)\exp(-t\xi)$. Hence

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} (\operatorname{Ad}_{\exp(t\xi)} \eta)^L &= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} R^*_{\exp(s \operatorname{Ad}_{\exp(t\xi)} \eta)} \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} R^*_{\exp(t\xi)\exp(s\eta)\exp(-t\xi)} \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} R^*_{\exp(t\xi)} R^*_{\exp(s\eta)} R^*_{\exp(-t\xi)} \\
&= \frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(t\xi)}\ \eta^L\ R^*_{\exp(-t\xi)} \\
&= [\xi^L, \eta^L] \\
&= [\xi, \eta]^L = (\operatorname{ad}_\xi \eta)^L,
\end{aligned}$$

proving $\frac{\partial}{\partial t}\Big|_{t=0} \operatorname{Ad}_{\exp(t\xi)} \eta = \operatorname{ad}_\xi \eta$. The last part follows, since the exponential map is functorial with respect to Lie group morphisms (in this case $\operatorname{Ad}\colon G \to \operatorname{End}(\mathfrak{g})^\times$). $\square$

Remark 6.3. As a special case, this formula holds for matrices. That is, for $B, C \in \operatorname{Mat}_n(\mathbb{R})$,

$$e^B\, C\, e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} [B, [B, \cdots [B, C] \cdots]].$$

The formula also holds in some other contexts, e.g. if B, C are elements of an algebra with B nilpotent (i.e. $B^N = 0$ for some N). In this case, both the exponential series for e^B and the series on the right hand side are finite. (Indeed, $[B, [B, \cdots [B, C] \cdots]]$ with n B's is a sum of terms $B^j C B^{n-j}$, and hence must vanish if $n \geq 2N$.)