LIE GROUPOIDS AND LIE ALGEBROIDS
LECTURE NOTES, FALL 2017

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Abstract. These notes are under construction. They contain errors and omissions, and **the references are very incomplete**. Apologies!

Contents

1. Lie groupoids

Symmetries in mathematics, as well as in nature, are often defined to be invariance properties under actions of groups. Lie groupoids are given by a manifold M of 'objects' together with a type of symmetry of M that is more general than those provided by group actions. For example, a foliation of M provides an example of such a generalized symmetry, but foliations need not be obtained from group actions in any obvious way.

1.1. **Definitions.** The groupoid will assign to any two objects $m_0,\ m_1 \in M$ a collection (possibly empty) of arrows from m_1 to m_0. These arrows are thought of as 'symmetries', but in contrast to Lie group actions this symmetry need not be defined for all $m \in M$ – only pointwise. On the other hand, we require that the collection of all such arrows (with arbitrary end points) fit together smoothly to define a manifold, and that arrows can be composed provided the end point (target) of one arrow is the starting point (source) of the next.

The formal definition of a *Lie groupoid* $\mathcal{G} \rightrightarrows M$ involves a manifold $\mathcal{G}$ of *arrows*, a submanifold $i\colon M \hookrightarrow \mathcal{G}$ of *units* (or *objects*), and two surjective submersions $\mathsf{s}, \mathsf{t}\colon \mathcal{G} \to M$ called *source* and *target* such that

$$\mathsf{t} \circ i = \mathsf{s} \circ i = \mathrm{id}_M\,.$$

One thinks of g as an arrow from its source $\mathsf{s}(g)$ to its target $\mathsf{t}(g)$, with M embedded as trivial arrows.

$$\begin{tikzcd} \mathsf{t}(g) & & \mathsf{s}(g) \arrow[ll, bend right, "g"'] \end{tikzcd} \tag{1}$$

Using that s, t are submersion, one finds (cf. Exercise 1.1 below) that for all $k = 1, 2, \ldots$ the set of k-*arrows*

$$\mathcal{G}^{(k)} = \{(g_1, \ldots, g_k) \in \mathcal{G}^k |\ \mathsf{s}(g_i) = \mathsf{t}(g_{i+1})\}$$

$$\begin{tikzcd} m_0 & m_1 \arrow[l, bend right, "g_1"'] & m_2 \arrow[l, bend right, "g_2"'] & \cdots \arrow[l, bend right, "g_3"'] & \cdots & m_{k-1} \arrow[l, bend right, "g_{k-1}"'] & m_k \arrow[l, bend right, "g_k"'] \end{tikzcd} \tag{2}$$

is a smooth submanifold of $\mathcal{G}^k$, and the two maps $\mathcal{G}^{(k)} \to M$ taking $(g_1, \ldots, g_k)$ to $\mathsf{s}(g_k)$, respectively to $\mathsf{t}(g_1)$, are submersions. For $k = 0$ one puts $\mathcal{G}^{(0)} = M$.

The definition of a Lie groupoid also involves a smooth multiplication map, defined on composable arrows (i.e., 2-arrows)

$$\mathrm{Mult}_{\mathcal{G}}\colon \mathcal{G}^{(2)} \to \mathcal{G}, \quad (g_1, g_2) \mapsto g_1 \circ g_2,$$

such that $\mathsf{s}(g_1 \circ g_2) = \mathsf{s}(g_2),\ \ \mathsf{t}(g_1 \circ g_2) = \mathsf{t}(g_1)$. It is thought of as a concatenation of arrows. Note that when picturing this composition rule, it is best to draw arrows from the right to the left.

$$\begin{tikzcd} m_0 & m_1 \arrow[l, bend right, "g_1"'] & m_2 \arrow[l, bend right, "g_2"'] & m_0 & & m_2 \arrow[ll, bend right, "g_1 \circ g_2"'] \end{tikzcd} \tag{3}$$

Definition 1.1. The above data define a *Lie groupoid* $\mathcal{G} \rightrightarrows M$ if the following axioms are satisfied:

1. **Associativity:** $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ for all $(g_1, g_2, g_3) \in \mathcal{G}^{(3)}$.
2. **Units:** $\mathsf{t}(g) \circ g = g = g \circ \mathsf{s}(g)$ for all $g \in \mathcal{G}$.
3. **Inverses:** For all $g \in \mathcal{G}$ there exists $h \in \mathcal{G}$ such that $\mathsf{s}(h) = \mathsf{t}(g),\ \mathsf{t}(h) = \mathsf{s}(g)$, and such that $g \circ h,\ h \circ g$ are units.

The inverse of an element is necessarily unique (cf. Exercise). Denoting this element by g^{-1}, we have that $g \circ g^{-1} = \mathsf{t}(g), \quad g^{-1} \circ g = \mathsf{s}(g)$. Inversion is pictured as reversing the direction of arrows.

From now on, when we write $g = g_1 \circ g_2$ we implicitly assume that g_1, g_2 are composable, i.e. $\mathsf{s}(g_1) = \mathsf{t}(g_2)$. Let

$$\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}}) = \{(g, g_1, g_2) \in \mathcal{G}^3 |\ g = g_1 \circ g_2\}.$$

be the graph of the multiplication map; we will think of $\mathrm{Mult}_{\mathcal{G}}$ as a smooth relation from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$.

Remark 1.2. A groupoid structure on a manifold $\mathcal{G}$ is completely determined by $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$, i.e. by declaring when $g = g_1 \circ g_2$. Indeed, the units are the elements $m \in \mathcal{G}$ such that $m = m \circ m$. Given $g \in \mathcal{G}$, the source $\mathsf{s}(g)$ and target $\mathsf{t}(g)$ are the unique units for which $g = g \circ \mathsf{s}(g) = \mathsf{t}(g) \circ g$. The inverse of g is the unique element g^{-1} such that $g \circ g^{-1}$ is a unit.

Remark 1.3. In the definition above, our manifolds are always assumed to satisfy the Hausdorff separation axiom. For a (possibly) *non-Hausdorff* Lie groupoid, we allow the space $\mathcal{G}$ to be a non-Hausdorff manifold, but still require that the fibers of the source and target maps, as well as the units M, are Hausdorff. [1] Non-Hausdorff Lie groupoids are very common in the theory of foliations; see below.

Remark 1.4. One may similarly consider 'set-theoretic' groupoids $\mathcal{G} \rightrightarrows M$, by taking s, t, and $\mathrm{Mult}_{\mathcal{G}}$ to be set maps (with s, t surjective). Such a set-theoretic groupoid is the same as a category for which the objects M and arrows $\mathcal{G}$ are sets, and with the property that every arrow is invertible.

Remark 1.5. Let $\mathrm{Inv}_{\mathcal{G}} \colon \mathcal{G} \to \mathcal{G}, \quad g \mapsto g^{-1}$ be the inversion map. As an application of the implicit function theorem, it is automatic that $\mathrm{Inv}_{\mathcal{G}}$ is a diffeomorphism.

Definition 1.6. A *morphism of Lie groupoids* $F \colon \mathcal{H} \to \mathcal{G}$ is a smooth map such that

$$F(h_1 \circ h_2) = F(h_1) \circ F(h_2)$$

for all $(h_1, h_2) \in \mathcal{H}^{(2)}$. If F is an inclusion as a submanifold, we say that $\mathcal{H}$ is a Lie subgroupoid of $\mathcal{G}$.

[1] One of the consequences of the Hausdorff property is the uniqueness of flows of vector fields. But in the theory to be developed below, the vector fields that we integrate are all tangent to source fibers, the target fibers, or the units.

By Remark 1.2, it is automatic that such a morphism takes units of $\mathcal{H}$ to units of $\mathcal{G}$, and that it intertwines the source, target, and inversion maps. We will often present Lie groupoid homomorphisms by diagrams, as follows:

$$\begin{tikzcd} \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M \arrow[d] \\ \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & N \end{tikzcd}$$

If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, and $m \in M$, the intersection of the source and target fibers

$$\mathcal{G}_m = \mathsf{t}^{-1}(m) \cap \mathsf{s}^{-1}(m)$$

is a Lie group, with group structure induced by the groupoid multiplication. (We will prove later that it is a submanifold.) It is called the *isotropy group of $\mathcal{G}$ at m.*

1.2. **Examples.**

Example 1.7 (Lie groups). A Lie group G is the same as a Lie groupoid with a unique unit, $G \rightrightarrows \mathrm{pt}$. For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the inclusion of isotropy groups define Lie subgroupoids

$$\begin{tikzcd} \mathcal{G}_m \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \{m\} \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

for all $m \in M$.

Example 1.8 (Manifolds). At the opposite extreme, every manifold M can be regarded as a trivial Lie groupoid $M \rightrightarrows M$ where all elements are units. The groupoid multiplication is trivial: One has that $m = m_1 \circ m_2$ if and only of $m = m_1 = m_2$. Given any Lie groupoid $\mathcal{G} \rightrightarrows M$, the units of M define a Lie subgroupoid

$$\begin{tikzcd} M \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Example 1.9 (Pair groupoid). For any manifold M, one has the *pair groupoid*

$$\mathrm{Pair}(M) = M \times M \rightrightarrows M,$$

with a unique arrow between any two points m', m (labeled by the pair itself). The composition is necessarily

$$(m', m) = (m'_1, m_1) \circ (m'_2, m_2) \quad \Leftrightarrow \quad m'_1 = m,\ m_1 = m'_2,\ m_2 = m.$$

The units are given by the diagonal embedding $M \hookrightarrow M \times M$, and the source and target of (m', m) are m and m', respectively. Note that the isotropy groups $\mathrm{Pair}(M)_m$ of the pair groupoid are trivial.

For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the target and source map combine into a Lie groupoid morphism

$$\begin{tikzcd} \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\mathsf{t},\mathsf{s})"'] & M \arrow[d] \\ \mathrm{Pair}(M) \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{4}$$

This groupoid morphism (t, s) is sometimes called the (groupoid) *anchor*; it is related to the anchor of Lie algebroids as we will see below.

Example 1.10 (Fundamental groupoid). Another natural Lie group associated to any manifold M is the *fundamental groupoid*

$$\Pi(M) \rightrightarrows M,$$

consisting of homotopy classes $[\gamma]$ of continuous paths $\gamma\colon [0,1] \to \mathcal{G}$, relative to fixed end points. The source and target maps are $\mathsf{s}([\gamma]) = \gamma(0)$, $\mathsf{t}([\gamma]) = \gamma(1)$, and the groupoid multiplication is concatenation of paths. The groupoid anchor $(\mathsf{t}, \mathsf{s})\colon \Pi(M) \to \mathrm{Pair}(M)$ is a local diffeomorphism; it is a global diffeomorphism if and only if M is 1-connected. [2]

Example 1.11 (Lie group bundles). Suppose $\pi\colon Q \to M$ is a Lie group bundle, i.e., a locally trivial fiber bundle whose fibers have Lie group structure, in such a way that the local trivializations respect these group structures. (As a special case, any vector bundle is a Lie group bundle, using the additive group structure on the fibers.) Then Q is a groupoid $Q \rightrightarrows M$, with $\mathsf{s} = \mathsf{t} = \pi$, and with the groupoid multiplication $g = g_1 \circ g_2$ if and only if $\pi(g) = \pi(g_1) = \pi(g_2)$ and $g = g_1 g_2$ using the group structure on the fiber.

In the opposite direction, any Lie groupoid $\mathcal{G} \rightrightarrows M$ with $\mathsf{s} = \mathsf{t}$ defines a *family of Lie groups*: A surjective submersion with a fiberwise group structure such that the fiberwise multiplication depends smoothly on the base point. In general, it is *not* a Lie group bundle since there need not be local trivializations. In fact, the groups for different fibers need not even be isomorphic as Lie groups, or even as manifolds.

Example 1.12 (Jet groupoids). Given points $m_0, m_1 \in M$ and a diffeomorphism ϕ from an open neighborhood of m_1 to an open neighborhood m_0, with $\phi(m_1) = m_0$, let $j_k(\phi)$ denote its *k-jet*. Thus, $j_k(\phi)$ is the equivalence class of ϕ among such diffeomorphisms, where $j_k(\phi) = j_k(\phi')$ if the Taylor expansions of ϕ, ϕ' in local coordinates centered m_1, m_0 agree up to order k. The set of such triples $(m_0, j_k(\phi), m_1)$ is a manifold $J_k(M, M)$, and with the obvious composition of jets it becomes a Lie groupoid

$$J_k(M, M) \rightrightarrows M.$$

For $k = 0$, this is just the pair groupoid; for $k = 1$, the elements of the groupoid $J_1(M, M) \rightrightarrows M$ are pairs of elements $m_0, m_1 \in M$ together with an isomorphism $T_{m_1}M \to T_{m_0}M$. The natural maps

$$\cdots \to J_k(M, M) \to J_{k-1}(M, M) \to \cdots \to J_0(M, M) = \mathrm{Pair}(M)$$

[2] If M is connected, and $\widetilde{M}$ is a simply connected covering space, with covering map $\widetilde{M} \to M$, one has a Lie groupoid homomorphism $\mathrm{Pair}(\widetilde{M}) = \Pi(\widetilde{M}) \to \Pi(M)$. By homotopy lifting, this map is surjective. Let Λ be the discrete group of deck transformations of $\widetilde{M}$, i.e., diffeomorphisms covering the identity map on M. Then $M = \widetilde{M}/\Gamma$, and $\Pi(M) = \Pi(\widetilde{M})/\Gamma = (\widetilde{M} \times \widetilde{M})/\Gamma$, a quotient by the diagonal action.

are morphisms of Lie groupoids. (These groupoids may be regarded as finite-dimensional approximations of the *Haefliger groupoid of* M, consisting of germs of local diffeomorphisms. The latter is not a Lie groupoid since it is not a manifold.)

Example 1.13 (Action groupopids). Given an smooth action of a Lie group G on M, one has the *action groupoid* or transformation groupoid $\mathcal{G} \rightrightarrows M$. It may be defined as the subgroupoid of the direct product of groupoids $G \rightrightarrows \mathrm{pt}$ and $\mathrm{Pair}(M) \rightrightarrows M$, consisting of all $(g, m', m) \in G \times (M \times M)$ such that $m' = g.m$. Using the projection $(g, m', m) \mapsto (g, m)$ to identify $\mathcal{G} \cong G \times M$, the product reads as

$$(g, m) = (g_1, m_1) \circ (g_2, m_2) \quad \Leftrightarrow \quad g = g_1 g_2,\ m = m_2,\ m_1 = g_2.m_2.$$

Note that the isotropy groups $\mathcal{G}_m$ of the action groupoid coincide with the stabilizer groups of the G-action, G_m.

Example 1.14 (Submersion groupoids). Given a surjective submersion $\pi \colon M \rightrightarrows N$, one has a *submersion groupoid*

$$M \times_N M \rightrightarrows M$$

given as the fiber product with itself over N. The groupoid structure is as a subgroupoid of the pair groupoid $\mathrm{Pair}(M)$. For the special case of a principal G-bundle $\pi \colon P \to N$, the submersion groupoid is identified with $P \times G$; the groupoid structure is that of an action groupoid.

Example 1.15 (Atiyah groupoids). Let $P \to M$ be a principal G-bundle. Let $\mathcal{G}(P)$ be the set of triples (m', m, ϕ) where $m, m' \in M$ and $\phi \colon P_m \to P_{m'}$ is a G-equivariant map between the fibers over $m, m' \in M$. Put $\mathsf{s}(m', m, \phi) = m$, $\mathsf{t}(m', m, \phi) = m'$, and define the composition by

$$(m_1', m_1, \phi_1) \circ (m_2', m_2, \phi_2) = (m', m, \phi)$$

whenever $\phi = \phi_1 \circ \phi_2$ and $(m_1', m_1) \circ (m_2', m_2) = (m', m)$ (as for the pair groupoid). We will call the resulting groupoid

$$\mathcal{G}(P) \rightrightarrows M$$

the *Atiyah algebroid* of $\mathcal{P}$; it is also known as the *gauge groupoid*. Equivalently, we may regard $\mathcal{G}(P)$ as the quotient

$$\mathcal{G}(P) = \mathrm{Pair}(P)/G,$$

of the pair groupoid by the diagonal action. We have the following sequence of groupoids and groupoid morphisms,

$$1 \to \mathrm{Gau}(P) \to \mathcal{G}(P) \to \mathrm{Pair}(M) \to 1,$$

where $\mathrm{Gau}(P)$ is the subgroupoid of elements having the same source and target. [3] As a special case, for a vector bundle $\mathcal{V} \to M$ one has an Atiyah algebroid $\mathcal{G}(\mathcal{V})$ of its frame bundle, given more directly as the set of linear isomorphisms from one fiber of $\mathcal{V}$ to another fiber. Note that the Atiyah algebroid of the tangent bundle TM is the same as the first jet groupoid $J_1(M, M)$.

[3]Perhaps, a better notation for this groupoid $\mathcal{G}(P)$ is $\mathrm{Aut}(P) \rightrightarrows M$. The automorphism group of P is then the group $\Gamma(\mathrm{Aut}(P))$ of bisections of $\mathrm{Aut}(P)$.

1.3. **Exercises.**

Exercise 1.1. Let $\phi_1\colon Q_1 \to M_1,\ \phi_2\colon Q_2 \to M_2$ be two submersions. Given any smooth map $F\colon Q_1 \to M_2$, show that the fiber product

$$Q_2 \ {}_{\phi_2}\!\times_F Q_1 = \{(q_2, q_1)|\ \phi_2(q_2) = F(q_1)\}$$

is a smooth submanifold of $Q_2 \times Q_1$, and the map $Q_2 \ {}_{\phi_2}\!\times_F Q_1 \to M_1$ induced by ϕ_1 is a submersion. Use this to verify that for a Lie groupoid $\mathcal{G}$, the spaces of k-arrows $\mathcal{G}^{(k)}$ are smooth manifolds.

Exercise 1.2. Using the definition, show that inverses of a (Lie) groupoid are unique. In fact, show that if $g \in \mathcal{G}$ is given, and $h_1, h_2 \in \mathcal{G}$ are such that $g \circ h_1$ and $h_2 \circ g$ are units, then $h_1 = h_2$.

Exercise 1.3. Show that the inversion map $\mathrm{Inv}_{\mathcal{G}}$ of a groupoid is a diffeomorphism.

Exercise 1.4. a) Given two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$, show that their direct product becomes a Lie groupoid

$$\mathcal{G} \times \mathcal{H} \rightrightarrows M \times N.$$

b) Show that a smooth map $F\colon \mathcal{H} \to \mathcal{G}$ between Lie groupoids is a Lie groupoid morphism if and only if its graph

$$\mathrm{Gr}(F) = \{(g, h) \in \mathcal{G} \times \mathcal{H}|\ F(h) = g\}$$

is a Lie subgroupoid of the direct product:

$$\begin{array}{ccc} \mathrm{Gr}(F) & \rightrightarrows & \mathrm{Gr}(F|_M) \\ \downarrow & & \downarrow \\ \mathcal{G} \times \mathcal{H} & \rightrightarrows & M \times N \end{array}$$

Exercise 1.5. Show that any morphism of Lie groupoids $\mathrm{Pair}(N) \to \mathrm{Pair}(M)$ is induced by a smooth map $f\colon N \to M$.

Exercise 1.6. Show that if J is an open interval around 0, and $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, then a morphism of Lie groupoids

$$\mathrm{Pair}(J) \to \mathcal{G}$$

is equivalent to a $\mathcal{G}$-*path*: That is, a path $\widetilde{\gamma}\colon J \to \mathcal{G}$ such that $\widetilde{\gamma}(0) =: m \in M$ and $\mathsf{t}(\widetilde{\gamma}(t)) = m$ for all $t \in J$. That is, any such morphism if of the form

$$(t', t) \mapsto \tilde{\gamma}(t')^{-1}\tilde{\gamma}(t).$$

Show that the corresponding base path $\gamma(t) = \mathsf{s}(\widetilde{\gamma}(t))$ lies in a fixed orbit $\mathcal{O}$ of $\mathcal{G}$.

Exercise 1.7. Let $\mathcal{G} \subseteq \mathbb{R} \times \mathrm{Mat}_{\mathbb{R}}(3)$ be the 4-dimensional submanifold consisting of all (t, B) such that

$$B + B^\top + t\ B^\top B = 0.$$

Show that

$$(t, B) = (t_1, B_1) \circ (t_2, B_2)$$

if and only if

$$t = t_1 = t_2, \quad B = B_1 + B_2 + t\ B_1 B_2$$

defines a Lie groupoid structure $G \rightrightarrows \mathbb{R}$, with $\mathsf{s} = \mathsf{t}$ given by projection $\mathcal{G} \to \mathbb{R}$. Identify the Lie groups $\mathcal{G}_t$ given as the fibers of this projection. (Hint: For $t \neq 0$, consider $A = tB + I$.)

Exercise 1.8. Let $\mathfrak{d}$ be a Lie algebra, and $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{d}$ two Lie subalgebras such that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Let D, G, H be the corresponding simply connected Lie groups, and $i\colon G \to D,\ j\colon H \to D$ the group homomorphisms exponentiating the inclusions of $\mathfrak{g}, \mathfrak{h}$ into $\mathfrak{d}$. Let

$$\Gamma = \{(h, g, g', h') \in H \times G \times G \times H |\ j(h)i(g) = i(g)j(h)\}.$$

In other words, $\mathcal{G}$ is the fiber product of $H \times G$ with $G \times H$ over D, relative the natural maps from these spaces to D. Put

$$(h, g, g', h') = (h_1, g_1, g_1', h_1') \circ (h_2, g_2, g_2', h_2')$$

if and only if

$$h_1' = h_2, \quad h = h_2, \quad h' = h_2', \quad g = g_1 g_2, \quad g' = g_1' g_2'.$$

Show that this defines the structure of a Lie groupoid $\mathcal{G} \rightrightarrows H$. Exchanging the roles of G and H the same space has a groupoid structure $\mathcal{G} \rightrightarrows G$, and the two structures are compatible in the sense that they define a *double Lie groupoid*. Try to invent such a compatibility condition of two groupoid structures, and verify that it is satisfied in this example. (See Lu-Weinstein [28])

Exercise 1.9. Let X be a vector field on a manifold M. If X is complete, then the flow $\Phi_t(m)$ of any $m \in M$ is defined for all t, and one obtains a group action $\Phi\colon \mathbb{R} \times M \to M$. For an incomplete vector field, Φ is defined on a suitable $U \subseteq \mathbb{R} \times M$. Show that U becomes a Lie groupoid $U \rightrightarrows \mathbb{R}$.

2. Foliation groupoids

2.1. **Definition, examples.** A foliation $\mathcal{F}$ of a manifold M may be defined to be a subbundle $E \subseteq TM$ satisfying the Frobenius condition: for any two vector fields X, Y taking values in E, their Lie bracket again takes values in E. Given such a subbundle, one obtains a decomposition of M into *leaves* of the foliation, i.e., maximal connected injectively immersed submanifolds. The quotient space $M/\sim$, where two points are considered equivalent if they lie in the same leaf, is called the *leaf space*. Locally, a foliation looks very simple: For every $m \in M$ there exists a chart (U, ϕ) centered at m, with $\phi\colon U \to \mathbb{R}^n$, such that the tangent map $T\phi$ takes $E|_U$ to the tangent bundle of the projection $\mathrm{pr}_{\mathbb{R}^q}\colon \mathbb{R}^n \to \mathbb{R}^q$ to the last q coordinates. Such an adapted chart is called a *foliation chart*. For a foliation chart, every $U_a = \phi^{-1}(\mathbb{R}^{n-q} \times \{a\})$ for $a \in \mathbb{R}^q$ is an open subset of a leaf. Globally, the situation can be much more complicated, since U_a, U_b for $a \neq b$ might belong to the same leaf. Accordingly, the leaf space of a foliation can be extremely complicated.

Example 2.1. For any surjective submersion $\pi\colon P \to B$, the bundle $\ker(T\pi) \subseteq TP$ defines a foliation, with leaves the fibers $\pi^{-1}(b)$. If the fibers are connected, then $P/\sim$ is just B itself. If the fibers are disconnected, the leaf space can be a non-Hausdorff manifold. (E.g., take $P = \mathbb{R}^2 \backslash \{0\}$ with π projection to the first coordinate; here $P/\sim$ is the famous 'line with two origins'.)

Example 2.2. Given a diffeomorphism $\Phi\colon M \to M$ of a manifold, one can form the mapping torus as the associated bundle

$$M_\Phi = \mathbb{R} \times_{\mathbb{Z}} M,$$

where $\mathbb{R}$ is regarded as a principal $\mathbb{Z}$-bundle over $\mathbb{R}/\mathbb{Z}$, and the action of $\mathbb{Z}$ is generated by Φ. That is, it is the quotient of $\mathbb{R} \times M$ under the equivalence relation generated by $(t, m) \sim (t+1, \Phi(m))$. The 1-dimensional foliation of $M \times \mathbb{R}$, given as the fibers under projection to M, is invariant under the $\mathbb{Z}$-action, and so it descends to a 1-dimensional foliation of the mapping torus. If some point $m \in M$ is fixed under some power Φ^N, then the corresponding leaf in the mapping torus is a circle winding N times around the mapping torus. But if m is not a fixed point, then the corresponding leaf is diffeomorphic to $\mathbb{R}$.

Example 2.3. Given a manifold M with a foliation $\mathcal{F}$, and any proper action of a discrete group Λ preserving this foliation, the quotient M/Λ inherits a foliation. For example, given a connected manifold B, with base point b_0, and any action of the fundamental group $\Lambda = \pi_1(B, b_0)$ on another manifold Q, the foliation of $M = \widetilde{B} \times Q$ given by the fibers of the projection to Q is Λ-invariant for the diagonal action, and hence the associated bundle

$$M/\Lambda = \widetilde{B} \times_\Lambda Q$$

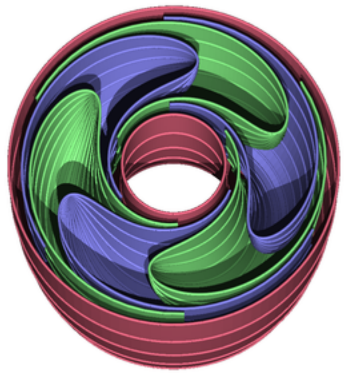
inherits a foliation. Note that the leaves of this foliations are coverings of B.

Example 2.4. Consider the foliation of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by the vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ whose lift to $\mathbb{R}^2$ is $\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$. If c is a rational number, then the flow of X is periodic, and space of leaves of the foliation is a manifold (a circle). If c is irrational, then the space $T^2/\sim$ of leaves is quite pathological: its only open subsets are the empty set and the entire space. (This example may also be regarded as a mapping torus, where Φ is given by a rotation of the standard circle by a fixed $2\pi c$.)

Example 2.5. We next describe a 2-dimensional foliation of the 3-sphere, known as the *Reeb foliation.* Consider S^3 as the total space of the Hopf fibration

$$\pi\colon S^3 \to S^2$$

(realized for example as the quotient map from $\mathrm{SU}(2) = S^3$ to its homogeneous space $\mathbb{C}P(1) \cong S^2$). The pre-image of the equator on S^2 is a 2-torus $T^2 \subseteq S^3$, and this will be one leaf of the foliation. The pre-image of the closed upper hemisphere is a solid 2-torus bounded by T^2, and similarly for the closed lower hemisphere. Thus, S^3 is obtained by gluing two solid 2-tori along their boundary. Note that this depends on the choice of gluing map: what is the 'small circle' with respect to one of the solid tori becomes the 'large circle' for the other, and vice versa. Now, foliate the interiors of these solid 2-tori as in the following picture (from wikipedia):



More specifically, this foliation of the interior of a solid torus is obtained from a translation invariant foliation of the interior of a cylinder $Z = \{(x,y,z)|\ x^2+y^2<1\}$, for example given by the hypersurfaces

$$z = \exp\left(\frac{1}{1-(x^2+y^2)}\right) + a,\ \ a \in \mathbb{R}$$

for $a \in \mathbb{R}$. The Reeb foliation has a unique compact leaf (the 2-torus), while all other leaves are diffeomorphic to $\mathbb{R}^2$.

2.2. **Monodromy and holonomy.** We will need the following notions.

Definition 2.6. Let $\mathcal{F}$ be a foliation of M, of codimension q.

(a) A path (resp. loop) in M that is contained in a single leaf of the foliation $\mathcal{F}$ is called a *foliation path* (resp., foliation loop).

(b) A q-dimensional submanifold N is called a *transversal* if N is transverse to all leaves of the foliation. That is, for all $m \in N$, the tangent space T_mN is a complement to the tangent space of the foliation.

Given points m, m' in the same leaf L, and transversals N, N' through these points, then any leaf path γ from m to m' determines the germ at m of a diffeomorphism

$$\phi_\gamma \colon N \to N',$$

taking m to m'. Indeed, given $m_1 \in N_1$ sufficiently close to m, there exists a foliation path γ_1 close to γ, and with end point in N'. This end point m_1' is independent of the choice of γ_1, as long as it stays sufficiently close to γ. This germ ϕ_γ is unchanged under homotopies of γ. It is also independent of the choice of transversals, since we may regard a sufficiently small neighborhood of m in N as the 'local leaf space' for M near m, and similarly for m'. One calls ϕ_γ the *holonomy* of the path γ. Intuitively, the holonomy of a path measures how the foliation 'twists' along γ. In the special case $m = m'$, we may take $N = N'$ and obtain a map from the fundamental group of the leaf, $\pi_1(L, m)$, to the group of germs of diffeomorphism of N fixing m.

2.3. **The monodromy and holonomy groupoids.** .

Definition 2.7. Let $\mathcal{F}$ be a foliation of M, and $m \in M$.

(a) The monodromy group of $\mathcal{F}$ at m is the fundamental group of the leaf $L \subseteq M$ through m:
$$\mathrm{Mon}(\mathcal{F}, m) = \pi_1(L, m).$$

(b) The holonomy group of $\mathcal{F}$ at m is the image of the homomorphism from $\mathrm{Mon}(\mathcal{F}, m)$ into germs of diffeomorphisms of a local transversal through m. It is denoted
$$\mathrm{Hol}(\mathcal{F}, m).$$

In other words, $\mathrm{Mon}(\mathcal{F}, m)$ consists of homotopy classes of foliation loops bases at m, while $\mathrm{Hol}(\mathcal{F}, m)$ consists of holonomy classes. $\mathrm{Hol}(\mathcal{F}, m)$ is the quotient of $\mathrm{Mon}(\mathcal{F}, m)$ by the classes of foliation loops having trivial holonomy.

Definition 2.8. Let $\mathcal{F}$ be a foliation of M.

(a) The *monodromy groupoid*
$$\mathrm{Mon}(\mathcal{F}) \rightrightarrows M,$$
consists of triples $(m', m, [\gamma])$, where $m, m' \in M$ and $[\gamma]$ is the homotopy class of a foliation path γ from $m = \gamma(0)$ to $m' = \gamma(1)$. The groupoid structure is induced by the concatenation of foliation paths.

(b) The *holonomy groupoid*
$$\mathrm{Hol}(\mathcal{F}) \rightrightarrows M,$$
is defined similarly, but taking $[\gamma]$ to be the holonomy class of a foliation path γ.

Proposition 2.9. $\mathrm{Mon}(\mathcal{F})$ *and* $\mathrm{Hol}(\mathcal{F})$ *are (possibly non-Hausdorff) manifolds.*

Sketch. Here is a sketch of the construction of charts, first for the monodromy groupoid. Given $(m', m, [\gamma]) \in \mathrm{Mon}(\mathcal{F})$, choose local transversals $N,\ N'$ through $m,\ m'$: that is, q-dimensional submanifolds transverse to the foliation, where q is the codimension of $\mathcal{F}$. Let $\phi_\gamma\colon N \to N'$ be the diffeomorphism germ determined by $[\gamma]$. Choosing a germ of a diffeomorphism $N \to \mathbb{R}^q$, which we may think of as *transverse* coordinates at m, we then also obtain transverse coordinates near m'. These sets of transverse coordinates may be completed to local foliation charts at m and m'. We hence obtain $2(n-q)+q = 2n-q$-dimensional charts for $\mathrm{Mon}(\mathcal{F})$. A similar construction works for $\mathrm{Hol}(\mathcal{F})$. □

Remark 2.10. (a) To see why the groupoids $\mathrm{Mon}(\mathcal{F})$ or $\mathrm{Hol}(\mathcal{F})$ are sometimes non-Hausdorff, suppose $g \in \mathrm{Mon}(M, m)$ is a non-trivial element of the monodromy group. It is represented by a non-contractible loop γ in the leaf through m. Then it can happen that γ is approached through loops γ_n in nearby leaves, but the γ_n are all contractible. Then the elements $g_n \in \mathrm{Mon}(M, m_n)$ (with $m_n = \gamma_n(0)$) satisfy $g_n \to g$, but since $g_n = m_n$ (constant loops) they also satisfy $g_n \to m$. This non-uniqueness of limits then implies that $\mathrm{Mon}(M)$ is not Hausdorff. Similarly phenomena appear for the holonomy groupoid.

(b) There is no simple relationship, in general, between the Hausdorff properties of the holonomy and monodromy groupoids of a foliation $\mathcal{F}$. Indeed, it can happen that two points of $\mathrm{Mon}(\mathcal{F})$ not admitting disjoint open neighborhoods get identified under the quotient map to $\mathrm{Hol}(\mathcal{F})$. On the other hand, it can also happen that two distinct points of $\mathrm{Hol}(\mathcal{F})$ do not admit disjoint open neighborhoods, even if they have pre-images in $\mathrm{Mon}(\mathcal{F})$ have disjoint open neighborhoods. (The images of the latter under the quotient map need no longer be disjoint.)

Exercise 2.1. (From Crainic-Fernandes [12].) Let $M = \mathbb{R}^3\backslash\{0\}$ be foliated by the fibers of the projection $(x, y, z) \mapsto z$. Is the monodromy groupoid Hausdorff? What about the holonomy groupoid?

Exercise 2.2. For the Reeb foliation of S^3, show that the holonomy groupoid coincides with the monodromy groupoid, and is non-Hausdorff.

Exercise 2.3. Think of S^3 has obtained by gluing two solid 2-tori as before, and let $M \subseteq S^3$ be the open subset obtained by removing the central circle of each of the solid 2-tori. Show that the monodromy groupoid is Hausdorff, but the holonomy groupoid is non-Hausdorff.

Exercise 2.4. Similar to S^3, the product $S^2 \times S^1$ is obtained by gluing to solid 2-tori, given as the pre-images of the closed upper/lower hemispheres under the projection to S^2. Foliate these solid 2-tori as for the Reeb foliation. Show that the monodromy groupoid is non-Hausdorff, but the holonomy groupoid is Hausdorff.

2.4. **Appendix: Haefliger's approach.** A cleaner definition of holonomy proceeds as follows (following Haefliger [24]): Let $\mathcal{F}$ be a given codimension q foliation of M. A foliated manifold can be covered by foliation charts $\phi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q$, i.e. the pre-images $\phi^{-1}(\mathbb{R}^{n-q} \times \{y_0\})$ for $y_0 \in \mathbb{R}^q$ are tangent to the leaves. There exists a topology on M, called the *foliation topology*, generated by such pre-images. Put differently, the foliation charts become local homeomorphisms if we give $\mathbb{R}^{n-q}$ its standard topology and $\mathbb{R}^q$ the discrete topology. The connected components of M for the foliation topology are exactly the leaves of M, and the continuous paths in M for the foliation topology are the foliation paths.

Let $\widetilde{M}$ be the set of all $(m, [\psi])$, where $m \in M$, and where $[\psi]$ is the germ of a smooth map $\psi\colon U \to \mathbb{R}^q$, where U is an open neighborhood of m, and ψ is a submersion whose fibers are tangent to leaves. Given a foliation chart (U, ϕ), one obtains a subset $\widetilde{U}$ of $\widetilde{M}$ consisting of germs of $[\psi]$ at points of U, where ψ is ϕ followed by projection. Give $\widetilde{M}$ the topology generated by all $\widetilde{U}$. Then the natural projection $\widetilde{M} \to M$ is a local homeomorphism relative to the foliation topology on M.

Let $\gamma\colon [0, 1] \to M$ be a continuous path for the foliation topology (a foliation path in M), with end points $m = \gamma(0)$ and $m' = \gamma(1)$. Since $\widetilde{M} \to M$ is a covering, γ to paths in $\widetilde{M}$, defining a map $\pi^{-1}(m) \to \pi^{-1}(m')$ between fibers of $\widetilde{M}$. Two such paths, with the same end points, are said to define the same *holonomy* if they determine the same map.

3. Properties of Lie groupoids

3.1. **Orbits and isotropy groups.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Define a relation $\sim$ on M, where

$$m \sim m' \Leftrightarrow \exists g \in \mathcal{G}\colon \mathsf{s}(g) = m,\ \mathsf{t}(g) = m'.$$

Lemma 3.1. *The relation $\sim$ is an equivalence relation.*

Proof. Transitivity follows from the groupoid multiplication, symmetry follows from the existence of inverses, reflexivity follows since elements of M are units (so, $m \circ m = m$) $\square$

The equivalence classes of the relation $\sim$ are called the *orbits* of the Lie groupoid. The equivalence class of the element $m \in M$ is denoted

$$\mathcal{G} \cdot m \subseteq M.$$

Let us also recall the definition of isotropy groups

$$\mathcal{G}_m = \mathsf{s}^{-1}(m) \cap \mathsf{t}^{-1}(m).$$

Example 3.2. For a Lie group G with a Lie group action on M, the orbits and isotropy groups of the action groupoid $\mathcal{G} = G \times M$ are just the usual ones for the G-action:

$$\mathcal{G} \cdot m = G \cdot m, \quad \mathcal{G}_m = G_m.$$

Example 3.3. For a foliation $\mathcal{F}$ of M, the orbits of both $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$ and $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ are the leaves of the foliation $\mathcal{F}$, while the isotropy groups $\mathcal{G}_m$ are the monodromy groups and holonomy groups, respectively.

We will show later in this section that the orbits are injectively immersed submanifolds of M, while the isotropy groups are embedded submanifolds of $\mathcal{G}$. Note that for any given m, the orbit can be characterized as

$$\mathcal{G} \cdot m = \mathsf{t}(\mathsf{s}^{-1}(m)),$$

(or also as $\mathsf{s}(\mathsf{t}^{-1}(m))$). The isotropy group $\mathcal{G}_m$ is the fiber of m under the map $\mathsf{s}^{-1}(m) \to G \cdot m$. Since is a submersion, the fibers $\mathsf{s}^{-1}(m)$ are embedded submanifolds of $\mathcal{G}$, of dimension $\dim \mathcal{G} - \dim M$. Hence, to show that the orbits and stabilizer group are submanifolds, it suffices to show that the restriction of t to any source fiber $\mathsf{s}^{-1}(m)$ has constant rank. This will be proved as Proposition 3.7 below.

3.2. **Bisections.** A *bisection* of a Lie groupoid is a submanifold $S \subseteq \mathcal{G}$ such that both t, s restrict to diffeomorphisms $S \to M$. For example, M itself is a bisection. The name indicates that S can be regarded as a section of both s and t. We will denote by

$$\Gamma(\mathcal{G})$$

the set of all bisections. It has a group structure, with the multiplication given by

$$S_1 \circ S_2 = \mathrm{Mult}_{\mathcal{G}}((S_1 \times S_2) \cap \mathcal{G}^{(2)}).$$

That is, $S_1 \circ S_2$ consists of all products $g_1 \circ g_2$ of composable elements with $g_i \in S_i$ for $i = 1, 2$. The identity element for this multiplication is the unit bisection M, and the inverse is given by $S^{-1} = \mathrm{Inv}_{\mathcal{G}}(S)$. This group of bisections comes with a group homomorphism

$$\Gamma(\mathcal{G}) \to \mathrm{Diff}(M), \quad S \mapsto \Phi_S$$

where $\Phi_S = \mathsf{t}|_S \circ (\mathsf{s}|_S)^{-1}$.

Remark 3.4. Alternatively, a bisection of $\mathcal{G}$ may be regarded as a section $\sigma\colon M \to \mathcal{G}$ of the source map s such that its composition with the target map t is a diffeomorphism of Φ. The definition as a submanifold has the advantage of being more 'symmetric'.

Examples 3.5. (a) For a Lie group $G \rightrightarrows \mathrm{pt}$, regarded as a Lie groupoid, a bisection is simply an element of G, and $\Gamma(G) = G$ as a group.

(b) For a vector bundle $V \to M$, regarded as groupoid $V \rightrightarrows M$, a bisection is the same as a section. More generally, this is true for any bundle of Lie groups.

(c) For the 'trivial' groupoid $M \rightrightarrows M$ the only bisection is M itself. The resulting group $\Gamma(M)$ consists of only the identity element.

(d) For the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$, a bisection is the same as the graph of a diffeomorphism of M. This identifies $\Gamma(\mathcal{G}) \cong \mathrm{Diff}(M)$.

(e) Let $P \to M$ be a principal G-bundle. A bisection of Atiyah groupoid $\mathcal{G}(P) \rightrightarrows M$ is the same as a principal bundle automorphism $\Phi_P\colon P \to P$. That is,

$$\Gamma(\mathcal{G}) = \mathrm{Aut}(P).$$

(f) Given a G-action on M, a bisection of the action groupoid is a smooth map $f\colon M \to G$ for which the map $m \mapsto f(m).m$ is a diffeomorphism.

The group of bisections has three natural actions on $\mathcal{G}$:

- Left multiplication:

$$\mathcal{A}^L_S(g) = h \circ g,$$

with the unique element $h \in S$ such that $\mathsf{s}(h) = \mathsf{t}(g)$. Namely, $h = ((\mathsf{s}|_S)^{-1} \circ \mathsf{t})(g)$. This has the property

$$\mathsf{s} \circ \mathcal{A}^L_S = \mathsf{s}, \quad \mathsf{t} \circ \mathcal{A}^L_S = \Phi_S \circ \mathsf{t}. \tag{5}$$

- Right multiplication:

$$\mathcal{A}^R_S(g) = g \circ (h')^{-1},$$

with the unique element $h' \in S$ such that $\mathsf{s}(h') = \mathsf{s}(g)$. Namely, $h' = (\mathsf{s}|_S)^{-1}(\mathsf{s}(g))$. We have that

$$\mathsf{t} \circ \mathcal{A}^R_S = \mathsf{t}, \quad \mathsf{s} \circ \mathcal{A}^R_S = \Phi_S \circ \mathsf{s}. \tag{6}$$

- Adjoint action:

$$\mathrm{Ad}_S(g) = h \circ g \circ (h')^{-1},$$

with h, h' as above. Note that the adjoint action is by groupoid automorphisms, restricting to the map Φ_S on units.

Examples 3.6. Diffeomorphisms of a manifold give a natural action on the pair groupoid $\mathrm{Pair}(M)$. For a principal G-bundle, the group $\mathrm{Aut}(P)$ of principal bundle automorphisms naturally acts by automorphisms of the Atiyah groupoid. In both cases, the natural action is the adjoint action.

3.3. **Local bisections.** In general, there may not exist a global bisection passing through a given point $g \in \mathcal{G}$. However, it is clear that one can always find a *local bisection* $S \subseteq M$, that is, t, s restrict to local diffeomorphisms to open subsets $\mathsf{t}(S) = V,\ \mathsf{s}(S) = U$ of M. Any local bisection defines a diffeomorphism between these open subsets:

$$\Phi_S = \mathsf{t}|_S \circ (\mathsf{s}|_S)^{-1} \colon U \to V,$$

with inverse defined by the local bisection $S^{-1} = \mathrm{Inv}_{\mathcal{G}}(S)$. We have the left, right, and adjoint actions defined as diffeomorphisms

$$\begin{aligned} \mathcal{A}^L_S &\colon \mathsf{t}^{-1}(U) \to \mathsf{t}^{-1}(V), \quad g \mapsto h \circ g, \\ \mathcal{A}^R_S &\colon \mathsf{s}^{-1}(U) \to \mathsf{s}^{-1}(V), \quad g \mapsto g \circ (h')^{-1}, \end{aligned}$$

where, for a given element $g \in \mathcal{G}$, we take h, h' to be the unique elements in S such that $\mathsf{s}(h) = \mathsf{t}(g),\ \mathsf{s}(h') = \mathsf{s}(g)$. These satisfy the relations (5), (6) as before, and hence we also have an adjoint action defined as a diffeomorphism

$$\mathrm{Ad}_S \colon \mathsf{s}^{-1}(U) \cap \mathsf{t}^{-1}(U) \to \mathsf{s}^{-1}(V) \cap \mathsf{t}^{-1}(V), \quad g \mapsto h \circ g \circ (h')^{-1},$$

extending the map Φ_S on units. As an application of local bisections, we can now prove

Proposition 3.7. *For any Lie groupoid $\mathcal{G} \rightrightarrows M$ and any $m \in M$, the restriction of t to the source fiber $\mathsf{s}^{-1}(m)$ has constant rank.*

Proof. To show that the ranks of

$$\mathsf{t}|_{\mathsf{s}^{-1}(m)} \colon \mathsf{s}^{-1}(m) \to M$$

at given points $g, g' \in \mathsf{s}^{-1}(m)$ coincide, let S be a local bisection containing the element $g' \circ g^{-1}$, and let $U = \mathsf{s}(S),\ V = \mathsf{t}(S)$. The diffeomorphism

$$\mathcal{A}^L_S \colon \mathsf{t}^{-1}(U) \to \mathsf{t}^{-1}(V)$$

takes g to g'. Since $\mathsf{s} \circ \mathcal{A}^L_S = \mathsf{s}$, it restricts to a diffeomorphism on each s fiber. Since furthermore $\mathsf{t} \circ \mathcal{A}^L_S = \Phi_S \circ \mathcal{A}^L_S$, we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{t}^{-1}(U) \cap \mathsf{s}^{-1}(m) \arrow[r, "\mathcal{A}^L_S|_{\mathsf{s}^{-1}(m)}"] \arrow[d, "\mathsf{t}|_{\mathsf{s}^{-1}(m)}"'] & \mathsf{t}^{-1}(V) \cap \mathsf{s}^{-1}(m) \arrow[d, "\mathsf{t}|_{\mathsf{s}^{-1}(m)}"] \\ U \arrow[r, "\Phi_S"'] & V \end{tikzcd}$$

where the horizontal maps are diffeomorphisms, and the upper map takes g to g'. Hence, the ranks of the vertical maps at g, g' coincide. $\square$

Corollary 3.8. *For every $m \in \mathcal{G}$, the orbit $\mathcal{G}\cdot m$ is an injectively immersed submanifold of M, while the isotropy group $\mathcal{G}_m$ is an embedded submanifold of $\mathcal{G}$, hence is a Lie group. In fact, all fibers of the map*

$$\mathsf{t},\mathsf{s}\colon \mathcal{G} \to \mathrm{Pair}(M)$$

are embedded submanifolds.

For the last part, we have that $(\mathsf{t},\mathsf{s})^{-1}(m',m)$ is a submanifold because it coincides with the fiber of m' under the surjective submersion $\mathsf{s}^{-1}(m) \to \mathcal{G}\cdot m$. (The fiber is empty if $m' \notin \mathcal{G}\cdot m$.) Note that (t,s) does not have constant rank, in general.

3.4. **Transitive Lie groupoids.** A G-action on a manifold is called *transitive* if it has only a single orbit: $G\cdot m$. The definition carries over to Lie groupoids:

Definition 3.9. A Lie groupoid is called *transitive* if it has only one orbit: $\mathcal{G}\cdot m = M$.

Here are some examples:

- The pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$ is transitive.
- The jet groupoids $J^k(M,M) \rightrightarrows M$ are transitive.
- The homotopy groupoid $\Pi(M) \rightrightarrows M$ is transitive if and only is M is connected.
- For an action of a Lie group G on M, the action groupoid $G \ltimes M \rightrightarrows M$ is transitive if and only if the G-action on M is transitive.
- For any Lie groupoid $\mathcal{G} \rightrightarrows M$, and any orbit $i\colon \mathcal{O} \hookrightarrow M$, the restriction of $\mathcal{G}|_{\mathcal{O}}$ to $\mathcal{O}$ is transitive. Here, the 'restriction' consists of all groupoid elements having source and fiber in $\mathcal{O}$. More precisely,
$$\mathcal{G}|_{\mathcal{O}} = \{(g,x',x) \in \mathcal{G}\times \mathrm{Pair}(\mathcal{O})|\ \mathsf{s}(g)=x,\ \mathsf{t}(g)=x'\},$$
with the groupoid structure as a subgroupoid of $\mathcal{G}\times\mathrm{Pair}(\mathcal{O})$. Note that $\mathcal{G}_{\mathcal{O}}$ comes with an injective immersion to $\mathcal{G}$, and $\mathcal{G}$ is a disjoint union of all such immersions.
- For any principal G-bundle $\pi\colon P \to M$, the Atiyah groupoid $\mathcal{G}(P)$ is transitive.

It turns out that all these examples are special cases of the last one. For the following, see e.g. [30].

Theorem 3.10. *Suppose $\mathcal{G} \rightrightarrows M$ is a transitive Lie groupoid. Then $\mathcal{G}$ is isomorphic to an Atiyah groupoid $\mathcal{G}(P)$, for a suitable principal G-bundle $P \to M$. The identification depends on the choice of a base point $m_0 \in M$.*

Proof. Given m_0, let $G = \mathcal{G}_{m_0}$ be the isotropy group at m_0, and $P = \mathsf{s}^{-1}(m_0)$ the source fiber. The target map gives a surjective submersion

$$\pi = \mathsf{t}|_{\mathsf{s}^{-1}(m_0)}\colon P \to M,\quad p \mapsto \mathsf{t}(p).$$

The group G acts on P by

$$g.p = p\circ g^{-1};$$

this is well-defined since $\mathsf{s}(p) = m_0 = \mathsf{s}(g)$ and $\mathsf{s}(g.p) = \mathsf{s}(g^{-1}) = \mathsf{t}(g) = m_0$. This action preserves fibers, since $\pi(g.p) = \mathsf{t}(p\circ g^{-1}) = \mathsf{t}(p) = \pi(p)$. The action is free, since $g.p = p$ means

$p \circ g^{-1} = p$, hence $g = m_0$ as an element of G, which is the identity of $G = \mathcal{G}_{m_0}$. Conversely, given two points $p', p \in P$ in the same fiber, i.e. $\mathsf{t}(p') = \mathsf{t}(p)$, the element $g = (p')^{-1} \circ p$ is well-defined, lies in $\mathcal{G}_{m_0} = G$, and satisfies $p' = p \circ g^{-1}$. This shows that P is a principal G-bundle.[4]

It remains to identify $\mathcal{G}$ with the Atiyah groupoid of P. Let $\phi \in \mathcal{G}$ be given. Left multiplication by ϕ gives a map

$$P_{\mathsf{s}(\phi)} \to P_{\mathsf{t}(\phi)}, \quad p \mapsto \phi \circ p,$$

which commutes with the principal G-action given by multiplication from the right. This defines an injective smooth map $F\colon \mathcal{G} \to \mathcal{G}(P)$. It is clear that F is a groupoid homomorphism. The inverse map is constructed as follows: given $\psi \in \mathcal{G}(P)$, choose $p \in P_{\mathsf{s}(\psi)}$, then the element $\phi = \psi(p) \circ p^{-1} \in \mathcal{G}$ is defined, and independent of the choice of p. Clearly, $F(\phi) = \psi$. $\square$

Example 3.11. For a homotopy groupoid $\Pi(M) \rightrightarrows M$ over a connected manifold M, the choice of a base point m_0 defines the fundamental group $\mathcal{G}_{m_0} \cong \pi_1(M, m_0)$. The bundle P is the universal covering $\widetilde{M}$ of M (with respect to m_0), regarded as a principal $\pi_1(M, m_0)$-bundle, and $\Pi(M)$ gets identified as its Atiyah groupoid. In particular, we see that the group $\Gamma(\Pi(M))$ of bisections is the group

$$\mathrm{Aut}(\widetilde{M}) = \mathrm{Diff}(\widetilde{M})^{\pi_1(M)}$$

of automorphisms of the covering space $\widetilde{M}$.

Example 3.12. Let $G \times M \rightrightarrows M$ be the action groupoid of a transitive G-action on M. The choice of $m_0 \in M$ identifies M with the homogeneous space

$$M \cong G/K.$$

where $K = G_{m_0}$ is the stabilizer. The principal bundle P for this transitive Lie groupoid is G itself, regarded as a principal K-bundle over M. The resulting identification of the action groupoid and the Atiyah groupoid is the map

$$G \times (G/K) \to (G \times G)/K \quad (g,\ aK) \mapsto (ga, a)K,$$

the inverse map is

$$(G \times G)/K \to G \times (G/K),\ (b, a)K \mapsto (ba^{-1}, aK).$$

Exercise 3.1. Let M be a connected manifold. Show (by giving a counter-example) that the map $\mathrm{Aut}(\widetilde{M}) \to \mathrm{Diff}(M)$ is not always surjective. **Hint:** You can take $M = S^1$.

Exercise 3.2. Show that a Lie groupoid is transitive if and only if the map (t, s) is surjective, and that it must be a submersion in that case.

4. More constructions with groupoids

In this section, we will promote the viewpoint of describing groupoid structures in terms of the graph of the groupoid multiplication. This will require some preliminary background material in differential geometry.

[4]The local triviality is automatic: given a free Lie group action on a manifold, and a surjective submersion onto another manifold such that the orbits are exactly the fibers of the action, the manifold is a principal bundle; local trivializations are obtained from local sections of the submersion.

4.1. **Vector bundles in terms of scalar multiplication.** It is a relatively recent observation that vector bundles are uniquely determined by the underlying manifold structures together with the scalar multiplications:

Proposition 4.1 (Grabowski-Rotkievicz)**.** [22]
(a) *A submanifold of the total space of a vector bundle $E \to M$ is a vector subbundle if and only if it is invariant under scalar multiplication by all $t \in \mathbb{R}$.*
(b) *A smooth map $E' \to E$ between the total spaces of two vector bundles $E \to M$, $E' \to M'$ is a vector bundle morphism if and only if it intertwines the scalar multiplications by all $t \in \mathbb{R}$.*

That is, the additive structure is uniquely determined by the scalar multiplication.

4.2. **Relations.** A linear relation from a vector space V_1 to a vector space V_2 is a subspace $R \subseteq V_2 \times V_1$. We will think of R as a generalized map from V_1 to V_2, and will write

$$R\colon V_1 \dashrightarrow V_2.$$

We define the kernel and range of R as

$$\ker(R) = \{v_1 \in V_1\colon (0, v_1) \in R\},$$
$$\mathrm{ran}(R) = \{v_2 \in V_2\colon \exists v_1 \in V_1, (v_2, v_1) \in R\}.$$

R is called surjective if $\mathrm{ran}(R) = V_2$, and injective if $\ker(R) = 0$.

An actual linear map $A\colon V_1 \to V_2$ can be viewed as a linear relation, by identifying A with its graph $\mathrm{Gr}(A)$; the kernel and range of A as a linear map coincide withe the kernel and relation as a relation. The identity map $\mathrm{id}_V\colon V \to V$ defines the relation

$$\Delta_V = \mathrm{Gr}(\mathrm{id}_V)\colon V \dashrightarrow V$$

given by the diagonal in $V \times V$. Any subspace $S \subseteq V$ can be regarded as a relation $S\colon 0 \dashrightarrow V$. Given a relation $R\colon V_1 \to V_2$, we define the transpose relation $R^\top\colon V_2 \dashrightarrow V_1$ by setting $(v_1, v_2) \in R^\top \Leftrightarrow (v_2, v_1) \in R$. Note that R is the graph of a linear map $A\colon V_1 \to V_2$ if and only if $\dim R = \dim V_1$ and $\ker(R^\top) = 0$. We also define a relation

$$\mathrm{ann}^\natural(R)\colon V_1^* \to V_2^*$$

by declaring that $(\mu_2, \mu_1) \in \mathrm{ann}^\natural(R)$ if and only if $\langle \mu_2, v_2\rangle = l\mu_1, v_1\rangle$ for all $(v_2, v_1) \in R$; equivalently, it is obtained from the annihilator of R by a sign change in one of the factors of $V_2^* \times V_1^*$. Note that

$$\mathrm{ann}^\natural(\Delta_V) = \Delta_{V^*}.$$

Also, if $A\colon V_1 \to V_2$ is a linear map, and $A^*\colon V_2^* \to V_1^*$ the dual map, then

$$\mathrm{ann}^\natural(\mathrm{Gr}(A)) = \mathrm{Gr}(A^*)^\top.$$

The composition of relations $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$ is the relation

$$R' \circ R\colon V_1 \dashrightarrow V_3,$$

where $(v_3, v_1) \in R' \circ R$ if and only if there exists $v_2 \in V_2$ such that $(v_3, v_2) \in R'$ and $(v_2, v_1) \in R$. This has the property

$$\mathrm{ann}^\natural(R' \circ R) = \mathrm{ann}^\natural(R') \circ \mathrm{ann}^\natural(R)$$

(see [27, Lemma A.2]). Note that in general, given smooth families of subspaces R_t, R'_t, the composition $R_t \circ R'_t$ need not have constant dimension, and even if it does it need not depend smoothly on t (as elements of the Grassmannian). For this reason, one often imposes transversality assumptions on the composition.

Definition 4.2. We say that R', R have *transverse composition* if
(a) $\ker(R') \cap \ker(R^\top) = 0$,
(b) $\operatorname{ran}(R) + \operatorname{ran}((R')^\top) = V_2$.

Notice that the first condition in (4.2) means that for $(v_3, v_1) \in R' \circ R$, the element $v_2 \in V_2$ such that $(v_3, v_2) \in R'$, $(v_2, v_1) \in R$ is *unique*. The second condition is equivalent to the condition that the sum $(R' \times R) + (V_3 \times \Delta_{V_2} \times V_1)$ equals $V_3 \times V_2 \times V_2 \times V_1$. The first condition is automatic if $\ker(R') = 0$ or $\ker(R^\top) = 0$, while the second condition is automatic if $\operatorname{ran}(R) = V_2$ or $\operatorname{ran}((R')^\top) = V_2$.

See e.g. [27, Appendix A] for further details, as well as the proof of the following dimension formula:

Proposition 4.3. *If $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$ have transverse composition, then*

$$\dim(R' \circ R) = \dim(R') + \dim(R) - \dim V_2.$$

Conversely, if this dimension formula holds, then the composition is transverse provided that at least one of the conditions in Definition 4.2 holds.

Lemma 4.4. *Let $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$ be* surjective *relations, whose transpose relations are* injective. *Then R', R have transverse composition, $R' \circ R$ is surjective, and $(R' \circ R)^\top$ is injective.*

Proof. Transversality of the composition is immediate from the definition 4.2: The first condition follows from injectivity of $R^\top$, the second condition from surjectivity of R. On the other hand, the composition of surjective relations is surjective, while the composition of injective relations is injective. □

More generally, we can consider smooth relations between manifolds. A *smooth relation*

$$\Gamma\colon M_1 \dashrightarrow M_2$$

from a manifold M_1 to a manifold M_2, is an (immersed) submanifold $\Gamma \subseteq M_2 \times M_1$. Any smooth $\Phi\colon M_1 \to M_2$ defines such a relation $\mathrm{Gr}(\Phi) \subseteq M_2 \times M_1$, and we have $\mathrm{Gr}(\Phi \circ \Psi) = \mathrm{Gr}(\Phi) \circ \mathrm{Gr}(\Psi)$ (composition of relations). Given another such relation $\Gamma'\colon M_2 \dashrightarrow M_3$, the set-theoretic composition of relations

$$\Gamma' \circ \Gamma = \{(m_3, m_1) |\ \exists m_2 \in M_2\colon (m_3, m_2) \in \Gamma',\ \ (m_2, m_1) \in \Gamma\}$$

is a smooth relation if the composition is *transverse*:

Definition 4.5. The composition of smooth relations $\Gamma\colon M_1 \dashrightarrow M_2$ and $\Gamma'\colon M_2 \dashrightarrow M_3$ is *transverse* if for all points of $\Gamma' \diamond \Gamma := (\Gamma' \times \Gamma) \cap (M_3 \times \Delta_{M_2} \times M_1)$ the composition of tangent spaces is transverse.

This assumption implies that $\Gamma' \circ \Gamma$ is a submanifold of dimension $\dim \Gamma + \dim \Gamma' - \dim M_2$, and the map to $\Gamma \circ \Gamma$ is a (local) diffeomorphism. It also follows that

$$T(\Gamma' \circ \Gamma) = T\Gamma \circ T\Gamma'.$$

The manifold counterpart to Lemma 4.4 reads as:

Lemma 4.6. *Let $\Gamma\colon M_1 \dashrightarrow M_2$ and $\Gamma'\colon M_2 \dashrightarrow M_3$ be smooth relations, with the property that the projections from Γ, Γ' to their targets is a surjective submersion, while their projection to the source is an injective immersion. Then Γ', Γ have a transverse composition, and the projections from $\Gamma' \circ \Gamma$ to the target and source are a surjective submersion and injective immersion, respectively.*

Finally, we can also consider relations in the category of vector bundles. A $\mathcal{VB}$-relation $\Gamma\colon E_1 \dashrightarrow E_2$ between vector bundles is a vector subbundle of $\Gamma \subseteq E_2 \times E_1$. By Grabowski-Rotkievicz, this is the same as a smooth relation that is invariant under scalar multiplication. The definition of $\mathrm{ann}^\natural(\Gamma)$ generalizes, and the property under compositions extends:

$$\mathrm{ann}^\natural(\Gamma' \circ \Gamma) = \mathrm{ann}^\natural(\Gamma') \circ \mathrm{ann}^\natural(\Gamma).$$

4.3. **Groupoid structures as relations.** The axioms of a Lie groupoid can be phrased in terms of smooth relations, as follows. Let $\Gamma = \mathrm{Gr}(\mathrm{Mult}_\mathcal{G}) \subseteq \mathcal{G} \times \mathcal{G} \dashrightarrow \mathcal{G}$ be the graph of the multiplication map. The projection of Γ onto $\mathcal{G}$ given as $(g; g_1, g_2) \mapsto g$ is a surjective submersion, while the map $\Gamma \to \mathcal{G} \times \mathcal{G}, (g; g_1, g_2) \mapsto (g_1, g_2)$ is an embedding (its image is the submanifold $\mathcal{G}^{(2)}$). By Lemma 4.6, it is automatic that the composition of Γ with $\Gamma \times \Delta_\mathcal{G}$ and also with $\Delta_\mathcal{G} \times \Gamma$ are smooth, and the associativity of the groupoid multiplication is equivalent to the equality

$$\Gamma \circ (\Gamma \times \Delta_\mathcal{G}) = \Gamma \circ (\Delta_\mathcal{G} \times \Gamma) \tag{7}$$

Similarly, regarding the submanifold of units as a relation $M\colon \mathrm{pt} \to \mathcal{G}$, the condition for units reads as

$$\Gamma \circ (M \times \Delta_\mathcal{G}) = \Delta_\mathcal{G} = \Gamma \circ (\Delta_\mathcal{G} \times M). \tag{8}$$

[5] In the next section, we will give a first application of this viewpoint.

4.4. **Tangent groupoid, cotangent groupoid.** For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the tangent bundle becomes a Lie groupoid

$$T\mathcal{G} \rightrightarrows TM,$$

by applying the tangent functor to all the structure maps. For example, the source map is $\mathsf{s}_{T\mathcal{G}} = T\mathsf{s}_\mathcal{G}$, and similarly for the target map; and the multiplication map is $\mathrm{Mult}_{T\mathcal{G}} = T\,\mathrm{Mult}_\mathcal{G}$ as a map from $(T\mathcal{G})^{(2)} = T(\mathcal{G}^{(2)})$ to $T\mathcal{G}$. The associativity and unit axioms are obtained from those of $\mathcal{G}$, by applying the tangent functor.

[5] `This last composition is not transverse, though -- FIX THIS`

In fact, $T\mathcal{G}$ is a so-called $\mathcal{VB}$-groupoid: It is a vector bundle, and all structure maps are vector bundle morphisms.

Definition 4.7. A $\mathcal{VB}$-groupoid is a groupoid $\mathcal{V} \rightrightarrows E$ such that $\mathcal{V} \to \mathcal{G}$ is a vector bundle, and $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{V}})$ is a vector subbundle of $\mathcal{V}^3$.

Using this result, it follows that the units of a $\mathcal{VB}$-groupoid are a vector bundle $E \to M$, and that all groupoid structure maps are vector bundle morphisms. Furthermore, the zero sections of $\mathcal{V}$ defines a subgroupoid $\mathcal{G} \rightrightarrows M$ of $\mathcal{V} \rightrightarrows E$.

Suppose that $\mathcal{W} \rightrightarrows F$ is a $\mathcal{VB}$-subgroupoid of $\mathcal{V} \rightrightarrows E$, with base $\mathcal{H} \rightrightarrows N$ a subgroupoid of $\mathcal{G} \rightrightarrows M$. Then we can form the *quotient $\mathcal{VB}$-groupoid*,

$$\mathcal{V}|_{\mathcal{H}}/\mathcal{W} \rightrightarrows E|_N/F$$

For example, if $\mathcal{H} \subseteq \mathcal{G}$ is a Lie subgroupoid with units $N \subseteq M$, then the normal bundle $\nu(\mathcal{G}, \mathcal{H}) = T\mathcal{G}|_{\mathcal{H}}/T\mathcal{H}$ becomes a Lie groupoid over $\nu(M, N)$,

$$\nu(\mathcal{G}, \mathcal{H}) \rightrightarrows \nu(M, N).$$

The dual of a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ is also a $\mathcal{VB}$-groupoid:

Theorem 4.8. *For any $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$, the dual bundle $\mathcal{V}^*$ has a unique structure of a $\mathcal{VB}$-groupoid such that $\mu = \mu_1 \circ \mu_2$ if and only if*

$$\langle \mu, v\rangle = \langle \mu_1, v_1\rangle + \langle \mu_2, v_2\rangle$$

whenever $v = v_1 \circ v_2$ in $\mathcal{V}$. (Here it is understood that $v, v_1, v_2 \in \mathcal{V}$ have the same base points as μ, μ_1, μ_2, respectively. In particular, these base points must satisfy $g = g_1 \circ g_2$.) The units for this groupoid structure is the annihilator bundle $\mathrm{ann}(E)$.

We call

$$\mathcal{V}^* \rightrightarrows \mathrm{ann}(E)$$

the *dual $\mathcal{VB}$-groupoid* to $\mathcal{V} \rightrightarrows E$.

Proof. Let $\Gamma_{\mathcal{V}} = \mathrm{Gr}(\mathrm{Mult}_{\mathcal{V}})$ be the graph of the groupoid multiplication of $\mathcal{V}$. Then the graph of the proposed groupoid multiplication of $\mathcal{V}^*$ is

$$\Gamma_{\mathcal{V}^*} = \mathrm{ann}^\natural(\mathrm{Gr}(\mathrm{Mult}_{\mathcal{V}})).$$

By applying $\mathrm{ann}^\natural$ the associativity and unit axioms of $\mathcal{V}$, given by (7) and (8) (with $\mathcal{G}$ replaced by $\mathcal{V}$), one obtains the corresponding axioms of $\mathcal{V}^*$. In particular, we see that the elements of $\mathrm{ann}(E)$ act as units. The inversion map for $\mathcal{V}^*$ is just the dual of that of $\mathcal{V}$; their graphs are related by $\mathrm{ann}^\natural$. $\square$

Remark 4.9. (Some details.) As usual, the units, as well as the source and target maps, are uniquely determined by the groupoid multiplication: Suppose $\mu \in \mathcal{V}^*$ is a unit. Then its base point must be a unit in $\mathcal{G}$. Let $v \in E$ (with the same base point), so that $v = v \circ v$. The multiplication rule tells us that $\langle \mu, v\rangle = \langle \mu, v\rangle + \langle \mu, v\rangle$, hence $\langle \mu, v\rangle = 0$. This shows that $\mu \in \mathrm{ann}(E)$. Conversely, if $v, v_1, v_2 \in \mathcal{V}|_M$ with $v = v_1 \circ v_2$ (in particular, all base points coincide) then $v = v_1 + v_2$ modulo E. (Exercise below.) Hence, for $\mu \in \mathrm{ann}(E)$ we obtain $\mu = \mu \circ \mu$, by definition of the multiplication.

Remark 4.10. We might call a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ a *$\mathcal{VB}$-group* if E is the zero vector bundle over pt. For example, the tangent bundle of a Lie group is a $\mathcal{VB}$-group. The dual bundle to a $\mathcal{VB}$-group need not be a group, in general, since $\mathrm{ann}(E) \cong (\mathcal{V}|_e)^*$ with e the group unit of the base $\mathcal{G} \rightrightarrows \mathrm{pt}$ is non-trivial unless $\mathcal{V}$ is the zero bundle over $\mathcal{G}$. For the case that $\mathcal{G} = G$ is a Lie group, and $\mathcal{V} = TG$, we find that $\mathrm{ann}(E) = \mathfrak{g}^*$.

Exercise 4.1. Show that if $\mathcal{V}$ is a $\mathcal{VB}$-groupoid, and $v = v_1 \circ v_2$ where the base points are in M, then these base points are all the same, and $v = v_1 + v_2 - \mathsf{s}(v_1)$.

Exercise 4.2. Using the preceding exercise, give explicit formulas for the source and target map of $\mathcal{V}^*$. (Start with $\mathcal{V}^*|_M$.)

4.5. **Prolongations of groupoids.** Given a groupoid $\mathcal{G} \rightrightarrows M$, one can define new groupoids

$$J_k(\mathcal{G}) \rightrightarrows M,$$

the so-called *k-th prolongation* of $\mathcal{G}$. The points of $J_k(\mathcal{G})$ are k-jets of bisections of $\mathcal{G}$. Thinking of a bisection as a section $\sigma\colon M \to \mathcal{G}$ whose composition with the target map is a diffeomorphism, the source fiber of $J_k(\mathcal{G})$ consists of all k-jets of such sections at m, with the property that the composition with t is the k-jet of a diffeomorphism.

- For $k = 0$, one recovers $J_0(\mathcal{G}) = \mathcal{G}$ itself.
- Elements of $J_1(\mathcal{G})$ are pairs (g, W), where $g \in \mathcal{G}$ is an arrow and $W \subseteq T_g\mathcal{G}$ is a subspace complementary to both the source and target fibers. In other words, W is the tangent space to some bisection passing through g. The composition is induced from the composition of bisections, that is,
$$(g, W) = (g_1, W_1) \circ (g_2, W_2)$$
if and only if
$$W = T\,\mathrm{Mult}_{\mathcal{G}}((W_1 \times W_2) \cap T\mathcal{G}^{(2)})$$
(the linearized version for multiplication of bisections.)

The successive prolongations define a sequence of Lie groupoids

$$\cdots \to J_k(\mathcal{G}) \to J_{k-1}(\mathcal{G}) \to \cdots \to J_0(\mathcal{G}) = \mathcal{G}.$$

By applying this construction to the pair groupoid, one recovers the groupoids $J_k(M, M)$ discussed earlier. Prolongations of groupoids were introduced by Ehresmann [20]; recently they have been used in the work of Crainic, Salazar, and Struchiner [13] on *Pfaffian groupoids* and *Spencer operators.*

4.6. **Pull-backs and restrictions of groupoids.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Given a submanifold $i\colon N \hookrightarrow M$ such that the map $(\mathsf{t}, \mathsf{s})\colon \mathcal{G} \rightrightarrows \mathrm{Pair}(M)$ is transverse to $\mathrm{Pair}(N) \subseteq \mathrm{Pair}(M)$, one obtains a new groupoid $i^!\mathcal{G} \rightrightarrows N$ by taking the pre-image

$$i^!\mathcal{G} = (\mathsf{t}, \mathsf{s})^{-1}(\mathrm{Pair}(N)).$$

The groupoid multiplication is simply the restriction of that of $\mathcal{G}$. More generally, suppose $f\colon N \to M$ is a smooth map such that the induced map $\mathrm{Pair}(f)\colon \mathrm{Pair}(N) \to \mathrm{Pair}(M)$ is transverse to (t, s). Then we define a pull-back groupoid $f^!\mathcal{G} \rightrightarrows N$ by

$$f^!\mathcal{G} = \{(g, n', n)|\ \mathsf{s}(g) = f(n),\ \mathsf{t}(g) = f(n')\}.$$

Its groupoid structure is that as a subgroupoid of $\mathcal{G}\times \mathrm{Pair}(N)$ over $N\cong \mathrm{Gr}(f)\subseteq M\times N$. Note

$$\dim f^!\mathcal{G} = \dim \mathcal{G} + 2\dim N - 2\dim M,$$

and also that

$$f^!\,\mathrm{Pair}(M) = \mathrm{Pair}(N).$$

By construction, the pull-back groupoid comes with a morphism of Lie groupoids

$$\begin{tikzcd} f^!\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & N \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Remark 4.11. In the definition of $f^!\mathcal{G}$, one can weaken the transversality assumption to *clean intersection* assumptions. However, the dimension formula for $f^!\mathcal{G}$ has to be modified in that case, adding the *excess* of the clean intersection to the right hand side.

Exercise 4.3. Show that under composition of maps,

$$(f_1\circ f_2)^!\mathcal{G} = f_2^! f_1^!\mathcal{G}.$$

provided that the clean intersection hypotheses are satisfied.

Exercise 4.4. Let $\pi\colon P\to M$ be a principal bundle, $f\colon N\to M$ a smooth map, and $f^*P\to N$ the pull-back bundle (this is the subbundle of $P\times N\to M\times N$ along $\mathrm{Gr}(f)\cong N$, consisting of all (p,n) such that $\pi(p)=f(n)$). Show that

$$f^!\mathcal{G}(P) = \mathcal{G}(f^*P).$$

4.7. **A result on subgroupoids.** .

Theorem 4.12. *Let $\mathcal{G}\rightrightarrows M$ be a Lie groupoid, and $\mathcal{H}\rightrightarrows N$ a set-theoretic subgroupoid. If $\mathcal{H}$ is a submanifold of $\mathcal{G}$, and the source fibers of $\mathcal{H}$ are connected, then $\mathcal{H}$ is a Lie subgroupoid.*

Our proof use the following Lemma from differential geometry (see e.g. [26])

Lemma 4.13 (Smooth retractions)**.** *Let Q be a manifold, and $p\colon Q\to Q$ a smooth map such that $p\circ p = p$. Then $p(Q)$ is a submanifold, and admits an open neighborhood in Q on which the map p is s surjective submersion onto $p(Q)$.*

Remark 4.14.
(a) If Q is connected, then $p(Q)$ is connected. If Q is disconnected, then $p(Q)$ can have several connected components of different dimensions.
(b) In general, the smooth retraction p need not be a submersion *globally*, even when Q is compact and connected.

Proof of Theorem 4.12. We denote by $\mathsf{s},\mathsf{t}\colon \mathcal{G}\to M$ the source and target map of $\mathcal{G}$, by $i\colon M\to\mathcal{G}$ the inclusion of units, and by Mult: $\mathcal{G}^{(2)}\to\mathcal{G}$ the groupoid multiplication. For the corresponding notions of $\mathcal{H}$, we will put a subscript $\mathcal{H}$. We have to show:

(a) N is a submanifold of $\mathcal{H}$,

(b) $\mathsf{s}_\mathcal{H}, \mathsf{t}_\mathcal{H} \colon \mathcal{H} \to N$ are submersions,
(c) $\mathrm{Mult}_\mathcal{H} \colon \mathcal{H}^{(2)} \to \mathcal{H}$ is smooth.

The map $i_\mathcal{H} \circ \mathsf{s}_\mathcal{H} \colon \mathcal{H} \to \mathcal{H}$ is a retraction to the subset $N \subseteq \mathcal{H}$. It is smooth, since it is the restriction of the smooth map $i \circ \mathsf{s} \colon \mathcal{G} \to \mathcal{G}$. Hence, by the Lemma, N is a submanifold. For the rest of this argument, we can and will assume that the $\mathcal{H}$-orbit space of N is connected; hence N (which may be disconnected) has constant dimension.

The Lemma also tells us that there exists an open neighborhood of $i_\mathcal{H}(N)$ in $\mathcal{H}$ on which $\mathsf{s}_\mathcal{H}$ is a submersions onto N. By using a similar argument for $\mathsf{t}_\mathcal{H}$, we see that the same is true for $\mathsf{t}_\mathcal{H}$. In particular, some neighborhood Ω of N in $\mathcal{H}$ becomes a 'local Lie algebroid'. Define a left-action of Ω on $\mathcal{H}$, by the map

$$\Omega \;{}_{\mathsf{s}_\mathcal{H}}\!\times_{\mathsf{t}_\mathcal{H}} \mathcal{H},\ (k, g) \mapsto k \circ g.$$

Note that $\Omega \;{}_{\mathsf{s}_\mathcal{H}}\!\times_{\mathsf{t}_\mathcal{H}} \mathcal{H}$ is a smooth submanifold of $\mathcal{G}^{(2)}$, due to the fact that $\mathsf{s}_\mathcal{H}$ is a submersion over Ω, and that the action map is smooth since it is the restriction of $\mathrm{Mult}_\mathcal{G}$ to this submanifold.

If $\mathsf{s}_\mathcal{H}$ is a surjective submersion at some point $g \in \mathcal{H}$, then it is also a submersion at $k \circ g$, for any $k \in \Omega$. Since $\mathcal{H}$ is assumed to be source connected, any $g \in \mathcal{H}$ can be written as a product $k_1 \circ \cdots \circ k_N$ with $k_j \in \Omega$. This shows that $\mathsf{s}_\mathcal{H}$ is a submersion, and similarly $\mathsf{t}_\mathcal{H}$ is a submersion. $\square$

If we drop the assumption that $\mathcal{H}$ is source-connected, it need no longer be true that $\mathsf{s}_\mathcal{H}$ is a submersion everywhere:

Example 4.15. Take $\mathcal{G} = \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$ be the 1-dimensional trivial vector bundle over $\mathbb{R}$, regarded as a groupoid with $\mathsf{s}(x, y) = \mathsf{t}(x, y) = x$. Pick a function $y = f(x)$, taking values in positive real numbers, such that the graph of f is a smooth submanifold, but has a vertical tangent at some point (x_0, y_0). Then $\mathcal{H} = \{(x, kf(x)) |\ x \in \mathbb{R},\ k \in \mathbb{Z}\}$ is a set-theoretic subgroupoid which is not a Lie subgroupoid, since $\mathsf{s}_\mathcal{H}$ is not surjective at (x_0, y_0).

Remark 4.16. If the submanifold $\mathcal{H}$ is a set-theoretic subgroupoid of the Lie groupoid $\mathcal{G}$, with possibly disconnected source fibers, the its 'source component' is still a Lie subgroupoid.

4.8. **Clean intersection of submanifolds and maps.** Some of the subsequent results will depend on intersection properties of maps that are weaker than transversality. The following notion of clean intersection goes back to Bott [5, Section 5].

Definition 4.17 (Clean intersections)**.** (a) Two submanifold S_1, S_2 of a manifold M *intersect cleanly* if $S_1 \cap S_2$ is a submanifold, with

$$T(S_1 \cap S_2) = TS_1 \cap TS_2.$$

(b) A smooth map $F \colon N \to M$ between manifolds has *clean intersection with a submanifold* $S \subseteq M$ if $F^{-1}(S)$ is a submanifold of N, with

$$T_n\left(F^{-1}(S)\right) = (T_nF)^{-1}(T_{F(n)}S), \quad n \in N.$$

(c) Two smooth maps $F_1 \colon N_1 \to M$ and $F_2 \colon N_2 \to M$ *intersect cleanly* if the map

$$F_1 \times F_2 \colon N_1 \times N_2 \to M \times M$$

is clean with respect to the diagonal $\Delta_M \subseteq M \times M$.

Remarks 4.18. (a) One can show (see e.g. [25]) that at any point of a clean intersection of submanifolds S_1, S_2, there exist local coordinates in which the submanifolds are vector subspaces. One consequence of this is that for any two functions $f_i \in C^\infty(S_i)$, with

$$f_1|_{S_1\cap S_2} = f_2|_{S_1\cap S_2},$$

there exists a smooth function $f \in C^\infty(M)$ with

$$f|_{S_1} = f_1,\ f|_{S_2} = f_2.$$

More generally, given a vector bundle $V \to M$, and two sections of $\sigma_i \in \Gamma(V|_{S_i})$ with $\sigma_1|_{S_1\cap S_2} = \sigma_2|_{S_1\cap S_2}$, there exists $\sigma \in \Gamma(V)$ with $\sigma|_{S_i} = \sigma_i$.

(b) Note that $F\colon N \to M$ is clean with respect to $S \subseteq M$ if and only if its graph $\mathrm{Gr}(F) \subseteq M \times N$ has clean intersection with $S \times N$.

(c) As a special case, if S_1, S_2 have transverse intersection, in the sense that

$$T_m S_1 + T_m S_2 = T_m M$$

for all $m \in S_1 \cap S_2$, then the intersection is clean: it is automatic in this case that the intersection is a submanifold. Similarly, transversality of a map $F\colon N \to M$ to a submanifold $S \subseteq M$, in the sense that

$$\mathrm{ran}(T_n F) + T_{F(n)} S = T_{F(n)} M$$

for all $n \in N$) implies cleanness.

(d) For a clean intersection of submanifolds S_1, S_2, one calls the quantity

$$e = \dim(S_1 \cap S_2) + \dim(M) - \dim(S_1) - \dim(S_2)$$

the *excess* of the clean intersection. Thus, $e = 0$ if and only if the intersection is transverse. Similarly, one defines the excess of a clean intersection of two maps, or of a map with a submanifold.

Given a vector bundle $V \to M$, with a subbundle $W \to N$, we denote by $\Gamma(V, W) \subseteq \Gamma(V)$ the sections of V whose restriction to N takes values in W. As a special case, if 0_N is the zero bundle over N, then $\Gamma(V, 0_N)$ are the sections of V vanishing along N. We have the exact sequence,

$$0 \to \Gamma(V, 0_N) \to \Gamma(V, W) \to \Gamma(W) \to 0.$$

For example, if N is a submanifold of M, then $\Gamma(TM, TN)$ are the vector fields on M that ar tangent to N. Later we will need the following fact:

Lemma 4.19. *Suppose $W_i \to N_i$ are two vector subbundles of a vector bundle $V \to M$. If W_1, W_2 intersect cleanly (as manifolds), then the zero sections intersect cleanly, and $W_1 \cap W_2 \to N_1 \cap N_2$ is a vector subbundle of V. Furthermore, the map*

$$\Gamma(V, W_1) \cap \Gamma(V, W_2) \to \Gamma(W_1 \cap W_2)$$

is surjective.

Proof. The first part follows by using the Grabowski-Rotkievicz theorem, since the intersection is a submanifold by assumption, and since it is invariant under scalar multiplication. In particular, its zero sections $N_1 \cap N_2$ is a submanifold, where the intersection is clean:

$$T(N_1 \cap N_2) = T(W_1 \cap W_2) \cap TM = TW_1 \cap TW_2 \cap TM = TN_1 \cap TN_2.$$

For the second part, given a section $\sigma_{12} \in \Gamma(W_1 \cap W_2)$, extend to sections $\sigma_1 \in \Gamma(W_1)$ and $\sigma_2 \in \Gamma(W_2)$, and use Remark 4.18a to extend to a section σ of V. Then $\sigma \in \Gamma(V, W_1) \cap \Gamma(V, W_2)$, and $\sigma|_{N_1 \cap N_2} = \sigma_{12}$. □

4.9. **Intersections of Lie subgroupoids, fiber products.** .

Theorem 4.20 (Clean intersection of Lie subgroupoids)**.** *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{H}_i \rightrightarrows N_i,\ i = 1, 2$ two Lie subgroupoids with clean intersection. Then $\mathcal{H}_1 \cap \mathcal{H}_2$ is a Lie subgroupoid.*

Proof. The cleanness assumption means that $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2$ is a submanifold, with

$$T\mathcal{H} = T\mathcal{H}_1 \cap T\mathcal{H}_2.$$

Let s be the source map of $\mathcal{G}$, and s' its restriction to $\mathcal{H}$. By Theorem 4.12, the source components of $\mathcal{H}$ are Lie subgroupoids; in particular the space of units $N = N_1 \cap N_2$ is a submanifold. Furthermore, s' is a surjective submersion from some open neighborhood of N inside $\mathcal{H}$ onto N. To see that it is a surjective submersion everywhere on $\mathcal{H}$, we use right translation. Given $g \in \mathcal{G}$, with $\mathsf{s}(g) = m,\ \mathsf{t}(g) = m'$, we have that

$$\mathcal{A}^R_{g^{-1}} \colon \mathsf{s}^{-1}(m) \to \mathsf{s}^{-1}(m').$$

In particular, the tangent map preserves the source fibers, and hence

$$T_g\mathcal{A}^R_{g^{-1}} \colon \ker(T_g\mathsf{s}) \to \ker(T_{m'}\mathsf{s}). \tag{9}$$

Given $g \in \mathcal{H}$, the map $T_g\mathcal{A}^R_{g^{-1}}$ restricts to isomorphisms

$$T_g\mathcal{A}^R_{g^{-1}} \colon T_g\mathcal{H}_i \cap \ker(T_g\mathsf{s}) \to T_{\mathsf{s}(g)}\mathcal{H}_i \cap \ker(T_{m'}\mathsf{s})$$

for $i = 1, 2$, since both $]\mathcal{H}_1, \mathcal{H}_2$ are Lie subgroupoids. Using the clean intersection condition, we see that it restricts to an isomorphism

$$T_g\mathcal{A}^R_{g^{-1}} \colon T_g(\mathcal{H}_1 \cap \mathcal{H}_2) \cap \ker(T_g\mathsf{s}) \to T_{\mathsf{s}(g)}(\mathcal{H}_1 \cap \mathcal{H}_2) \cap \ker(T_{m'}\mathsf{s}),$$

that is,

$$T_g\mathcal{A}^R_{g^{-1}} \colon \ker(T_g\mathsf{s}') \to \ker(T_{m'}\mathsf{s}').$$

This shows that s' has constant rank globally, and likewise for t'. □

Corollary 4.21. *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Lie groupoids, and $F \colon \mathcal{G} \to \mathcal{H}$ a morphism of Lie groupoids. Suppose that $\mathcal{H}' \rightrightarrows N'$ is a Lie subgroupoid of $\mathcal{H}$, and let $\mathcal{G}' \rightrightarrows M'$ be its pre-image. If F is clean with respect to $\mathcal{H}'$, then $\mathcal{G}'$ is a Lie subgroupoid. In particular, this is true if F is transverse to $\mathcal{H}'$.*

Proof. We may regard the pre-image as the intersection of two Lie subgroupoids, $\mathrm{Gr}(F)\cap(\mathcal{H}'\times\mathcal{G})\subseteq\mathcal{H}\times\mathcal{G}$. ☐

Corollary 4.22. *Suppose $\mathcal{G}\rightrightarrows M$ and $\mathcal{H}\rightrightarrows N$ are Lie groupoids, and that $F\colon\mathcal{G}\to\mathcal{H}$ is a morphism of Lie groupoids. If F is clean with respect to N, then the kernel*

$$\ker(F)=F^{-1}(N)$$

is a Lie subgroupoid. In particular, this is true if F is transverse to N.

Corollary 4.23. *Let $\mathcal{G}_1\rightrightarrows M_1, \mathcal{G}_2\rightrightarrows M_2,\ \mathcal{H}\rightrightarrows N$ be Lie groupoids, and $F_i\colon\mathcal{G}_i\to\mathcal{H},\ i=1,2$ morphisms of Lie groupoids. Let*

$$\mathcal{G}=\mathcal{G}_1\times_{\mathcal{H}}\mathcal{G}_2\subseteq\mathcal{G}_1\times\mathcal{G}_2.$$

be the fiber product. If $\mathcal{G}$ is a submanifold, and if, for all $g=(g_1,g_2)\in\mathcal{G}$, the tangent space of $\mathcal{G}$ is the fiber product of $T_{g_1}\mathcal{G}_1$ and $T_{g_2}|G_2$ under the tangent maps of the F_i, then $\mathcal{G}$ is a Lie groupoid. In particular, this is true if F_1,F_2 are transverse.

Proof. We may interpret the fiber product as the pre-image of the diagonal $\Delta_{\mathcal{H}}$ under the map $F_1\times F_2$. The given assumption just means that $F_1\times F_2$ is clean with respect to $\Delta_{\mathcal{H}}$. ☐

Remark 4.24. This result, in the strong version stated here, is due to Bursztyn-Cabrera-del Hoyo [6]. Note that conversely, the result for fiber products implies Theorem **??**. Indeed, our proof of the Theorem **??** was motivated by the argument in [6].

Example 4.25. The pull-back construction may be re-phrased in these terms: Given $\mathcal{G}\rightrightarrows M$ and $f\colon N\to M$ such that $\mathrm{Pair}(f)$ is transverse to (t,s), we have that

$$f^!\mathcal{G}=\mathrm{Pair}(N)\times_{\mathrm{Pair}(M)}\mathcal{G}.$$

More generally, this holds if the two maps are clean with respect to each other.

4.10. **The universal covering groupoid.** Let $\mathcal{G}\rightrightarrows M$ be a source connected Lie groupoid. The source map $\mathsf{s}\colon\mathcal{G}\to M$ is a surjective submersion. Let $\widetilde{\mathcal{G}}$ be obtained by replacing each source fiber by its universal covering. That is,

$$\widetilde{\mathcal{G}}=\{[\gamma]|\ \gamma\colon[0,1]\to\mathcal{G} \text{ is an } \mathsf{s}\text{-foliation path with } \gamma(0)\in M\},$$

where $[\gamma]$ stands for homotopy classes of s-foliation paths, with fixed end points. (Put differently, letting $\mathcal{F}$ be the s-foliation of $\mathcal{G}$, we take the pre-image of $M\subseteq\mathcal{G}$ under the source map of $\mathrm{Mon}(\mathcal{F})\rightrightarrows\mathcal{G}$.) The space $\widetilde{\mathcal{G}}$ has a natural structure of a (possibly non-Hausdorff) manifold. The manifold structure is obtained from the inclusion as a submanifold of $\mathrm{Mon}(\mathcal{F})$. We emphasize that $\widetilde{\mathcal{G}}$ may be non-Hausdorff even if $\mathcal{G}$ is Hausdorff. Define source and target maps of $\widetilde{\mathcal{G}}$ by

$$\mathsf{s}([\gamma])=\gamma(0),\quad \mathsf{t}([\gamma])=\mathsf{t}(\gamma(1)),$$

and define the groupoid multiplication of composable elements as

$$[\gamma']\circ[\gamma]=[\gamma'']$$

where γ'' is a concatenation of the path γ with the $\mathcal{A}^R_{\gamma(1)}$-translate of the path γ': Thus

$$\gamma''(t) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2, \\ \gamma'(2t-1) \circ \gamma(1)^{-1} & 1/2 \le t \le 1 \end{cases}$$

Using these definitions, the space $\widetilde{\mathcal{G}}$ is a (possibly non-Hausdorff) Lie groupoid

$$\widetilde{\mathcal{G}} \rightrightarrows M.$$

It comes with a local diffeomorphism

$$\pi \colon \widetilde{\mathcal{G}} \to \mathcal{G},\ [\gamma] \mapsto \gamma(1),$$

which is a morphism of Lie groupoids.

5. Groupoid actions, groupoid representations

5.1. **Actions of Lie groupoids.** Generalizing Lie group actions, one can also consider *groupoid actions* of $\mathcal{G} \rightrightarrows M$ on other manifolds.

Definition 5.1. An action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a manifold Q is given by a map $\Phi \colon Q \to M$ together with an *action map*

$$\mathcal{A} \colon \mathcal{G} \times_M Q \to Q,\ (g, q) \mapsto \mathcal{A}_g(q) = g \cdot q$$

where $\mathcal{G} \times_M Q := \{(g, q) |\ \mathsf{s}(g) = \Phi(q)\}$. These are required to satisfy $\Phi(g \cdot q) = \mathsf{t}(g)$, as well as

$$(g_1 \circ g_2) \cdot q = g_1 \cdot (g_2 \cdot q), \quad m \cdot q = q$$

for $g_i \in \mathcal{G},\ q \in Q,\ m \in M$, and whenever these are defined.

The map Φ is sometimes called a *moment map* of the action (due to some relationship with the moment map in symplectic geometry), sometimes it is called an *anchor*. We will say that $\mathcal{G} \rightrightarrows M$ *acts on Q along M*.

For any $\mathcal{G}$-action, one can define its *orbits in Q* as the equivalence classes under the relation

$$q \sim q' \Leftrightarrow \exists g \in \mathcal{G} \colon q' = g \cdot q.$$

Also, for $q \in Q$ we can define its *isotropy group*

$$\mathcal{G}_q = \{g \in \mathcal{G} |\ g \cdot q = q\}.$$

Example 5.2. For a group G, one always has the trivial action on point. The generalization to groupoids is its action on the units $M = \mathcal{G}^{(0)}$; here $\Phi = \mathrm{id}_M$, and the action is $g \cdot m = m'$ for $\mathsf{s}(g) = m,\ tz(g) = m'$. Note that in general, there is no action of a groupoid on a point.

Example 5.3. Every Lie groupoid $\mathcal{G} \rightrightarrows M$ acts on itself by left multiplication $g \cdot a = l_g(a) = g \circ a$ (here $\Phi = \mathsf{t}$), and by right multiplication $g \cdot a = r_{g^{-1}}(a) = a \circ g^{-1}$ (here $\Phi = \mathsf{s}$). These two actions commute, and combine into an action of $\mathcal{G} \times \mathcal{G} \rightrightarrows M \times M$ on $\mathcal{G}$ (with $(\Phi = (\mathsf{t}, \mathsf{s}))$. On the other hand, there is no natural *adjoint action* of $\mathcal{G}$ on itself (although we do have an 'adjoint action' of the group $\Gamma(\mathcal{G})$ of bisections on $\mathcal{G}$).

Remark 5.4. Given an action of $\mathcal{G} \rightrightarrows M$ on $\Phi\colon Q \to M$, and a morphism of Lie groupoids from $\mathcal{H} \rightrightarrows N$ to $\mathcal{G} \rightrightarrows M$, there is no natural way, in general, of producing an $\mathcal{H}$-action on Q (unless the map on units $N \to M$ is a diffeomorphism). For example, in the case of the $\mathcal{G} \times \mathcal{G}$-action on $\mathcal{G}$, there is no natural way of passing to a diagonal action, in general.

Remark 5.5. Note that any $\mathcal{G}$-action on Q determines an action of the group of bisections $\Gamma(\mathcal{G})$ on Q, where the bisection S takes $q \in Q$ to $g \times q$, for the unique $g \in S$ such that this composition is defined (i.e., $\mathsf{s}(g) = \Phi(q)$). More generally, there is an action of the local bisections,

$$\mathcal{A}_S\colon \Phi^{-1}(U) \to \Phi^{-1}(V)$$

where $U = \mathsf{s}(S),\ V = \mathsf{t}(S)$.

Remark 5.6. A groupoid action is fully determined by its graph $\mathrm{Gr}(\mathcal{A}_Q) \subseteq Q \times (\mathcal{G} \times Q)$; for example, $\Phi(q) \in M$ is the unique unit such that $q = \Phi(q) \cdot q$.

Given a groupoid action of $\mathcal{G} \rightrightarrows M$ on $Q \to M$, one can again form an action groupoid

$$\mathcal{G} \ltimes Q \rightrightarrows Q$$

as the subgroupoid of $\mathcal{G} \times \mathrm{Pair}(Q) \rightrightarrows M \times Q$ consisting of all (g, q', q) such that $q' = g \circ q$. The action groupoid comes with a groupoid morphism

$$\begin{array}{ccc} \mathcal{G} \ltimes Q & \rightrightarrows & Q \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}$$

where the left vertical map is $(g, q) \mapsto g$. The orbits and isotropy groups of the action groupoid are the orbits and isotropy groups for the $\mathcal{G}$-action on Q.

Remark 5.7. Suppose conversely that we are given a groupoid morphism ϕ from $\mathcal{H} \rightrightarrows Q$ to $\mathcal{G} \rightrightarrows M$ such that the map $(\phi, \mathsf{s})\colon \mathcal{H} \to \mathcal{G} \times_M Q$ is a diffeomorphism. Then $\mathcal{H}$ is the action groupoid for a $\mathcal{G}$-action on Q, in such a way that the action map is identified with the target map for $\mathcal{H}$. The stabilizers for the action are the isotropy groups of $\mathcal{H}$.

5.2. **Principal actions.** A special case of a groupoid action is a *principal action.* If G is a Lie group, a principal bundle is a G-manifold P for which there exists a surjective submersion $\kappa\colon P \to B$ onto another manifold B, such that the fibers of κ are exactly the G-orbits, and the action is free. These condition may be restated as the assertion that the map

$$G \times P \to P \times_B P,\ \ (g, p) \mapsto (g \cdot p, p)$$

is a diffeomorphism. (In other words, the action groupoid $G \ltimes P \rightrightarrows P$ is identified with the foliation groupoid). This definition has a direct analogue for groupoids:

Definition 5.8 (Moerdijk-Mcrun [31])**.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A principal $\mathcal{G}$-bundle is given by a manifold P with a surjective submersion $\kappa\colon P \to B$, together with a $\mathcal{G}$-action on P along a map $\Phi\colon P \to M$, such that

(a) $\kappa(g \cdot p) = \kappa(p)$ whenever $\mathsf{s}(g) = \Phi(p)$,
(b) the map
$$\mathcal{G} \ltimes P \to P \times_B P, \quad (g, p) \mapsto (g \cdot p, p)$$
is a diffeomorphism.

Morphisms of principal $\mathcal{G}$-bundles $\kappa_i P_i \to B_i$ are $\mathcal{G}$-equivariant maps $P_1 \to P_2$, intertwining the maps κ_i with respect to some base map $B_1 \to B_2$.

As a consequence of (b), the isotropy groups for a principal action are trivial: $\mathcal{G}_p = \{e\}$ for all $p \in P$.

Example 5.9. $\mathcal{G}$ itself is a principal $\mathcal{G}$-bundle for
$$\Phi(h) = \mathsf{s}(h),\ \kappa(h) = \mathsf{t}(h),\ g.h = hg^{-1}.$$

Given a principal $\mathcal{G}$-bundle $\kappa \to B$, with moment map $\Phi\colon P \to \mathcal{G}^{(0)}$, and any smooth map $f\colon B' \to B$, the fiber product of P with B' over B becomes a principal $\mathcal{G}$-bundle, called the *pull-back*:
$$\kappa'\colon f^*P = B' \times_B P \to B'.$$
Here $\kappa'(b', p) = b',\ \Phi'(b', p) = \Phi(p),\ g.(b', p) = (b', g \cdot p)$.

Example 5.10. As a special case, every smooth map $f\colon B \to \mathcal{G}^{(0)}$ defines a *trivial principal G-bundle* by pulling back $\mathcal{G}$ itself:
$$B \times_{\mathcal{G}^{(0)}} \mathcal{G},$$
with
$$\Phi(b, h) = \mathsf{t}(h), \quad \kappa(b, h) = b, \quad g \cdot (b, h) = (b, hg^{-1}).$$
Principal bundles of this type are regarded as *trivial*.

As in the case of principal bundles for Lie groups, if a principal $\mathcal{G}$-bundle admits a section $\sigma\colon B \to P$, then P is identified with the trivial bundle relative to the map $f = \mathsf{t} \circ \sigma$. Explicitly, this map is
$$B \times_{\mathcal{G}^{(0)}} \mathcal{G} \to P,\ (b, h) \mapsto h^{-1} \cdot \sigma(b).$$

For principal bundles of Lie groups, one has a notion of *associated bundle* $P \times_G S$ for any G-manifold S on which G acts. It is the quotient of $P \times S$ under the diagonal action. For general Lie groupoids $\mathcal{G} \rightrightarrows M$, there is no notion of diagonal action, in general. However, given a $\mathcal{G}$-manifold Q with a $\mathcal{G}$-equivariant submersion $Q \to P$, one has that Q is a principal $\mathcal{G}$-bundle, and its space of orbits may be regarded as an analogue of the associated bundle construction. See Moerdijk-Mcrun for more details.

5.3. **Representations of Lie groupoids.** A (linear) *representation* of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a vector bundle $\pi \colon V \to M$ is a $\mathcal{G}$-action on V along π such that the action map

$$\mathcal{G} \times_M V \to V,\ (g, v) \mapsto g \cdot v$$

is a vector bundle map. Equivalently, the action groupoid

$$\mathcal{G} \ltimes V \rightrightarrows V$$

is a $\mathcal{VB}$-groupoid. A representation of $\mathcal{G}$ on V gives a family of linear isomorphisms between fibers

$$\phi_g \colon V_{\mathsf{s}(g)} \to V_{\mathsf{t}(g)}$$

with the property that $\phi_{g_1 \circ g_2} = \phi_{g_1} \circ \phi_{g_2}$. Recall that for any vector bundle, we defined the Atiyah groupoid $\mathcal{G}(V) \rightrightarrows M$ to consist of triples (m', m, ϕ) with $\phi \colon V_m \to V_{m'}$. We may hence also regard a representation to be a Lie groupoid morphism

$$\mathcal{G} \to \mathcal{G}(V).$$

Examples:

- A linear representation of the pair groupoid $\mathrm{Pair}(M)$ on $\mathcal{V}$ is the same as a trivialization of V. Indeed, the action map gives consistent identifications of the fibers.
- A representation of the homotopy groupoid $\Pi(M)$ on $\mathcal{V}$ is the same as a flat connection on $\mathcal{V}$: Any element of $\Pi(M)$ gives a 'parallel transport'.
- Given a G-action on M, a representation of the action groupoid $G \ltimes M$ is the same as G-equivariant vector bundle $V \to M$ (lifting the given action on the base).
- Given a foliation $\mathcal{F}$ of M, let $\nu(M, \mathcal{F})$ be the normal bundle of the foliation. This normal bundle comes with a natural representation of the holonomy groupoid $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$: Given $g = (m', m, [\gamma])$, and given transversals N, N' at m, m', the tangent map to the holonomy gives an isomorphism $T_m N \to T_{m'} N'$; under the identification with normal bundles this is the desired representation.

6. Lie algebroids

The infinitesimal counterpart to Lie groupoids was introduced by Pradines in the 1960s.

6.1. **Definitions.** .

Definition 6.1. A *Lie algebroid over M* is a vector bundle $A \to M$, together with a Lie bracket $[\cdot, \cdot]$ on its space of sections, such that there exists a vector bundle map

$$\mathsf{a} \colon A \to TM$$

called the *anchor map* satisfying the *Leibniz rule*

$$[\sigma, f\tau] = f[\sigma, \tau] + (\mathsf{a}(\sigma) f)\ \tau$$

for all $\sigma, \tau \in \Gamma(A)$ and $f \in C^\infty(M)$.

Remark 6.2. (a) If an anchor map satisfying the Leibniz rule exists, it is unique. (Exercise.)

(b) In the original definition, it was also assumed that the anchor map a induces a Lie algebra morphism
$$\mathsf{a}\colon \Gamma(E) \to \mathfrak{X}(M).$$
Later, it was noticed that this is automatic. (Exercise.)

6.2. **Examples.**

Example 6.3. A Lie algebroid over a point $M = \mathrm{pt}$ is the same as a Lie algebra.

Example 6.4. The tangent bundle, with its usual bracket on sections, is a Lie algebroid with a the identity.

Example 6.5. Suppose E is a Lie algebra bundle, that is, a vector bundle whose fibers have Lie algebra structures, and with local trivializations respecting the Lie algebra structures. Then E with the pointwise Lie bracket and zero anchor is a Lie algebroid. Conversely, if E is a Lie algebroid with zero anchor, then the fibers inherit Lie brackets such that $[\sigma, \tau](m) = [\sigma(m), \tau(m)]$ for all $m \in M$. However, the Lie algebras for different fibers need not be isomorphic.

Example 6.6. Given a foliation $\mathcal{F}$ of a manifold M, one has the Lie algebroid $T_{\mathcal{F}}M$ given by the tangent bundle of the foliation.

Example 6.7. An action of a Lie algebra $\mathfrak{k}$ on M is, by definition, a vector bundle homomorphism $\mathfrak{k} \to \mathfrak{X}(M),\ X \mapsto X_M$ such that the action map $M \times \mathfrak{k} \to TM,\quad (m, X) \mapsto X_M(m)$ is smooth. It defines an *action Lie algebroid*
$$A = \mathfrak{k} \ltimes M,$$
where the anchor map is given by the action map, and the bracket is the unique extension of the given Lie bracket on constant section determined by the Leibnitz rule. That is, if $X, Y\colon M \to \mathfrak{k}$ (viewed as sections of A)
$$[X, Y] = [X, Y]_{\mathfrak{k}} + \mathcal{L}_{\mathsf{a}(X)}Y - \mathcal{L}_{\mathsf{a}(Y)}X.$$
Here the subscript $\mathfrak{k}$ indicates the pointwise bracket $[X, Y]_{\mathfrak{k}}(m) = [X(m), Y(m)]$.

Example 6.8. For any principal K-bundle $\kappa\colon P \to M$, the bundle
$$A(P) = (TP)/K \to M$$
is a Lie algebroid, called the *Atiyah algebroid* of P. The bracket on the space of sections $\Gamma(A)$ is induced from its identification with K-invariant vector field on P, while the anchor map is induced by the bundle projection $T\kappa\colon TP \to TM$. This Lie algebroid fits into an exact sequence of vector bundles (in fact, of Lie algebroids), called the *Atiyah sequence*
$$0 \to \mathfrak{gau}(P) \to A(P) \to TM \to 0$$
where $\mathfrak{gau}(P) = (P \times \mathfrak{k})/K$ is the quotient of the vertical bundle $T_vP \cong P \times \mathfrak{k}$ by K; it is the bundle of infinitesimal gauge transformations. A splitting $j\colon TM \to A(P)$ of the Atiyah sequence is equivalent to a principal bundle connection. Given two vector fields X, Y, since $[j(X), j(Y)]$ is a lift of $[X, Y]$, the difference $[j(X), j(Y)] - j([X, Y])$ is a section of the bundle $\mathfrak{gau}(P)$. The resulting 2-form
$$F \in \Omega^2(M, \mathfrak{gau}(P)), \quad F(X, Y) = [j(X), j(Y)] - j([X, Y])$$
is the curvature form of the connection.

A Lie algebroid whose anchor map is a surjection onto TM is called a *transitive Lie algebroid*. Thus, Atiyah algebroids are transitive. Not every transitive Lie algebroid corresponds to a globally defined principal bundle, though.

Example 6.9. Let $\omega \in \Omega^2(M)$ be a 2-form on M, and let $A = TM \times \mathbb{R}$ with the following bracket on sections $\Gamma(A) = \mathfrak{X}(M) \oplus C^\infty(M)$,

$$[X + f, Y + g] = [X, Y] + X(g) - Y(f) + \omega(X, Y),$$

and with anchor the projection to TM. Then A is a Lie algebroid if and only if ω is closed. Note that it is a transitive Lie algebroid. As we will explain later, this Lie algebroid corresponds to a principal $\mathbb{R}/\Lambda$-bundle (for Λ a discrete subgroup of $\mathbb{R}$) if and only if for all $f\colon S^2 \to M$, the integral

$$\int_{S^2} f^*\omega$$

lies in Λ. Hence, if the set of all such integrals (for all maps $f\colon S^2 \to \mathbb{R}$) is dense in $\mathbb{R}$, then no such principal bundle, for *any* Λ, can exist.

Example 6.10. Given a hypersurface $N \subseteq M$, there is a Lie algebroid $A \to M$ whose space of sections $\Gamma(A)$ are the vector fields tangent to N. (In local coordinates $x^1, \ldots, x^n$, with N corresponding to $x^n = 0$, the space of such vector fields is generated as a module over $C^\infty(M)$ by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}},\ x^n \frac{\partial}{\partial x^{n-1}}.$$

This Lie algebroid was considered by Melrose for manifolds with boundary, in his so-called b-calculus.

Example 6.11 (Jet prolongations). For any vector bundle $V \to M$, one has its k-th jet prolongation $J^k(V) \to M$. The fiber of $J^k(V)$ at m consists of k-jets of sections $\sigma \in \Gamma(V)$, that is, equivalence classes of sections, where two sections are equivalent if their Taylor expansions up to order k agree. The jet bundles come with a tower of bundle maps

$$\cdots J^k(V) \to J^{k-1}(V) \to \cdots \to J^0(V) = V.$$

For any section σ of V, one obtains a section $j^k(\sigma)$ of $J^k(V)$, and these generate $J^k(V)$ as a $C^\infty(M)$-module. Taking V to be a Lie algebroid (for example, $A = TM$), one finds that $J^k(A)$ has a unique Lie algebroid structure such that

$$[j^k(\sigma), j^k(\tau)] = j^k([\sigma, \tau]), \quad \mathsf{a}(j^k(\sigma)) = \mathsf{a}(\sigma)$$

for all sections $\sigma, \tau \in \Gamma(A)$.

Remark 6.12. Suppose $A \to M$ is a vector bundle with an anchor map $\mathsf{a}\colon A \to TM$, and with a skew-symmetric 'bracket' $[\cdot, \cdot]$ on $\Gamma(A)$ satisfying the Leibnitz rule. Then the 'Jacobiator'

$$\mathrm{Jac}(\sigma_1, \sigma_2, \sigma_3) = [\sigma_1, [\sigma_2, \sigma_3]] + [\sigma_2, [\sigma_3, \sigma_1]] + [\sigma_3, [\sigma_1, \sigma_2]]$$

is $C^\infty(M)$-linear in each entry. (Exercise.) That is, it defines a tensor $\mathrm{Jac} \in \Gamma(\wedge^3 A^*)$. In particular, to verify whether $\mathrm{Jac} = 0$ over some open subset $U \subseteq M$, it suffices to check on generators for $\Gamma(A|_U)$.

6.3. **Lie subalgebroids.** We will make use of the following notation, for a vector bundle $V \to M$ with given subbundle $W \to N$:

$$\Gamma(V,W) = \{\sigma \in \Gamma(V)|\ \sigma|_N \in \Gamma(W)\}.$$

The map $\Gamma(V,W) \to \Gamma(W)$ is surjective, with kernel $\Gamma(V,0_N)$ the sections of V whose restriction to N vanishes.

Definition 6.13. A subbundle $B \to N$ of a Lie algebroid $A \to M$ is called a *Lie subalgebroid* if it has the following two properties:

(a) $\mathsf{a}(B) \subseteq TN$,
(b) $\Gamma(A,B) \subseteq \Gamma(A)$ is a Lie subalgebra.

Proposition 6.14. *A Lie subalgebroid B of A inherits a unique Lie algebroid structure, in such a way that the restriction map $\Gamma(A,B) \to \Gamma(B)$ preserves Lie brackets.*

Proof. We have to show that $\Gamma(A,0_N)$ is an ideal in $\Gamma(A,B)$. Let $\sigma \in \Gamma(A,B)$. The space $\Gamma(A,0_N)$ is spanned by products $f\tau$, where $\tau \in \Gamma(A)$ and where $f \in C^\infty(M)$ vanishes along N. The Leibnitz rule

$$[\sigma, f\tau] = f[\sigma,\tau] + (\mathsf{a}(\sigma)f)\ \tau$$

shows that the bracket lies in $\Gamma(A,0_N)$, since both f and $\mathsf{a}(\sigma)f$ vanish along N. This proves the claim, and hence $\Gamma(B) = \Gamma(A,B)/\Gamma(A,0_N)$ inherits a Lie bracket. The Leibnitz rule for $\Gamma(A)$ implies a Leibnitz rule for B, with anchor the restriction of a to B. $\square$

Remark 6.15. A similar argument shows that for $\mathrm{rank}(B) < \mathrm{rank}(A)$, the first condition in Definition 6.13 is redundant. (On the other hand, for $\mathrm{rank}(B) = \mathrm{rank}(A)$, the second condition is redundant.)

Example 6.16. If $N \subseteq M$ is a submanifold, then $TN \subseteq TM$ is a sub-Lie algebroid.

Example 6.17. The tangent bundle to a foliation $\mathcal{F}$ is a Lie subalgebroid $T_\mathcal{F}M \subseteq TM$.

Example 6.18. Let $A \to M$ be a Lie algebroid, and $N \subseteq M$ a submanifold. If $B := \mathsf{a}^{-1}(TN)$ is a submanifold, then it is a vector subbundle (by the GR Lemma), and is in fact a Lie subalgebroid. Its sections are all restrictions $\sigma|_N$ of sections $\sigma \in \Gamma(A)$ such that $\mathsf{a}(\sigma)$ is tangent to N. That is, it is the pre-image of $\Gamma(TM,TN)$ under a. Letting $i\colon N \to M$ be the inclusion, we will also use the notation

$$i^! A \to N$$

for this Lie algebroid. The condition that $\mathsf{a}^{-1}(TN)$ be a subbundle holds true, for example, if N is transverse to the anchor. (I.e., $\mathsf{a}(A_m) + T_mN = T_mM$ for all $m \in N$.) In this case, we have that

$$\mathrm{rank}(i^!A) = \mathrm{rank}(A) - \dim M + \dim N.$$

The transversality conditions is automatic if A is a transitive Lie algebroid. In the special case $A = TM$, we obtain

$$i^!TM = TN.$$

More generally, for an Atiyah groupoid $A = A(P)$ of a principal K-bundle,

$$i^! A(P) = A(P|_N).$$

Example 6.19. If $A \to M$ is any Lie algebroid, and $m \in M$, the kernel of the anchor at m

$$\mathfrak{g}_m := \ker(a)|_m$$

is a Lie algebroid over $\{m\}$. As a Lie algebroid over a point, it is a Lie algebra. This is the *isotropy Lie algebra* of A at m.

Example 6.20. More generally, if $N \subseteq M$ is a submanifold such that $\ker(\mathsf{a})|_N$ is a submanifold of A, then this submanifold (the union of isotropy Lie algebras of points in N) is a Lie subalgebroid. In particular, for any transitive Lie algebroid $A \to M$ the kernel of the anchor map is a Lie subalgebroid $\ker(\mathsf{a}) \subseteq A$.

Suppose $A \to M$ is a Lie algebroid, and $B \subseteq A$ is an anchored subbundle along $N \subseteq M$, that is, $\mathsf{a}(B) \subseteq TN$. Then B is a Lie subalgebroid if and only if the bracket of any two sections of A which restrict to sections of B, is again a section which restricts to a section of B. Fortunately, it is not necessary to check this condition for *all* sections.

Lemma 6.21. *Suppose $A \to M$ is a Lie algebroid, and $B \subseteq A$ is an anchored subbundle along $N \subseteq M$. Suppose that we are given a subset $\mathcal{R} \subseteq \Gamma(A, B)$, whose image in $\Gamma(B)$ generates $\Gamma(B)$ as a $C^\infty(N)$-module. Then B is a Lie subalgebroid if and only if*

$$[\mathcal{R}, \mathcal{R}] \subseteq \Gamma(A, B). \tag{10}$$

Proof. By the Leibnitz rule, the condition (10) on $\mathcal{R}$ is equivalent to a similar property for the $C^\infty(M)$-submodule generated by $\mathcal{R}$. We may hence assume that $\mathcal{R}$ is a $C^\infty(M)$-submodule, which hence surjects onto all of $\Gamma(B)$. Consequently,

$$\Gamma(A, B) = \mathcal{R} + \Gamma(A, 0_N).$$

Since $\Gamma(A, B), \Gamma(A, 0_N) \subseteq \Gamma(A, 0_N)$ (which holds regardless of whether $\Gamma(A, B)$ is a Lie subalgebra), we see that $\Gamma(A, B)$ is a Lie subalgebra if and only if (10) holds. ☐

Example 6.22. If A is a Lie algebroid, and B is a subbundle of rank 1 along $N \subseteq M$, then B is a Lie subalgebroid if and only if $\mathsf{a}(B) \subseteq TN$. Indeed, compatibility with the bracket is automatic, since we may (locally) take $\mathcal{R}$ to consist of a single section.

6.4. **Intersections of Lie subalgebroids.** In general, the intersection of two Lie subalgebroids need not be a Lie subalgebroid, even if the intersection is smooth:

Example 6.23. Consider two foliations $\mathcal{F}_\pm$ of $\mathbb{R}^2$, given by the curves $y = a \pm x^2$ with $a \in \mathbb{R}$. Let $B_\pm \subseteq T\mathbb{R}^2$ be the tangent bundles of these foliations. Then

$$B_+ \cap B_- = B_+|_S = B_-|_S$$

where $S \subseteq \mathbb{R}^2$ is the y-axis. However, this restriction is not a Lie subalgebroid, since $\mathsf{a}(B_+|_S) \not\subseteq TS$.

However, a clean intersection assumption is all that is needed:

Theorem 6.24 (Clean intersection of Lie subalgebroids)**.** *Suppose $A \to M$ is a Lie algebroid, and $B_1 \to N_1,\ B_2 \to N_2$ are two Lie subalgebroids. If B_1, B_2 intersect cleanly, then $B_1 \cap B_2$ is again a Lie subalgebroid of A.*

Proof. As discussed in Section 4.8, as a clean intersection of two vector subbundles, $B_1 \cap B_2$ is again a vector subbundle, and the map

$$\Gamma(A, B_1) \cap \Gamma(A, B_2) \to \Gamma(B_1 \cap B_2)$$

is *surjective*. Furthermore,

$$\mathsf{a}(B_1 \cap B_2) \subseteq \mathsf{a}(B_1) \cap \mathsf{a}(B_2) \subseteq TN_1 \cap TN_2 = T(N_1 \cap N_2).$$

Since both $\Gamma(A, B_i)$ are Lie subalgebras, their intersection is a Lie subalgebra. Hence, Remark 6.21 applies, and shows that $B_1 \cap B_2$ is a Lie subalgebroid. □

6.5. **Direct products of Lie algebroids.** .

Lemma 6.25. *Given two Lie algebroids $A \to M$ and $B \to N$, their direct product*

$$A \times B \to M \times N$$

has a unique Lie algebroid structure, with anchor the direct product of the anchors, in such a way that the map

$$\Gamma(A) \oplus \Gamma(B) \to \Gamma(A \times B),\ (\sigma, \tau) \mapsto \mathrm{pr}_M^* \sigma + \mathrm{pr}_N^* \tau$$

is a Lie algebra homomorphism. Here $\mathrm{pr}_M \colon M \times N \to M$ and $\mathrm{pr}_N \colon M \times N \to N$ are the two projections, and $\mathrm{pr}_M^* \colon \Gamma(A) \to \Gamma(\mathrm{pr}_M^* A) \subseteq \Gamma(A \times B)$ and $\mathrm{pr}_N^* \colon \Gamma(B) \to \Gamma(\mathrm{pr}_N^* B) \subseteq \Gamma(A \times B)$ the pull-back maps.

Proof. Uniqueness is clear since $\Gamma(A) \oplus \Gamma(B)$ generates $\Gamma(A \times B)$ as a module over $C^\infty(M \times N)$.

To show existence, we write the bracket locally: Let $\sigma_1, \ldots, \sigma_k$ be a local basis of sections of A, $\tau_1, \ldots, \tau_l$ a local basis of sections of B. Together, this form a local basis ϵ_i of sections of $A \times B$. Then all brackets $[\epsilon_i, \epsilon_j]$ are determined, and the Leibniz rule forces us to put

$$[\sum_i f_i \epsilon_i, \sum_j g_j \epsilon_j] = \sum_{ij} \big(f_i g_j\, [\epsilon_i, \epsilon_j] + f_i\, (\mathsf{a}(\epsilon_i) g_j) \epsilon_j - (\mathsf{a}(\epsilon_j) f_i) g_j \epsilon_i \big)$$

for all $f_i, g_j \in C^\infty(M \times N)$. By Remark 6.12, this is a Lie bracket. □

7. Morphisms of Lie algebroids

7.1. **Definition of morphisms.**

The definition of a morphism of Lie algebroids is not entirely obvious, due to the fact that a vector bundle map does not induce a map of sections, in general. Instead, we will characterize Lie algebroid morphisms in terms of their graphs.

Definition 7.1. Let $A \to M$ and $B \to N$ be two Lie algebroids. A morphism of Lie algebroids $\phi\colon B \to A$ is a vector bundle morphism whose graph,

$$\mathrm{Gr}(\phi) \subseteq A \times B$$

is a Lie subalgebroid of the direct product.

Note that a Lie algebroid morphism is in particular a morphism of anchored vector bundles. That is, the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ TN \arrow[r, "TF"'] & TM \end{tikzcd}$$

where $F\colon N \to M$ is the base map of ϕ.

Example 7.2. For any smooth map $F\colon N \to M$ of manifolds, the tangent map

$$TF\colon TN \to TM$$

is a morphism of Lie algebroids. To see this, it suffices to note that $\mathrm{Gr}(TF) = T\,\mathrm{Gr}(F)$, under the identification $TM \times TN \cong T(M \times N)$. Conversely, if $\phi\colon TN \to TM$ is a morphism of Lie algebroids, then $\phi = TF$, just from the compatibility with anchor maps.

Example 7.3. If $B \to N$ is a Lie subalgebroid of $A \to M$, the inclusion map $\phi\colon B \to A$ is a morphism of Lie algebroids. To see that $\mathrm{Gr}(\phi)$ is a Lie subalgebroid, note that

$$\mathcal{R} = \{\sigma \times \sigma|_N |\ \sigma \in \Gamma(A, B)\} \subseteq \Gamma(A \times B, \mathrm{Gr}(\phi))$$

surjects onto $\mathrm{Gr}(\phi)$, and is closed under the Lie bracket. As a special case, the diagonal inclusion $A \to A \times A$ is a Lie algebroid morphism.

Example 7.4. Given two Lie algebroids A, B the two projections

$$\mathrm{pr}_A\colon A \times B \to A, \quad \mathrm{pr}_B\colon A \times B \to B$$

are Lie algebroid morphisms. For pr_A this follows because $\Delta_A \times B$ is a Lie subalgebroid of $A \times A \times B$; the argument for pr_B is similar.

Example 7.5. Let $A \to M,\ B \to N$ be Lie algebroids, and $\phi\colon B \to A$ a vector bundle morphism intertwining anchors. If the base map $F\colon N \to M$ is a diffeomorphism, then we have an induced map on sections

$$\phi_*\colon \Gamma(B) \to \Gamma(A), \quad \phi_*\tau = \phi \circ \tau \circ F^{-1}.$$

ϕ is a Lie algebroid morphism if and only if this map on sections intertwines the Lie brackets. This follows from the criterion, by taking

$$\mathcal{R} = \{\phi_*(\tau) \times \tau |\ \tau \in \Gamma(B)\} \subseteq \Gamma(A \times B, \mathrm{Gr}(\phi))$$

Example 7.6. In particular, the anchor map $\mathsf{a}\colon A \to TM$ is a morphism of Lie algebroids. Also, the natural maps between jet prolongations $J^k(A) \to J^{k-1}(A)$ are morphisms of Lie algebroids.

Example 7.7. Let $A \to M$ be a Lie algebroid. If N is a 1-dimensional manifold, then a Lie algebroid morphism $\phi\colon TN \to A$ is just a morphism of anchored vector bundles: Compatibility with brackets is automatic. Indeed, since the graph of ϕ is a rank 1 subbundle, we may take $\mathcal{R}$ in our criterion to consist of just one section.

Let us consider the case that N is an open interval $J \subseteq \mathbb{R}$. Given a Lie algebroid morphism

$$\phi\colon TJ \to \mathcal{A},$$

the base map defines a smooth curve $\gamma\colon J \to M$. Consider the coordinate vector field $\frac{\partial}{\partial t}$ as a section of TJ. Compatibility with the anchor $\mathsf{a}\colon A \to TM$ means that a takes the section along γ,

$$\phi(\frac{\partial}{\partial t}) \in \Gamma(\gamma^* A) \tag{11}$$

to the vector field along γ,

$$(T\gamma)(\frac{\partial}{\partial t}) \in \Gamma(\gamma^* TM). \tag{12}$$

More intuitively, we may interpret (11) as a path $\tilde{\gamma}\colon J \to A$, with base path γ, and (12) as the path

$$\dot{\gamma} = \frac{\partial \gamma}{\partial t}\colon J \to TM.$$

The compatibility condition is simply that

$$\mathsf{a} \circ \tilde{\gamma} = \dot{\gamma}. \tag{13}$$

Definition 7.8. A path $\widetilde{\gamma}\colon J \to A$ in a Lie algebroid, with base path $\gamma\colon J \to M$, is called a *Lie algebroid path* in A, or simply an *A-path* if it satisfies (13).

To summarize, the Lie algebra morphisms $\phi\colon TJ \to A$ are in 1-1 correspondence with Lie algebroid paths in A.

Example 7.9. Let M be a manifold, and $\mathfrak{g}$ a Lie algebra. A vector bundle morphism

$$\theta\colon TM \to \mathfrak{g},$$

may be regarded as a $\mathfrak{g}$-valued 1-form, $\theta \in \Omega^1(M, \mathfrak{g})$. One finds that θ is a Lie algebroid morphism if and only if is a solution of the Maurer-Cartan equation,

$$\mathrm{d}\theta + \frac{1}{2}[\theta, \theta] = 0.$$

Indeed, the graph of θ is spanned by sections $\sigma_X = \theta(X) + X \in \Gamma(\mathfrak{g} \times TM) = \mathfrak{g} \oplus \mathfrak{X}(M)$, and the condition that the bracket of two such sections σ_X, σ_Y is again tangent to the graph gives the desired identity.

7.2. **Morphisms and sections.** Let $A \to M$ and $B \to N$ be two Lie algebroids, and $\phi\colon B \to A$ a morphism of Lie algebroids. Sections $\sigma \in \Gamma(A)$ and $\tau \in \Gamma(B)$ are called ϕ-related,

$$\tau \sim_\phi \sigma$$

if $\phi(\tau(n)) = \sigma(\phi_0(n))$ for all $n \in N$. In particular, the vector fields $\mathsf{a}(\tau)$, $\mathsf{a}(\sigma)$ are then ϕ_0-related. Observe that

$$\tau \sim_\phi \sigma \Leftrightarrow \sigma \times \tau \in \Gamma(A \times B, \,\mathrm{Gr}(\phi)).$$

Hence, the following is immediate:

Proposition 7.10. *Let $A \to M$ and $B \to N$ be two Lie algebroids, and $\phi\colon B \to A$ a morphism of Lie algebroids. Given $\sigma, \sigma' \in \Gamma(A)$ and $\tau, \tau' \in \Gamma(B)$ we have that*

$$\tau \sim_\phi \sigma, \quad \tau' \sim_\phi \sigma' \quad \Rightarrow [\tau, \tau'] \sim_\phi [\sigma, \sigma'].$$

Our definition of Lie algebroid morphisms circumvents the problem that, in general, a vector bundle map does not induce a map on sections, unless the vector bundles have the same base. However, the bundle map $\phi\colon B \to A$, with base map $F\colon N \to M$, does give a map

$$\phi\colon \Gamma(B) \to \Gamma(F^*A), \;\tau \mapsto \phi(\tau),$$

to sections of the *pull-back* vector bundle. The space $\Gamma(F^*A)$ is generated, as a $C^\infty(M)$-module, by pull-backs $F^*\sigma$ of sections $\sigma \in \Gamma(A)$. Its general sections are thus of the form $\sum_i f_i F^*\sigma_i$ where $f_i \in C^\infty(N)$ and $\sigma_i \in \Gamma(A)$.

Proposition 7.11. *Let $A \to M$, $B \to N$ be Lie algebroids, and $\phi\colon B \to A$ a vector bundle morphism intertwining the anchor maps, with base map $F\colon N \to M$. Then ϕ is a Lie algebroid morphism if and only if for all $\tau, \tau' \in \Gamma(B)$, with*

$$\phi(\tau) = \sum_i f_i \; F^*\sigma_i, \quad \phi(\tau') = \sum_j f'_j \; F^*\sigma'_j, \tag{14}$$

we have that

$$\phi([\tau, \tau']) = \sum_{ij} f_i f_j F^*[\sigma_i, \sigma'_j] + \sum_j \mathsf{a}(\tau)(f'_j) \; F^*\sigma'_j - \sum_i \mathsf{a}(\tau')(f_i) \; F^*\sigma_i. \tag{15}$$

Proof. Given $f_i \in C^\infty(N)$ and $\sigma_i \in \Gamma(A)$, the first formula in (14) is equivalent to stating that

$$\widetilde{\tau} := (0 \times \tau) + \sum_i (\mathrm{pr}_M^* f_i)(\sigma_i \times 0)$$

lies in $\Gamma(A \times B, \mathrm{Gr}(\phi))$. Define $\widetilde{\tau}'$ similarly in terms of f'_j, σ'_j. Then

$$\begin{aligned}[\widetilde{\tau}, \widetilde{\tau}'] =&(0 \times [\tau, \tau']) + \sum_j \mathrm{pr}_M^*(\mathsf{a}(\tau) f'_j)(\sigma'_j \times 0) - \sum_i \mathrm{pr}_M^*(\mathsf{a}(\tau') f_i)(\sigma_i \times 0) \\ &+ \sum_{ij} \mathrm{pr}_M^*(f_i f'_j) \; ([\sigma_i, \sigma'_j] \times 0).\end{aligned}$$

ϕ is a Lie algebroid morphism if and only if, for all $\tau, \tau' \in \Gamma(B)$, this section again lies $\Gamma(A \times B, \mathrm{Gr}(\phi))$. But this says precisely that $\phi([\tau, \tau'])$ is given by the right hand side of (15). ☐

Remark 7.12. In [29], the formula (15) is used as the *definition* of Lie algebroid morphism. However, in this approach one has to check that the bracket is well-defined, i.e. independent of the choice of the expressions (14) for $\phi(\tau), \phi(\tau')$.

7.3. **Fibered products, pre-images.** .

Proposition 7.13. *Let $\phi\colon B \to A$ be a morphism of Lie algebroids $A \to M$, $B \to N$, with base map $F\colon N \to M$, and let $A' \subseteq A$ be a Lie subalgebroid. If $\phi\colon B \to A$ is clean with respect to A', then $B' = \phi^{-1}(A')$ is a Lie subalgebroid.*

Proof. This follows from the identification

$$B' \cong \mathrm{Gr}(\phi) \cap (B \times A')$$

where the intersection is clean. ☐

Proposition 7.14. *Let $\phi_i\colon B_i \to A$ be two Lie algebroid morphisms. If ϕ_1, ϕ_2 intersect cleanly, then the fiber product $B_1 \times_A B_2$ is a Lie subalgebroid of $B_1 \times B_2$.*

Proof. This follows by interpreting the fiber product as the clean intersection of two Lie algebroids

$$\mathrm{Gr}(\phi_1 \times \phi_2) \cap (\Delta_A \times B_1 \times B_2) \subseteq A \times A \times B_1 \times B_2.$$

☐

7.4. **Pull-backs.** Suppose $A \to M$ is a Lie algebroid, and that $F \in C^\infty(N, M)$ is a smooth map. Suppose that the anchor $\mathsf{a}\colon A \to TM$ has clean intersection with $TF\colon TN \to TM$. Then the fiber product

$$F^! A := A \times_{TM} TN, \tag{16}$$

has a natural structure as a Lie algebroid over N, with anchor induced by the natural projection $A \times TN \to TN$. To define the Lie bracket on sections, note that under the identification of N with $\mathrm{Gr}(F) \subseteq M \times N$, the fiber product $F^! A$ is just the pre-image of $T\,\mathrm{Gr}(F) = \mathrm{Gr}(TF)$ under the anchor $A \times TN \to TM \times TN = T(M \times N)$. The pull-back Lie algebroid comes with a Lie algebroid morphism

$$F^! A \to A \tag{17}$$

with base map F. Under composition of maps we have that

$$(F_1 \circ F_2)^! A = (F_2)^! (F_1)^! A,$$

provided that all the pullbacks are clean. If F is *transverse* to a, then the dimension count gives

$$\mathrm{rank}(F^! A) = \mathrm{rank}(A) - \dim M + \dim N.$$

Example 7.15. For the tangent bundle, we find that

$$F^!(TM) = TN.$$

Example 7.16. If $F\colon N \to M$ is a submersion, the transversality is automatic, hence $F^!A$ is defined. For the case of product $N = M \times Q$, with $F\colon N \to M$ projection to M, then $F^!A = A \times TQ$. For an arbitrary submersion F, this describes $F^!A$ in local trivializations.

Example 7.17. Likewise, if $A \to M$ is a *transitive* Lie algebroid, the transversality is automatic, and $F^!A \to N$ is again transitive. In particular, if $A = A(P)$ is the Atiyah algebroid of a principal G-bundle, then

$$F^!A(P) = A(F^*P),$$

the Atiyah algebroid of the pullback bundle.

Example 7.18. Suppose $\mathcal{F}$ is a foliation of M. If $F\colon N \to M$ has clean intersection with all leaves of the foliation, then by taking pre-images of leaves we obtain a foliation $F^!\mathcal{F}$ of N. We have

$$F^!T_{\mathcal{F}}M = T_{F^!\mathcal{F}}N.$$

Example 7.19. If $F\colon N \to M$ is an injective immersion, then $F^!A = \mathsf{a}^{-1}(TN)$. In particular, is $\mathsf{a}(A)|_N \subseteq TN$ everywhere, then $F^!A$ is the usual pull-back as a vector bundle; the bracket on sections of $F^!A$ is given by 'restriction'.

7.5. **Further Constructions.**

7.5.1. *Homotopy.* Using pull-backs, we can define a notion of *homotopy* between two Lie algebroids morphisms. Recall that a homotopy between two maps of manifolds $F_0, F_1\colon N \to M$ is a smooth map $F\colon N \times [0,1] \to M$ such that

$$F_0 = F \circ j_0, \quad F_1 = F \circ j_1$$

where $j_t\colon N \to N \times [0,1]$ is the map $n \mapsto (n,t)$. To get avoid smoothness problems under composition, one sometimes imposes the extra condition of *sitting end points*, that is, F being t-independent for t close to the end points. [6]

Definition 7.20. Let $B \to N$, $A \to M$ be two Lie algebroids. A homotopy between two $\mathcal{LA}$ morphisms $\varphi_0, \varphi_1\colon B \to A$ (with base maps F_0, F_1) to be an $\mathcal{LA}$ morphism

$$\varphi\colon B \times T[0,1] \to A$$

whose composition with the $\mathcal{LA}$ morphisms $B \to B \times T[0,1]$ is the identity. We say that φ has sitting end points if the base map has sitting end points, and $\varphi|_{(n,t)}(\frac{\partial}{\partial t}) = 0$ for t close 0 or close to 1.

Write $\varphi_1 \sim \varphi_0$ for the relation of homotopy of $\mathcal{LA}$ morphisms.

Lemma 7.21. *If $\varphi_1 \sim \varphi_0$, then there exists a homotopy φ between φ_0, φ_1 with sitting end points.*

[6] A slightly different version of this condition is to assume 'flatness at the end points', in the sense that the extension of F to all of $\mathbb{R} \times N$, by extending as a constant map on $(-\infty, 0] \times M$ and on $[1, \infty)$, be smooth.

Proof. Let $\chi\colon [0,1] \to [0,1]$ be a smooth function with $\chi(0) = 0$, $\chi(1) = 1$, and such that χ is constant near the end points. Given an arbitrary homotopy between φ_0, φ_1, its composition with $\mathrm{id}_A \times T\chi$ has sitting end points. ☐

Since $\mathcal{LA}$ homotopies with sitting end points can be composed, the Lemma implies that $\mathcal{LA}$ homotopy is an equivalence relation.

Given a Lie subalgebroid $B_1 \subseteq B$ along a submanifold $N_1 \subseteq N$, we can define a notion of *relative homotopy*, where we require that the homotopy φ is constant when restricted to $B_1 \times T[0,1]$.

7.5.2. *A-connections.* Let $A \to M$ be a Lie algebroid. An A-connection on a vector bundle $V \to M$ is a bilinear map

$$\nabla\colon \Gamma(A) \times \Gamma(V) \to \Gamma(V), \quad (\sigma, \tau) \mapsto \nabla_\sigma \tau,$$

with the property that

$$\nabla_{f\sigma} = f\nabla_\sigma \tau, \quad \nabla_\sigma(f\tau) = f\nabla_\sigma(\tau) + \mathcal{L}_{\mathsf{a}(\sigma)}(f)\,\tau.$$

The notion of A-connections generalizes that of ordinary connections. to which it reduced for $A = TM$. On the other hand, every TM-connection ∇ determines an A-connection by setting $\nabla_\sigma := \nabla_{\mathsf{a}(\sigma)}$.

Example 7.22. Suppose A is a regular Lie algebroid, so that a has constant rank, and defines a foliation $\mathcal{F}$ with $TF = \mathrm{ran}(\mathsf{a})$. One obtains an A-connection on the vector bundle $\ker(\mathsf{a})$ by

$$\nabla_\sigma \tau = [\sigma, \tau],$$

as well as a *Bott A-connection* on $\mathrm{ann}(T\mathcal{F})$, the conormal bundle to the leaves:

$$\nabla_\sigma \beta = \mathcal{L}_{\mathsf{a}(\sigma)} \beta.$$

(Sections of the conormal bundle are 1-forms on M that vanish on tangent vectors to the leaves.) More generally, if A is not necessarily regular, and $\mathcal{O} \subseteq M$ is any leaf, one has natural $A_{\mathcal{O}}$-connections on $\ker(\mathsf{a})|_{\mathcal{O}}$ as well as on $\mathrm{ann}(T\mathcal{O})$.

The *curvature* of an A-connection is a tensor field

$$F^\nabla \in \Gamma(\wedge^2 A^* \otimes \mathrm{End}(V)),$$

defined on sections $\sigma_1, \sigma_2 \in \Gamma(A)$ by

$$F^\nabla(\sigma_1, \sigma_2) = [\nabla_{\sigma_1}, \nabla_{\sigma_2}] - \nabla_{[\sigma_1, \sigma_2]}.$$

Thus, $F^\nabla = 0$ if and only if the map $\sigma \mapsto \nabla_\sigma$ preserves brackets. For the special case $V = A$, one can also introduce a *torsion* of an A-connection on A, as the tensor field $T^\nabla \in \Gamma(\wedge^2 A^* \otimes A)$ given on sections by

$$T^\nabla(\sigma_1, \sigma_2) = \nabla_{\sigma_1}\sigma_2 - \nabla_{\sigma_2} - [\sigma_1, \sigma_2].$$

7.5.3. *Prolongations, $\mathcal{VB}$-algebroids, tangent algebroids.*

7.6. **Lie algebroid actions, representations of Lie algebroids.** Recall that an action of a Lie algebra $\mathfrak{g}$ on a manifold Q is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(Q)$ such that the action map $\mathfrak{g} \times Q \to TQ$, $(\xi, q) \mapsto \xi_Q(q)$ is smooth. Given such an action, one can then form the *action Lie algebroid* $\mathfrak{g} \ltimes Q \rightrightarrows Q$ as discussed in section **??**. Note that the projection $\mathfrak{g}\ltimes \to \mathfrak{g}$ is a morphism of Lie algebroids, due to the fact that the bracket on constant sections is the given one on $\mathfrak{g}$. In fact, Lie algebra actions can be defined in these terms, as Lie algebroid structures on the trivial bundle $Q \times \mathfrak{g}$, such that the projection to $\mathfrak{g}$ is a Lie algebroid morphism.

This suggests the following generalization to actions of arbitrary Lie algebroids.

Definition 7.23. An action of a Lie algebroid $A \to M$ on a manifold Q is given by a smooth map

$$\Phi \colon Q \to M$$

called the *moment map* (or anchor) of the action, together with a Lie algebroid structure on the pull-back bundle $\Phi^* A$, such that the natural bundle map

$$\Phi^* A \to A$$

is a morphism of Lie algebroids. We define the generating vector fields of the action to be

$$\sigma_Q = \mathsf{a}(\Phi^*\sigma), \quad \sigma \in \Gamma(A).$$

Remarks 7.24.
(a) In this context, $\Phi^* A$ becomes the *action Lie algebroid*; it is also denoted $A \ltimes Q$.
(b) Every section $\sigma \in \Gamma(A)$ defines a *generating vector field*

$$\sigma_Q = \mathsf{a}_{\Phi^* A}(\Phi^*\sigma).$$

The map $\sigma \mapsto \sigma_Q$ is a Lie algebra morphism.
(c) The assumption that $\Phi^* A \to A$ is a Lie algebroid morphism means in particular that the following diagram commutes:

$$\begin{tikzcd} \Phi^* A \arrow[r] \arrow[d, "\mathsf{a}_{\Phi^* A}"'] & A \arrow[d, "\mathsf{a}_A"] \\ TQ \arrow[r, "T\Phi"'] & TM \end{tikzcd}$$

(d) We may equivalently define a Lie algebroid action of $A \to M$ on $\Phi \colon Q \to M$ to be a Lie algebra homomorphism

$$\Gamma(A) \to \mathfrak{X}(Q), \quad \sigma \mapsto \sigma_Q$$

such that $(f\sigma)_Q = (\Phi^* f)\sigma_Q$ for all sections $\sigma \in \Gamma(A)$ and functions $f \in C^\infty(M)$, and such that the resulting map $\Phi^* A \to TQ$ defined by $\sigma_{\Phi(q)} \mapsto (\sigma_Q)_q$ is smooth.

Examples 7.25. Every Lie algebroid $A \to M$ as a natural action on its base M, with generating vector fields $\sigma_M = \mathsf{a}(\sigma)$.

We will encounter more interesting examples of Lie algebroid actions later on, as derivatives of Lie groupoid actions. At this point, we are mainly interested in linear actions on vector bundles.

Definition 7.26. A representation of a Lie algebroid $A \to M$ on a vector bundle $\pi\colon V \to M$ is a Lie algebroid action of A on V, with moment map π, such that the generating vector fields $\sigma_V \in \mathfrak{X}(V)$ are linear, i.e., homogeneous of degree 0.

At this point it is useful to recall the following fact.

Proposition 7.27. *Let $\pi\colon V \to M$ be a vector bundle. There is a 1-1 correspondence between:*

(a) *Linear vector fields on $V \to M$.*

(b) *Linear operators $D\colon \Gamma(V) \to \Gamma(V)$ such that there exists a vector field $X \in \mathfrak{X}(M)$ such that*

$$D(f\tau) = f\,D(\tau) + X(f)\,\tau, \quad f \in C^\infty(M),\ \tau \in \Gamma(V).$$

(Thus, D is a first order linear differential operator on V with scalar principal symbol.)

The correspondence takes a linear vector field $\widetilde{X}$ on V, with restriction $X = \widetilde{X}|_M$, to the differential operator on $\Gamma(V)$ defined by the infinitesimal flow of $\widetilde{X}$.

Using this result, one can rephrase the definition of a Lie algebroid representation as follows.

Proposition 7.28. *A representation of a Lie algebroid A on a vector bundle V is equivalent to a flat A-connection on V.*

Example 7.29. Given a Lie algebroid $A \to M$, and a Lie subalgebroid $B \subseteq A$, with base manifold $N = M$, one obtains a representation of B on A/B by

$$\nabla_\tau \overline{\sigma} = \overline{[\tau, \sigma]}$$

where $\overline{\sigma} \in \Gamma(A/B)$ is the equivalence class of a section $\sigma \in \Gamma(A)$.

8. The generalized foliation of a Lie algebroid

8.1. **Integral submanifolds.** Let $A \to M$ be a Lie algebroid. If the Lie algebroid A is *regular*, in the sense that the anchor map has constant rank, then $\mathsf{a}(A) \subseteq TM$ is a subbundle, with sections $\Gamma(\mathsf{a}(A)) = \mathsf{a}(\Gamma(A))$. Since a induces a Lie algebra morphism on sections, it hence follows by Frobenius' theorem that $\mathsf{a}(A)$ is the tangent bundle of a foliation $\mathcal{F}$ on M.

For a non-regular Lie algebroid, $\mathsf{a}(A) \subseteq TM$ is not a subbundle since it does not have constant rank. One still has a notion of integral submanifolds:

Definition 8.1. An *integral submanifold* of A is an injectively immersed submanifold $i\colon N \hookrightarrow M$ with the property that

$$Ti(T_nN) = \mathsf{a}(A_{i(n)})$$

for all $n \in N$. A connected integral submanifold is called a *leaf* of A if it is not properly contained in any larger connected integral submanifold.

Note that since the notion of integral submanifolds does not involve the bracket on $\Gamma(A)$, it is defined more generally for *anchored* vector bundles $A \to M$. But we will use the Lie algebroid structure to prove the following result:

Theorem 8.2. *Given a Lie algebroid $A \to M$, there is a unique leaf passing through any given point $m \in M$. That is, M acquires a unique decomposition into leaves of A.*

The decomposition of M into leaves of A gives a 'generalized foliation' in the sense of Stefan-Sussmann. One can prove the result above by referring to an appropriate version of the Stefan-Sussmann theorem, as for example in [2]. Instead, we will follow a different approach, due to [7] where we first prove a *splitting theorem* for Lie algebroids. This will require some background information on normal bundles and tubular neighborhood embeddings.

8.2. **Normal bundles and tubular neighborhoods.** Let M be a manifold, and $N \subseteq M$ an embedded submanifold. The *normal bundle* of N in M is the vector bundle

$$\nu(M,N) := TM|_N/TN \to N.$$

We will denote by $i\colon N \to M$ the inclusion map and by $p\colon \nu(M,N) \to N$ the projection:

$$\begin{tikzcd} \nu(M,N) \arrow[d, "p"'] & \\ N \arrow[r, "i"'] & M. \end{tikzcd} \tag{18}$$

Given a smooth map of pairs $\varphi\colon (M',N') \to (M,N)$ (that is, $\varphi\colon M' \to M$ is a smooth map with $\varphi(N') \subseteq N$), one obtains a vector bundle morphism

$$\nu(\varphi)\colon \nu(M',N') \to \nu(M,N) \tag{19}$$

over $\varphi|_{N'}\colon N' \to N$, with the obvious functorial property under composition of such maps. If φ is transverse to N, and $N' = \varphi^{-1}(N)$, then $\nu(\varphi)$ is a fiberwise isomorphism.

Remarks 8.3.
(a) The *conormal bundle* of $i\colon N \to M$ is the subbundle $\operatorname{ann}(TN) \subseteq T^*M|_N$. It is canonically isomorphic to the dual of $\nu(M,N)$.
(b) Of course, normal bundles are defined more generally for *immersions* $i\colon N \to M$, as $\nu(M,N) = i^*TM/TN$.
(c) The normal bundle functor is compatible with the tangent functor: There is a canonical isomorphism

$$\nu(TM,TN) \xrightarrow{\cong} T\nu(M,N) \tag{20}$$

identifying the structures as vector bundles over $\nu(M,N)$ and also as vector bundles over TN. See Appendix **??** in [**?**] for a detailed discussion.

For a vector bundle $\pi\colon E \to M$, the restriction of $TE \to E$ to $M \subseteq E$ has a canonical decomposition

$$TE|_M = E \oplus TM,$$

where $E \subseteq TE|_M$ is identified with the vectors tangent to fibers, and $TM \subseteq TE|_M$ with vectors tangent to the base. Taking a quotient by TM, this gives a canonical isomorphism

$$\nu(E,M) \cong E. \tag{21}$$

As a special case, for any submanifold $N \subseteq M$, we have

$$\nu(\nu(M,N),N) = \nu(M,N).$$

Using this identification, we define:

Definition 8.4. Let $i\colon N \to M$ be a submanifold. A *tubular neighborhood embedding* of $\nu(M,N)$ is an embedding $\phi\colon \nu(M,N) \to M$, taking $N \subseteq \nu(M,N)$ to $N \subseteq M$, such that the map $\nu(\phi)$ induced by

$$\phi\colon (\nu(M,N),N) \to (M,N)$$

is the identity map on $\nu(M,N)$.

Thus, a tubular neighborhood embedding induces the identity map on the base N, as well as in directions normal to N. It is well-known that every submanifold $N \subseteq M$ admits a tubular neighborhood embedding of its normal bundle. (The standard construction uses a Riemannian metric on M.) One may think of $\nu(M,N)$ as a 'model' for a neighborhood of N insider M: For instance, it follows that if $i_1\colon N_1 \to M_1$ and $i_2\colon N_i \to M_2$ are two embeddings, then a diffeomorphism $N_1 \to N_2$ extends to a diffeomorphism of open neighborhoods if and only if $\nu(M_1,N_1) \to N_1$ is isomorphic to the pullback of $\nu(M_2,N_2) \to N_2$.

Remark 8.5. One might call the above a 'complete' tubular neighborhood embedding. One could also consider 'incomplete' tubular neighborhood embeddings, where ϕ is only defined on some open neighborhood of the zero section $N \subseteq \nu(M,N)$.

Note that a tubular neighborhood embedding ϕ specifies a complement to TN in $TM|_N$, namely the image of $\nu(M,N) \subseteq T\nu(M,N)|_N$ under $T\phi$. Much more than that, the image $U = \phi(\nu(M,N))$ acquires the structure of a vector bundle over N.

Remark 8.6. A closely related construction for pairs (M,N) is the *deformation to the normal cone* $\mathcal{D}(M,N)$. This is a manifold of dimension $\dim(M)+1$, together with a surjective submersion $\mathcal{D}(M,N) \to \mathbb{R}$ such that the fibers $\mathcal{D}(M,N)_t$ at $t \in \mathbb{R}$ are given by M itself for $t \neq 0$, and by $\nu(M,N)$ for $t = 0$. That is,

$$\mathcal{D}(M,N) = \nu(M,N) \amalg (M \times \mathbb{R}^\times)$$

where $\nu(M,N)$ is embedded as a codimension 1 submanifold, with complement the open subset $M \times \mathbb{R}^\times$. The smooth structure on $\mathcal{D}(M,N)$ is defined in such a way that (i) the map to $M \times \mathbb{R}$, given by the obvious inclusion on $M \times \mathbb{R}^\times$ and by bundle projection to N on $\nu(M,N)$, is smooth, (ii) for any $f \in \mathcal{I}(M,N)$ (the ideal of smooth functions on M that vanish on N), the function $\widetilde{f}\colon \mathcal{D}(M,N) \to \mathbb{R}$ given on the two pieces by

$$M \times \mathbb{R}^\times \to \mathbb{R}, \quad (m,t) \mapsto t^{-1} f(m)$$

and

$$\nu(M,N) \to \mathbb{R}, \quad [v] \mapsto v(f)$$

(where $[v]$ is represented by $v \in TM|_N$), is smooth. Intuitively, as one approaches $t = 0$ the normal directions get 'magnified'. [7] Again, this construction is functorial; for example the map

[7] For details on why this defines a smooth structure, see e.g. [**?**], or the appendix.

described in (i) above is the map

$$\mathcal{D}(M,N) \to D(M,M) = M \times \mathbb{R},$$

induced by $(M,N) \to (M,M)$, while $\widetilde{f}$ is the first component of the map induced by $\mathcal{D}(f)\colon (M,N) \to (\mathbb{R},\{0\})$.

8.3. **Euler-like vector fields.** The Euler vector field on $\mathbb{R}^k$ is the vector field

$$\mathcal{E} = \sum_{j=1}^{k} y^j \frac{\partial}{\partial y^j}. \tag{22}$$

It has the property that $\mathcal{E}(f) = f$ whenever $f \in C^\infty(\mathbb{R}^k)$ is linear (i.e., homogeneous of degree 1), and is in fact uniquely determined by this property. For any vector bundle $\pi\colon V \to M$ we may define the *Euler vector field* $\mathcal{E} \in \mathfrak{X}(V)$ uniquely by its property that $\mathcal{E}(f) = f$ for all linear functions $f \in C^\infty(M)$. Indeed, this property implies that $\mathcal{E}$ must be tangent to the fibers of V, and is given fiberwise by (22).

Definition 8.7. Let $N \subseteq M$ be a submanifold, and $\mathcal{I}(M,N) \subseteq C^\infty(M)$ the ideal of functions vanishing along N. A vector field $X \in \mathfrak{X}(M)$ is called *Euler-like along* N if it is complete, and has the property

$$X(f) = f \mod \mathcal{I}(M,N)^2$$

for all $f \in \mathcal{I}(M,N)$.

The condition means that whenever f vanishes along N, then $X(f) - f$ vanishes to second order along N. (In particular, X itself must vanish along N.)

The definition may be expressed in several equivalent ways. Note that since X vanishes along N, it is in particular tangent to N, and so it defines a map of pairs $X\colon (M,N) \to (TM,TN)$. Applying the normal functor, we obtain

$$\nu(X)\colon \nu(M,N) \to \nu(TM,TN) \cong T\nu(M,N),$$

which we may think of as a vector field on $\nu(M,N)$. The Euler-like property of a complete vector field X is then equivalent to stating that $\nu(X)$ is *the* Euler vector field on $\nu(M,N)$. That is, X is Euler-like if it vanishes along N and its linear approximation is Euler-like.

We may also express the Euler-like condition in local coordinates. Suppose $x^1,\ldots,x^r,y^1,\ldots,y^k$ are local coordinates on M such that N is given by the vanishing of the y-coordinates. A vector field in these local coordinates is Euler-like if and only if it has the form

$$X = \sum_i a^i(x,y) \frac{\partial}{\partial x^i} + \sum_j (y^j + b^j(x,y)) \frac{\partial}{\partial y^j} \tag{23}$$

where the a^i vanish for $y = 0$, and b^j vanishes to second order for $y = 0$.

Exercise 8.1. Verify that this condition in local coordinates is equivalent to Definition 8.7.

Remark 8.8. The interpretation in terms of the deformation space $\mathcal{D}(M,N)$ is as follows. Given a vector field X that is tangent to N, the resulting map $X\colon (M,N) \to (TM,TN)$ gives, by functoriality, a vertical vector field $\mathcal{D}(X)$ on $\mathcal{D}(M,N)$, such that $\mathcal{D}(X)$ is given by X on fibers $\mathcal{D}(M,N)_t$ for $t\neq 0$, and by $\nu(X)$ on $\mathcal{D}(M,N)_0=\nu(M,N)$.

A tubular neighborhood embedding $\phi\colon \nu(M,N)\to U\subseteq M$ gives rise to an Euler-like vector field on U, by taking the image of the Euler vector field $\mathcal{E}$ on $\nu(M,N)$. Somewhat surprisingly, the converse is true:

Theorem 8.9. [7] *Suppose $N\subseteq M$ is a submanifold, and X is Euler-like along N. Then there is a unique tubular neighborhood embedding $\phi\colon \nu(M,N)\to M$ such that*

$$\mathcal{E}\sim_\phi X$$

where $\mathcal{E}$ is the Euler vector field on $\nu(M,N)$.

Proof. The key step is to show that X is *linearizable* along N. Choose an initial tubular neighborhood embedding to identify M with a neighborhood of N inside $\nu(M,N)$. Let $\mathcal{E}$ be the Euler vector field on $\nu(M,N)$, and write

$$X=\mathcal{E}+Z$$

where $\mathcal{E}=\sum y^j\frac{\partial}{\partial y^j}$ is the standard Euler vector field. Our goal is to modify the tubular neighborhood embedding to arrange that $Z=0$. In local coordinates, we have that

$$Z=\sum_i a^i(x,y)\frac{\partial}{\partial x^i}+\sum_j b^j(x,y)\frac{\partial}{\partial y^j}$$

where a^i vanish to first order for $y=0$, while the b^j vanish to second order for $y=0$. Let $\kappa_t\colon \nu(M,N)\to\nu(M,N)$ be scalar multiplication by t. For $t\neq 0$, this is a diffeomorphism. Observe that the family of vector fields

$$Z_t=t^{-1}\kappa_t^*Z,$$

which a priori is defined only for $t\neq 0$, extends smoothly to a family of vector fields defined for all $t\in\mathbb{R}$. To see this, note that in local coordinates

$$Z_t=\sum_i\frac{a^i(x,ty)}{t}\frac{\partial}{\partial x^i}+\sum_j\frac{b^j(x,ty)}{t^2}\frac{\partial}{\partial y^j},$$

which has good limits, as claimed. We see furthermore that Z_0 vanishes along N, since all Z_t do. (In coordinates, the coefficients in front of $\frac{\partial}{\partial x^i}$ become linear in y, while those in front of $\frac{\partial}{\partial y^j}$ become quadratic in y.) It follows that on a sufficiently small open neighborhood of N inside $\nu(M,N)$, the flow φ_t of the time dependent vector field Z_t, with initial condition $\varphi_0=\mathrm{id}$, is defined for all $|t|\leq 1$. By the scaling property $\kappa_a^*Z_t=aZ_{at}$ for $0<a<1$, this neighborhood is invariant under κ_t for $0\leq t\leq 1$. Using that $\kappa_t^*\mathcal{E}=\mathcal{E}$, and $t\frac{d}{dt}\kappa_t^*Y=\kappa_t^*[\mathcal{E},Y]$ for all vector fields Y, we obtain

$$\begin{aligned}\frac{d}{dt}\varphi_t^*(\mathcal{E}-t\,Z_t)&=\frac{d}{dt}\varphi_t^*(\mathcal{E}-\kappa_t^*Z)\\&=\varphi_t^*\Big(-[Z_t,\mathcal{E}-\kappa_t^*Z]-\frac{1}{t}\kappa_t^*[\mathcal{E},Z]\Big)\\&=\varphi_t^*(-[Z_t,\mathcal{E}]-[\mathcal{E},Z_t])=0.\end{aligned}$$

Hence $\varphi_t^*(\mathcal{E}-tZ_t)$ does not depend on t. Equality of the values at $t=1$ and $t=0$ gives $\varphi_1^*(X)=\mathcal{E}$. Hence, any tubular neighborhood embedding that agrees with φ_1 near N will give the desired linearization. This shows the existence of a possibly incomplete tubular neighborhood embedding taking $\mathcal{E}$ to X. Using the flows of $\mathcal{E}$ and of X, it extends to a complete tubular neighborhood embedding $\psi\colon \nu(M,N)\to U\subseteq M$.

This proves the existence part. For the uniqueness part, let Φ_s be the flow of X, and $\lambda_t=\Phi_{-\log(t)}$. Then the image $U\subseteq M$ of the tubular neighborhood embedding constructed above is characterized as the set of all $m\in M$ such that $\lim_{\to 0}\lambda_t(m)$ exists and lies in N. The inverse map $\psi^{-1}\colon U\to\nu(M,N)$ is explicitly given as

$$\psi^{-1}(m)=\frac{d}{dt}|_{t=0}\lambda_t(m) \mod TN.$$

(The element $\frac{d}{dt}|_{t=0}\lambda_t(m)$ is a tangent vector at $\lambda_0(m)$, and we take its image in $\nu(M,N)$.) $\square$

Remark 8.10. The question of linearizability of vector fields is subtle, and has been extensively studied. (See e.g. [3] for a quick overview and recent results.) The classical result of Sternberg [38, 39] gives C^∞-linearizability of vector fields at critical points m, provided the endomorphism of T_mM describing this linear approximation has *non-resonant* eigenvalues. If the linear approximation is the Euler vector field, then this endomorphism is $-\,\mathrm{id}$, and the non-resonance condition is satisfied. Thus, for $N=\{m\}$, Theorem 8.9 is essentially a very special case of Sternberg's theorem. (Note however that the theorem gives more information than only the existence of a linearization.) To give some concrete examples, the vector field

$$x\frac{\partial}{\partial x}+(y+x^2)\frac{\partial}{\partial y}$$

is linearizable at 0. One can show [8] that the vector field

$$x\frac{\partial}{\partial x}+(2y+x^2)\frac{\partial}{\partial y}$$

is *not* linearizable at 0, while on the other hand[9]

$$2x\frac{\partial}{\partial x}+(y+x^2)\frac{\partial}{\partial y}$$

is linearizable at 0.

Remark 8.11. In [36], the authors give the following nice explanation of Theorem 8.9 in terms of the deformation to the normal cone. We will explain this proof in the appendix.

8.4. **Some applications of Theorem 8.9.** Before using this result to prove the splitting theorem for Lie algebroids, let us very quickly sketch some other applications of the theorem.

(a) Many 'normal form' results can be phrased in terms of Euler-like vector fields:

[8] The solution curves of this vector field are $x=0$ and curves of the form $y=(c+\log(|x|)x^2,\ c\in\mathbb{R}$. These are not smooth at the origin, unlike the solution curves of the linearized vector field. See Robert Bryant's explanation on https://mathoverflow.net/questions/76971/nice-metrics-for-a-morse-gradient-field-counterexample-request

[9] An argument similar to the proof of Theorem 8.9 applies.

(i) The linearizability of a vector field $Y \in \mathfrak{X}(M)$ with critical point at $0 \in M$ is equivalent to the existence of an Euler-like vector field X with respect to $\{m\}$ with $[X, Y] = 0$. Indeed, any such X determines coordinates in which it becomes the Euler vector field. But then $[X, Y] = 0$ tells us that in these coordinates, Y is homogeneous of degree 0, i.e. linear. This remark was already made (in the analytic context) by Guillemin-Sternberg in [23, Proposition 1.1].

(ii) Darboux's theorem for symplectic 2-forms amounts to the fact that for every $m \in M$, there exists an Euler-like vector field X with respect to $\{m\}$ such that $\mathcal{L}_X\omega = 2\omega$ near X. Indeed, any such X determines coordinates in which it becomes the Euler vector field. But then $\mathcal{L}(X)\omega = 2\omega$ tells us that in these coordinates, ω is homogeneous of degree 2, i.e. is a constant 2-form.

(iii) The Morse lemma for a function $f \in C^\infty(M)$ with non-degenerate critical point at $m \in M$ amounts to the existence of an Euler-like vector field X with respect to $\{m\}$ such that $\mathcal{L}(X)f = 2(f - f(m))$. Indeed, any such X determines coordinates in which it becomes the Euler vector field. But then $\mathcal{L}(X)f = 2(f - f(m))$ tells us that in these coordinates, $f - f(m)$ is quadratic.

(iv) Recall that the normal form theorem for cleanly intersecting manifolds $N_1, N_2 \subseteq M$ states that around any point $m \in N_1 \cap N_2$, one can choose local coordinates in which N_1, N_2 become subspaces. This is equivalent to the existence of an Euler-like vector field X with respect to $\{m\}$ such that X is tangent to both N_1 and N_2.

One advantage of these coordinate-free formulations is that the equivariant versions, given a compact Lie group G acting on M and preserving the given structures, are automatic: Given an Euler-like vector field solving the non-equivariant problem, one simply takes its G-average to obtain an Euler-like vector field solving the equivariant problem.

(b) As remarked in [36], we may use this perspective to give a quick proof of Darboux's theorem in the version stated above. Given a symplectic manifold (M, ω) and any $m \in M$, choose local coordinates centered at m to write

$$\omega = \frac{1}{2} \sum A_{ij} \mathrm{d}x^i \mathrm{d}x^j + \text{ higher order },$$

with a non-degenerate skew-symmetric matrix A_{ij}, where the higher order terms are of order ≥ 3 in x. [10] Applying the standard homotopy operator, we $\omega = \mathrm{d}\alpha$, with

$$\alpha = \frac{1}{2} \sum_{ij} A_{ij} x^i \mathrm{d}x^j + \text{ higher order },$$

where the higher order are again of order ≥ 3 in x, or higher. Define a vector field X by

$$\iota(X)\omega = 2\alpha.$$

[10] Here our notion of order includes the $\mathrm{d}x^i$, not only the coefficients. Thus, a k-form whose coefficients are homogeneous of degree l is homogeneous of degree $k + l$.

Then $\mathcal{L}(X)\omega = \mathrm{d}\iota(X)\omega = 2\mathrm{d}\alpha = 2\omega$. But

$$X = \sum_i x_i \frac{\partial}{\partial x^i} + \text{ higher order },$$

where the higher order term are of order ≥ 1 in x. Thus X is Euler-like. As stated above, $\mathcal{L}(X)\omega = 2\omega$ for an Euler-like vector field is equivalent to Darboux.

(c) Consider similarly the Morse Lemma: Suppose $f \in C^\infty(M)$ has a nondegenerate critical point at m, with $f(m) = 0$. We would like to construct an Euler-like X with $\mathcal{L}(X) = 2f$. After choice of local coordinates, Taylor's theorem allows us to write

$$f = \frac{1}{2}\sum_{ij} S_{ij}(x)x^i x^j, \quad \mathrm{d}f = \sum_{ij} R_{ij}(x)x^i \mathrm{d}x^j$$

where $S_{ij}(x) = S_{ji}(x)$, and $S_{ij}(0) = R_{ij}(0)$. In matrix notation,

$$f = \frac{1}{2}x \cdot (S(x)\, x), \quad \mathrm{d}f = x \cdot (R(x)\, \mathrm{d}x).$$

Since the matrix $R(0)$ is invertible, it remains invertible for small x. Put

$$X = \sum_r (R(x)^{-1}S(x) \cdot x) \cdot \frac{\partial}{\partial x}.$$

Then X is Euler-like, and satisfies $\mathcal{L}(X)f = \iota(X)\mathrm{d}f = 2f$ as desired.

(d) The Grabowski-Rotkievicz theorem is itself a consequence of Theorem 8.9. To recover the additive structure on a vector bundle $V \to M$ from the scalar multiplication, note that the scalar multiplication determines the Euler vector field of V. The latter (being Euler-like) gives an isomorphism $\nu(V, M) \to V$ which allows us to recover the additive structure on V from that on the normal bundle $\nu(V, M)$.

8.5. **The splitting theorem for Lie algebroids.** Let $A \to M$ be a Lie algebroid, and $i\colon N \hookrightarrow M$ an embedded submanifold, with normal bundle $p\colon \nu(M, N) \to N$. We assume that N is transverse to the anchor, so that $i^! A$ is a well-defined Lie algebroid.

Lemma 8.12. *Let $A \to M$ be an anchored vector bundle over M. If a submanifold $N \subseteq M$ is transverse to the anchor $\mathsf{a}\colon A \to TM$, then there exists a section $\sigma \in \Gamma(A)$ such that the vector field $X = \mathsf{a}(\sigma)$ is Euler-like along N.*

Proof. Suppose that in local coordinates $x^1, \ldots, x^r, y^1, \ldots, y^k$, the submanifold N is given by the vanishing of y-coordinates. Since a is transverse to N, it induces a surjective bundle map $A|_N \to \nu(M, N)$. Hence, there exist sections σ_i such that $\mathsf{a}(\sigma_j) = \frac{\partial}{\partial y^j}$ modulo TN. Then the vector field $\mathsf{a}(\sum_j y^j \sigma_j)$ is an (incomplete) Euler-like vector field. We may patch the local definitions by using partitions of unity, to obtain a section whose image under the anchor is an (incomplete) Euler-like vector field on M. Multiplying by a bump function, supported near N and equal to 1 on a smaller neighborhood of N, we can achieve that the vector field is complete. $\square$

We will use in the discussion below that for any section σ of a Lie algebroid, the operator $[\sigma,\cdot]$ is an infinitesimal Lie algebroid automorphism. Indeed, it is an infinitesimal vector bundle automorphism due to the Leibnitz rule, it is preserves the anchor [11] due to $\mathsf{a}\circ[\sigma,\cdot]=[\mathsf{a}(\sigma),\cdot]\circ\mathsf{a}$, and it preserves the bracket itself due to the Jacobi identity. Put differently, $[\sigma,\cdot]$ defines a vector field $\widetilde{Y}$ on A, homogeneous of degree 0, and restricting to Y on M, such that the local flow of $\widetilde{Y}$ on A is by Lie algebroid automorphisms. The compatibility with the anchor is the fact that

$$\widetilde{Y}\sim_{\mathsf{a}} Y_T$$

where $Y_T\in\mathfrak{X}(TM)$ is the tangent lift of Y.

Theorem 8.13 (Splitting theorem for Lie algebroids)**.** *Suppose $A\to M$ is a Lie algebroid, and let $N\subseteq M$ be a transversal. Choose $\sigma\in\Gamma(A)$ with $\sigma|_N=0$, such that $X=\mathsf{a}(\sigma)$ is Euler-like. Then the choice of σ determines an isomorphism of Lie algebroids*

$$\begin{tikzcd} p^!i^!A \arrow[r, "\widetilde{\psi}"] \arrow[d] & A|_U \arrow[d] \\ \nu(M,N) \arrow[r, "\psi"'] & U \end{tikzcd}$$

where the base map is the tubular neighborhood embedding defined by X.

Proof. Let $\widetilde{X}\in\mathfrak{X}(A)$ be the vector field on A defined by the Lie algebroid derivation $[\sigma,\cdot]$. It is homogeneous of degree 0, and restricts to $X=\mathsf{a}(\sigma)$ along M. We claim that $\widetilde{X}$ is Euler-like with respect to the submanifold $i^!A=\mathsf{a}^{-1}(TN)\subseteq A$. To see this, observe that under the anchor map,

$$\widetilde{X}\sim X_T,$$

where $X_T\in\mathfrak{X}(TM)$ is the tangent lift of X. This tangent lift is Euler-like with respect to $TN\subseteq TM$, as one checks in local coordinates. (E.g., if κ_t is scalar multiplication by t for a vector bundle $V\to M$, then $T\kappa_t$ is scalar multiplication for $TV\to TM$. Hence, the Euler vector field for TV is the tangent lift of that of V.) On the other hand, a gives a map of pairs

$$\mathsf{a}\colon (A,i^!A)\to(TM,TN).$$

By transversality, the resulting map $\nu(\mathsf{a})\colon\nu(A,i^!A)\to\nu(TM,TN)=T\nu(M,N)$ is a fiberwise isomorphism. Together with $\widetilde{X}\sim_{\mathsf{a}} X_T$, this implies that $\widetilde{X}$ is Euler-like, as claimed.

Let Φ_s, $\widetilde{\Phi}_s$ be the flow of $X,\widetilde{X}$, respectively. (The flow of $\widetilde{X}$ is complete, since $\widetilde{X}$ is homogeneous of degree 0, and the flow of X is complete.) As explained above, $\widetilde{\Phi}_s$ are Lie algebroid automorphisms. Let

$$\lambda_t=\Phi_{-\log(t)},\ \widetilde{\lambda}_t=\widetilde{\Phi}_{-\log(t)}.$$

[11] A vector bundle automorphism $\widetilde{\Phi}$ of A, with base map Φ, preserves the anchor if and only if $\mathsf{a}\circ\widetilde{\Phi}=T\Phi\circ\mathsf{a}$. In other words, $\widetilde{\Phi}$ is a-related to the tangent lift of Φ. Similarly, an infinitesimal automorphism $\widetilde{Y}$ preserves the anchor if it is a-related to the tangent lift of Y. For the corresponding operator D on sections, this means $\mathsf{a}\circ D=[Y,\cdot]\circ\mathsf{a}$.

(In terms of the tubular neighborhood embeddings defined by $X, \widetilde{X}$, these correspond to multiplication by t. Since $\widetilde{\lambda}_t$ covers the flow of λ_t, it is defined over $A|_U$ even for $t = 0$. Consider the following diagram, defined for all $0 \le t \le 1$,

$$\begin{tikzcd} A|_U \arrow[r, "\cong"'] \arrow[d] & \lambda_t^!(A|_U) \arrow[r, "\cong"'] \arrow[d] & \psi^!\lambda_t^!(A|_U) \arrow[d] \\ U \arrow[r, "\mathrm{id}"'] & U \arrow[r, "\psi^{-1}"'] & \nu(M,N) \end{tikzcd}$$

Here the first upper horizontal arrow is given by the Lie algebroid morphism

$$A|_U \to \lambda_t^!(A|_U) \subseteq TU \times A|_U, \quad v \mapsto (\mathsf{a}(v),\ \widetilde{\lambda}_t(v)).$$

This map is an isomorphism for all t: If $t > 0$, this is clear since $\widetilde{\lambda}_t$ is an isomorphism then. If $t = 0$ we note that it is an isomorphism along $N \subseteq U$ (using that $\widetilde{\lambda}_0\colon A|_N \to A_N$ is a projection to i^A), hence also on some neighborhood of N, and using e.g. $\lambda_0 = \lambda_0 \circ \lambda_t$ we conclude that it is an isomorphism over all of U. We hence obtain a family of Lie algebroid isomorphisms

$$\begin{tikzcd} \psi^!\lambda_t^!(A|_U) \arrow[r] \arrow[d] & A|_U \arrow[d] \\ \nu(M,N) \arrow[r, "\psi"'] & U \end{tikzcd}$$

all with the base map ψ. For $t = 0$, we have that $\lambda_0 \circ \psi = \psi \circ \kappa_0 = i \circ p$, so we obtain the desired Lie algebroid isomorphism

$$\widetilde{\psi}\colon p^! i^! (A|_U) \to A|_U,$$

with base map ψ. $\square$

Remark 8.14. One can show that $p^! i^! A$ is canonically isomorphic to $\nu(A, i^! A)$, and that $\widetilde{\psi}$ is the tubular neighborhood embedding defined by $\widetilde{X}$.

If the normal bundle is trivial, $\nu_N = N \times S$, then we obtain the simpler model

$$p^! i^! A = i^! A \times TS$$

as Lie algebroids. In particular, we obtain:

Corollary 8.15 (Local splitting of Lie algebroids)**.** *Let $(A, \mathsf{a}, [\cdot,\cdot])$ be a Lie algebroid over M, and $m \in M$. Let $i\colon N \hookrightarrow M$ be a submanifold containing m, such that $T_m N$ is a complement to $S = \mathsf{a}_m(A_m)$ in $T_m M$. Then Lie algebroid A is isomorphic, near m, to the direct product of Lie algebroids $i^! A \times TS$. If a compact Lie group G acts on A by Lie algebroid automorphisms, such that the action on M fixes m and preserves N, this isomorphism can be chosen G-equivariant.*

For $G = \{1\}$ this result is due to Weinstein [40], Fernandes [21], and Dufour [18].

Remark 8.16. There is a more elegant proof of Theorem 8.13, using the deformation to the normal cone $\mathcal{D}(M,N)$. We will explain this proof in the appendix.

8.6. **The generalized foliation.** An important consequence of the local splitting theorem is:

Proposition 8.17. *Let $A \to M$ be a Lie algebroid. Through every $m \in M$ there passes a unique maximal connected integral submanifold of A.*

Proof. In the local model given by Corollary 8.15, it is immediate that $\{m\} \times S$ is an integral submanifold, and the germ of such an integral submanifold is unique. Patching the local integral submanifolds, one obtains a maximal integral submanifold. □

Thus, M acquires a generalized foliation, with these maximal integral submanifolds as its leaves. The leaf through $m \in M$ is called the *orbit* of the Lie algebroid M.

Example 8.18. For any $\mathfrak{g}$-action, we obtain a decomposition of M into $\mathfrak{g}$-orbits. Note that we did not have to assume any special properties of the action.

9. The Lie functor

9.1. **The Lie algebra of a Lie group.** Before discussing the case of Lie algebroids, let us quickly review the construction of the Lie algebra of a Lie group G. As a vector space,

$$g = T_e G$$

is the tangent space to the identity. The Lie bracket of $\mathfrak{g}$ can be defined in several equivalent ways:

9.1.1. *Bracket via conjugation.* Consider the conjugation action of group elements $a \in G$:

$$\mathrm{Ad}_a\colon G \to G,\ g \mapsto \mathrm{Ad}_a(g) = aga^{-1}.$$

This action fixes the group unit, hence its tangent map at the identity is a linear map, $T_e \mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}$ It is common to denote this map again by

$$\mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}.$$

It may be regarded as a Lie group morphism $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}),\ a \mapsto \mathrm{Ad}_a$, where $\mathrm{GL}(\mathfrak{g})$ is the group of invertible linear transformations of $\mathfrak{g}$. Taking the differential at the identity, we obtain a linear map $T_e \mathrm{Ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. It is common to denote this map by

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

In terms of this map, one defines the Lie bracket as

$$[X, Y] = \mathrm{ad}_X(Y),$$

and one proves that it is skew-symmetric and satisfies the Jacobi identity.

If G is a *matrix Lie group* i.e. if it is a closed subgroup of the group $\mathrm{GL}(n, \mathbb{R}) = \mathrm{Mat}_{\mathbb{R}}(n)^{\times}$ of invertible $n \times n$-matrices for some $n \in \mathbb{N}$, then $\mathfrak{g} = T_e G$ is identified as a subspace of $\mathrm{Mat}_{\mathbb{R}}(n)$. The map $\mathrm{Ad}_a\colon G \to G$ is conjugation of matrices, hence $\mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}$ is again just conjugation $\mathrm{Ad}_a(Y) = aYa^{-1}$. Taking the derivative with respect to a, one obtains $\mathrm{ad}_X(Y) = XY - YX$, the usual commutator of matrices.

9.1.2. *Bracket via BCH series.* For any Lie group G and any $X \in \mathfrak{g}$, there is a unique group homomorphism $\phi_X : \mathbb{R} \to G$ whose differential at zero is the map $T_0\phi_X : T_0\mathbb{R} = \mathbb{R} \to \mathfrak{g},\ t \mapsto tX$. One defines an *exponential map* $\exp: \mathfrak{g} \to G,\ X \mapsto \phi_X(1)$. It is easy to see that exp is a local diffeomorphism near 0, hence its inverse $\log = \exp^{-1}$ is defined close to $e \in G$. One defines $[X, Y]$ by `check`

$$\log(\exp(tX)\exp(tY)) = t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3),$$

and proves that $[X, Y]$ is a Lie bracket. For a matrix Lie group, exp is just the usual exponential of matrices, and this formula (up to quadratic terms) is an easy consequence of Taylor's theorem.

9.1.3. *Bracket via left-invariant vector fields.* Every $a \in G$ acts on G by left translation $\mathcal{A}^L_a$. A vector field is left-invariant if it is invariant under $\mathcal{A}^L_a$ for all $a \in G$. Such a vector field is determined by its restriction to the group unit; conversely, each $X \in \mathfrak{g}$ has a unique extension to a left-invariant vector field. The Lie bracket of left-invariant vector fields is again left-invariant, and one uses this to define the bracket on $\mathfrak{g}$ by requiring that

$$[X, Y]^L = [X^L, Y^L].$$

9.1.4. *Bracket via Maurer-Cartan forms.* The left-invariant Maurer-Cartan form is the unique left-invariant $\mathfrak{g}$-valued 1-form such that

$$\iota(X^L)\theta^L = X$$

for all $X \in \mathfrak{g}$. For a matrix Lie group, one has that $\theta^L = g^{-1}\mathrm{d}g$, and since

$$\mathrm{d}(g^{-1}\mathrm{d}g) = -g^{-1}\mathrm{d}g\ g^{-1}\mathrm{d}g = -(g^{-1}\mathrm{d}g)^2 = -\frac{1}{2}[g^{-1}\mathrm{d}g,\ g^{-1}\mathrm{d}g$$

these satisfy the Maurer-Cartan equation,

$$\mathrm{d}\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0.$$

For general Lie groups, one can take this equation to be the *definition* of the Lie bracket on $\mathfrak{g}$. Let us verify that this convention is consistent with the definition via left-invariant vector fields: Using Cartan's calculus,

$$\begin{aligned}\iota(Y^L)\iota(X^L)\mathrm{d}\theta &= \iota(Y^L)\mathcal{L}(X^L)\theta^L - \iota(Y^L)\mathrm{d}\iota(X^L)\theta^L \\ &= \mathcal{L}(X^L)\iota(Y^L)\theta^L - \iota([X^L, Y^L])\theta^L - \iota(Y^L)\mathrm{d}X \\ &= \mathcal{L}(X^L)Y - \mathcal{L}(Y^L)X - \iota([X^L, Y^L])\theta^L \\ &= -\iota([X^L, Y^L])\theta^L\end{aligned}$$

where we used that X, Y are constant $\mathfrak{g}$-valued functions, and

$$\iota(Y^L)\iota(X^L)(-\frac{1}{2}[\theta^L, \theta^L]) = -\iota(Y^L)[X, \theta^L] = -[X, Y] = -\iota([X, Y]^L)\theta^L.$$

Comparing, we find $[X^L, Y^L] = [X, Y]^L$ as required.

Remark 9.1. Note if we were to use right-invariant vector fields to define the bracket, we would end up with the *opposite bracket* (unless we also change the definition of commutator of vector fields, as done in some of the literature). However, this would then result in sign changes also for the other formulas involving the bracket!

9.2. **The Lie algebroid of a Lie groupoid.** We now define the Lie algebroid $A = \mathrm{Lie}(\mathcal{G})$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$. As a vector bundle, we take

$$\mathrm{Lie}(\mathcal{G}) = \nu(\mathcal{G}, M) = T\mathcal{G}|_M/TM$$

to be the normal bundle of M in $\mathcal{G}$. To define the anchor map, note that t and s coincide on $M \subseteq \mathcal{G}$, hence the difference $T\mathsf{s} - T\mathsf{t}\colon T\mathcal{G} \to TM$ vanishes on $TM \subseteq T\mathcal{G}$, and hence descends to a map $\nu(\mathcal{G}, M) \to TM$ which we take to be the anchor,

$$\mathsf{a}\colon \mathrm{Lie}(\mathcal{G}) \to TM.$$

For the definition of the Lie bracket on $\Gamma(\mathrm{Lie}(\mathcal{G}))$, we may use any of the standard approaches for Lie groups. For example, we may observe that any 1-parameter family of bisections $S_t \subseteq \mathcal{G}$, with $S_0 = M$, has differential at $t = 0$ a section $\sigma \in \Gamma(A)$. In this sense, we may regard $\Gamma(A)$ as the tangent space to the infinite-dimensional group $\Gamma(\mathcal{G})$ at the identity, and define the bracket on $\Gamma(\mathrm{Lie}(\mathcal{G}))$ in such a way that

$$\mathrm{Lie}(\Gamma(\mathcal{G})) = \Gamma(\mathrm{Lie}(\mathcal{G}))$$

as Lie algebras. Concretely, one has an adjoint action of $\Gamma(\mathcal{G})$ on $\Gamma(\nu(\mathcal{G}, M))$, given by

$$\mathrm{Ad}_S(\tau) = \frac{\partial}{\partial t}|_{t=0}(SR_tS^{-1})$$

whenever R_t is a 1-parameter family of bisections such that $\tau = \frac{\partial}{\partial t}|_{t=0}R_t$. One may use this to define an infinitesmal adjoint action,

$$\mathrm{ad}_\sigma(\tau) = \frac{\partial}{\partial t}|_{t=0}\,\mathrm{Ad}_{S_t}\tau$$

whenever S_t is a 1-parameter family of bisections such that $\sigma = \frac{\partial}{\partial t}|_{t=0}S_t$. The Lie algebroid bracket is then $[\sigma, \tau] = \mathrm{ad}_\sigma \tau$. Parallel to the case of Lie groups, one may verify that this defines a Lie bracket, which furthermore satisfies the Leibnitz rule. Rather than pursuing this approach further, we turn to the alternative definition in terms of left-invariant vector fields.

9.3. **Left-and right-invariant vector fields.** For a general action of a groupoid $\mathcal{G} \rightrightarrows M$ on a manifold Q, there is no natural'lift' of the $\mathcal{G}$-action to the tangent bundle TQ. Suppose however that the moment map $\Phi\colon Q \to M$ is a submersion (or, more generally, has constant rank). Then the fibers of Φ are submanifolds, and every $g \in \mathcal{G}$, with source $m = \mathsf{s}(g)$ and target $m' = \mathsf{t}(g)$, gives a diffeomorphism

$$g\colon \Phi^{-1}(m) \to \Phi^{-1}(m'), \quad q \mapsto g \cdot q, \tag{24}$$

and hence by differentiation a bundle map $T(\Phi^{-1}(m)) \to T(\Phi^{-1}(m'))$. In this way, one obtains an action of $\mathcal{G}$ on $\ker(T\Phi) \subseteq TQ$, with moment map the bundle projection $TQ \to Q$ followed by Φ.

Definition 9.2. Suppose Q is a $\mathcal{G}$-manifold such that the moment map $\Phi\colon Q \to M$ for the action has constant rank. A vector field $X \in \mathfrak{X}(Q)$ is called $\mathcal{G}$-invariant if it is tangent to Φ-fibers, and if for all $q \in Q$ and $g \in G$ such that $\mathsf{s}(g) = \Phi(q)$,

$$X_{g.q} = g.X_q.$$

The space of $\mathcal{G}$-invariant vector fields is denoted $\mathfrak{X}(Q)^{\mathcal{G}}$.

Exercise 9.1. Show that a vector field $X \in \mathfrak{X}(Q)$ is $\mathcal{G}$-invariant if and only if for all local bisections S of $\mathcal{G}$, with $\mathsf{s}(S) = U$ and $\mathsf{t}(S) = V$, the diffeomorphism $\Phi^{-1}(U) \to \Phi^{-1}(V)$ takes X to itself. In particular, X must be tangent to the Φ-fibers.

Lemma 9.3. *The invariant vector fields on Q form a Lie subalgebra.*

Proof. By definition, a vector field X is $\mathcal{G}$-invariant if and only if it is tangent to Φ-fibers, and for all $g \in \mathcal{G}$, with $\mathsf{s}(g) = m,\ tz(g) = m'$, the restrictions $X|_{\Phi^{-1}(m)}$ and $X|_{\Phi^{-1}(m')}$ are related under (24). But if two pairs of vector fields are related under a diffeomorphism, then their Lie brackets are also related. $\square$

As a special case, a vector field X on $\mathcal{G}$ itself is called *left-invariant* if it is tangent to the source fibers, and $g \cdot X_h = X_{g \circ h}$ under the action $\mathcal{A}^L$. Similarly, X is called right invariant if it is tangent to s-fibers, and satisfies $g \cdot X = X_{h \circ g^{-1}}$ where the action is $\mathcal{A}^R$. The spaces of left-invariant, respectively right-invariant vector fields, are denoted

$$\mathfrak{X}^L(\mathcal{G}),\ \mathfrak{X}^R(\mathcal{G}).$$

By construction, both are Lie subalgebras. Now, using the left multiplication, we may identify $\ker(T_h \mathsf{t}) \cong \ker(T_{\mathsf{s}(h)}\mathsf{t})$, thus

$$\ker(T\mathsf{t}) = \mathsf{s}^* \ker(T\mathsf{t})|_M.$$

A vector field is left-invariant of and only if it is 'constant' under this identification. Similarly, the right multiplication identifies

$$\ker(T\mathsf{s}) = \mathsf{t}^* \ker(T\mathsf{s})|_M,$$

and a vector field is right-invariant if and only if it is constant under this isomorphism. Since both $\ker(T\mathsf{t})|_M$, $\ker(T\mathsf{s})|_M$ are complements to TM in $T\mathcal{G}|_M$, each of these bundles may be identified with the normal bundle. That is, we have

$$\ker(T\mathsf{t})|_M \cong \nu(\mathcal{G}, M) \cong \ker(T\mathsf{s})|_M. \tag{25}$$

Definition 9.4. For $\sigma \in \Gamma(\mathrm{Lie}(\mathcal{G}))$, we denote by

$$\sigma^L \in \mathfrak{X}(\mathcal{G})^L, \quad \sigma^R \in \mathfrak{X}(\mathcal{G})^R$$

the unique left-invariant, right-invariant vector fields mapping to σ under restriction to M followed by the quotient map $T\mathcal{G}|_M \to \nu(\mathcal{G}, M)$.

Lemma 9.5. *For all $\sigma \in \Gamma(\mathrm{Lie}(\mathcal{G}))$, we have that*
$$\sigma^L \sim_{\mathsf{t}} 0, \quad \sigma^L \sim_{\mathsf{s}} \mathsf{a}(\sigma), \quad \sigma^R \sim_{\mathsf{t}} -\mathsf{a}(\sigma), \quad \sigma^R \sim_{\mathsf{s}} 0,$$
and
$$\mathsf{a}(\sigma) \sim_i \sigma^L - \sigma^R$$
where $i\colon M \to \mathcal{G}$ is the inclusion of units. Furthrmore,
$$\sigma^L \sim_{\mathrm{Inv}_{\mathcal{G}}} -\sigma^R.$$

Proof. Since σ^L is tangent to the t-fibers, we have that $\sigma^L \sim_{\mathsf{t}} 0$. Since s is invariant under the left-action we see that
$$\sigma^L \sim_{\mathsf{s}} (T\mathsf{s})(\sigma^L|_M) \in \mathfrak{X}(M).$$
But
$$(T\mathsf{s})(\sigma^L|_M) = (T\mathsf{s} - T\mathsf{t})(\sigma^L|_M) = \mathsf{a}(\sigma).$$
This proves $\sigma^L \sim_{\mathsf{s}} \mathsf{a}(\sigma)$; similarly $\sigma^R \sim_{\mathsf{t}} -\mathsf{a}(\sigma)$. Finally, note that $\sigma^L - \sigma^R$ is tangent to M, hence restricts to a vector field on M. Since
$$T\mathsf{s}((\sigma^L - \sigma^R)|_M) = \mathsf{a}(\sigma),$$
we see that this restriction is $\mathsf{a}(\sigma)$. The last claim follows since $\mathrm{Inv}_{\mathcal{G}}$ interchanges source and target maps, and the induced map on the normal bundle is multiplication by -1. $\square$

Lemma 9.6. *We have, for all $\sigma, \tau \in \Gamma(\mathrm{Lie}(\mathcal{G}))$,*
$$[\sigma^L, \tau^L] = [\sigma, \tau]^L, \quad [\sigma^L, \tau^R] = 0, \quad [\sigma^R, \tau^R] = -[\sigma, \tau]^R. \tag{26}$$
The (local) flow of σ^L is by right translations by (local) bisections, the flow of $-\sigma^R$ is left translation by the same (local) bisections.

Proof. The first formula follows from the definition of the bracket, the third formula since inversion takes σ^L to $-\sigma^R$. We will now show that the (local) flow of $-\sigma^R$ is by left translations by (local) bisections. Applying the inversion, this will imply that the flow of σ^L is by right translations by local bisections, hence the flows of left-and right-invariant vector fields commute, so $[\sigma^L, \tau^R] = 0$,

The vector field $-\sigma^R$ is tangent to s-fibers, and satisfies
$$(\mathcal{A}^R_g)_*(\sigma^L|_{\mathsf{s}^{-1}(m)}) = \sigma^L|_{\mathsf{s}^{-1}(m')}$$
for $\mathsf{s}(g) = m,\ \mathsf{t}(g) = m'$. Hence, the flow ϕ_t of σ^L has the property $\mathsf{s} \circ \phi_t = \mathsf{s}$, and
$$\phi_t(h \circ g^{-1}) = \phi_t(h) \circ g^{-1}.$$
Letting $a = g^{-1}$ and $h = \mathsf{s}(g^{-1}) = \mathsf{t}(a)$, this shows
$$\phi_t(a) = \phi_t(\mathsf{t}(a)) \circ a.$$
Hence, the flow is left translation by a local bisection S_t, consisting of all $\phi_t(m)$ for all $m \in M$ such that the time t flow is defined. $\square$

9.4. **The Lie functor from Lie groupoids to Lie algebroids.** Following the standard definition for Lie groups, we will use the identification of $\Gamma(\mathrm{Lie}(\mathcal{G}))$ with *left-invariant* vector fields on $\mathcal{G}$ to define a Lie bracket on $\Gamma(\mathrm{Lie}(\mathcal{G}))$. In other words, the sign conventions are such that

$$[\sigma^L, \tau^L] = [\sigma, \tau]^L.$$

Lemma 9.7. *With these data,* $(\mathrm{Lie}(\mathcal{G}), \mathsf{a}, [\cdot,\cdot])$ *is a Lie algebroid.*

Proof. We only need to verify the Leibnitz rule. Let $f \in C^\infty(M)$ and $\sigma, \tau \in \Gamma(\mathrm{Lie}(\mathcal{G})))$. Using $(f\tau)^L = (\mathsf{s}^* f)\,\tau^L$ and $\sigma^L(\mathsf{s}^* f) = \mathsf{s}^*\big(\mathsf{a}(\sigma) f\big)$ (since $\sigma^L \sim_{\mathsf{s}} \mathsf{a}(\sigma)$) we compute,

$$[\sigma, f\tau]^L = [\sigma^L,\ \mathsf{s}^* f\ \tau^L] = \mathsf{s}^* f\,[\sigma^L, \tau^L] + \mathsf{s}^*(\mathsf{a}(\sigma) f)\tau^L = \big(f[\sigma,\tau] + \big(\mathsf{a}(\sigma) f\big)\,\tau\big)^L,$$

proving $[\sigma, f\tau] = f[\sigma,\tau] + \big(\mathsf{a}(\sigma) f\big)\,\tau$. $\square$

One calls the Lie algebroid $\mathrm{Lie}(\mathcal{G})$ the differentiation of the Lie groupoid $\mathcal{G}$; conversely, if $A \to M$ is a given Lie algebroid, a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a Lie algebroid isomorphism $A \cong \mathrm{Lie}(\mathcal{G})$ is called an *integration* of A.

Example 9.8. For the pair groupoid $\mathrm{Pair}(M) = M \times M$, a left -invariant vector field is of the form $(0, X)$ where X is a vector field on M, while right-invariant vector fields are of the form $(X, 0)$. The flow of a left-invariant vector field is $(m', m) \mapsto (m', \Phi_t(m))$ where Φ_t is the flow of X. This is right multiplication by the element $(m, \Phi_t(m)) = (\Phi_t(m), m)^{-1}$. Similarly, a right invariant vector field has flow $(m', m) \mapsto (\Phi_t(m'), m)$. From the definitions, it is clear that

$$\mathrm{Lie}(\mathrm{Pair}(M)) = TM$$

as a Lie algebroid. Likewise, the fundamental groupoid $\Pi(M)$ has TM as its associated Lie algebroid. Hence, both $\Pi(M)$ and $\mathrm{Pair}(M)$ integrate the tangent bundle.

Lemma 9.9. *If* $\mathcal{H} \rightrightarrows N$ *is a Lie subgroupoid of* $\mathcal{G} \rightrightarrows M$*, then the natural map of normal bundles, induced functorially by the map of pairs* $(\mathcal{H}, N) \to (\mathcal{G}, M)$*, defines a morphism of Lie algebroids*

$$\mathrm{Lie}(\mathcal{H}) \to \mathrm{Lie}(\mathcal{G}).$$

Proof. It is immediate that $\mathrm{Lie}(\mathcal{H}) \to N$ is a subbundle of $\mathrm{Lie}(\mathcal{G}) \to M$. Furthermore, $\sigma \in \Gamma(\mathrm{Lie}(\mathcal{G}))$ restricts to a section of $\mathrm{Lie}(\mathcal{H})$, if and only if the vector field σ^L is tangent to $\mathcal{H}$. If σ, τ are two such sections, then $[\sigma^L, \tau^L] = [\sigma, \tau]^L$ again is tangent to $\mathcal{H}$. That is, the space

$$\Gamma(\mathrm{Lie}(\mathcal{G}), \mathrm{Lie}(\mathcal{H}))$$

of sections of $\mathrm{Lie}(\mathcal{G})$ that restrict to sections of $\mathrm{Lie}(\mathcal{H})$, is a Lie subalgebra. $\square$

More generally, we have:

Theorem 9.10. *If $\mathcal{H} \rightrightarrows N$ and $\mathcal{G} \rightrightarrows M$ are Lie groupoids, then the map on normal bundles defined by the map of pairs $F\colon (\mathcal{H}, N) \to (\mathcal{G}, M)$, is a morphism of Lie algebroids*

$$\mathrm{Lie}(F)\colon\ \mathrm{Lie}(\mathcal{H}) \to \mathrm{Lie}(\mathcal{G}).$$

Proof. F is a morphism of Lie groupoids if and only if $\mathrm{Gr}(F) \subseteq \mathcal{G} \times \mathcal{H}$ is a Lie subgroupoid. But then $\mathrm{Lie}(\mathrm{Gr}(F)) \subseteq \mathrm{Lie}(\mathcal{G} \times \mathcal{H}) = \mathrm{Lie}(\mathcal{G}) \times \mathrm{Lie}(\mathcal{H})$ is a Lie subalgebroid. □

In summary, we have constructed a *Lie functor* from the category of Lie groupoids (and their morphisms) to Lie algebroids (and their morphisms).

9.5. **Examples.** We list the Lie algebroids for our main examples of Lie groupoids.

(a) For a Lie groupoid over a point $\mathcal{G} \rightrightarrows \mathrm{pt}$, one recovers the usual notion of the Lie algebra of a Lie group.

(b) For the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$, on obtains
$$\mathrm{Lie}(\mathrm{Pair}(M)) = TM,$$
the tangent bundle. The homotopy groupoid $\Pi(M)$ has the same Lie algebroid.

(c) For the Atiyah groupoid $\mathcal{G}(P) \rightrightarrows M$ of a principal K-bundle $P \to M$, we obtain the Atiyah Lie algebroid:
$$\mathrm{Lie}(\mathcal{G}(P)) = A(P).$$

(d) Given a K-action on a manifold, with corresponding action groupoid $K \ltimes M \rightrightarrows M$, one has
$$\mathrm{Lie}(K \ltimes M) = \mathfrak{k} \ltimes M,$$
the action Lie algebroid for the infinitesimal action.

(e) For the k-th jet prolongation $J_k(\mathcal{G})$ of a Lie groupoid, we obtain the k-th jet prolongation of the corresponding Lie algebroid:
$$\mathrm{Lie}(J_k(\mathcal{G})) = J_k(\mathrm{Lie}(\mathcal{G})).$$

(f) For the monodromy groupoid $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$ of a foliation, we obtain the tangent bundle of the foliation:
$$\mathrm{Lie}(\mathrm{Mon}(\mathcal{F})) = T_{\mathcal{F}}M.$$
The holonomy groupoid has the same Lie algebroid.

9.6. **Groupoid multiplication via σ^L, σ^R.** `we probably won't discuss this in class` For $i = 0, 1, 2$ and $\sigma \in \Gamma(A)$, the vector fields on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$,

$$X^0_\sigma = (-\sigma^R, -\sigma^R, 0),\quad X^1_\sigma = (0, \sigma^L, -\sigma^R),\quad X^2_\sigma = (\sigma^L, 0, \sigma^L). \tag{27}$$

are all tangent to $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$. For instance, the invariance of the graph under the (local) flow of X^1_σ follows from the fact that $g_1 \circ g_2 = (g_1 \circ h^{-1}) \circ (h \circ g_2)$ whenever $\mathsf{s}(h) = \mathsf{t}(g_2)$. The vector fields satisfy bracket relations
$$[X^i_\sigma,\, X^j_\tau] = X^i_{[\sigma,\tau]}\ \delta_{i,j}$$
for $\sigma, \tau \in \Gamma(A)$ and $i, j = 0, 1, 2$. If $\mathcal{G}$ is t-connected, then the graph is generated from $M \subseteq \Lambda$ (embedded as $m \mapsto (m, m, m)$) by the flow of these vector fields. In fact, it is already obtained using the flows of the X^i_σ's for any two of the indices $i \in \{0, 1, 2\}$. For reference in Section **??**, let us note the following partial converse.

Proposition 9.11. *Let $(A, \mathsf{a}, [\cdot,\cdot])$ be a Lie algebroid over M. Suppose $i\colon M \to P$ is an embedding, with normal bundle $\nu(P,M) \cong A$, and suppose σ^L, $\sigma^R \in \mathfrak{X}(P)$ are vector fields on P, mapping to $\sigma \in \Gamma(A)$ under the quotient map $TP|_M \to A$, with $\mathsf{a}(\sigma) \sim_i \sigma^L - \sigma^R$, and satisfying the bracket relations (26). Then a neighborhood of M in P inherits a structure of a local Lie groupoid integrating A, in such a way that σ^L, σ^R are the left, right invariant vector fields.*

Sketch of proof. Since $\sigma^L|_M$ maps to σ under $T\mathcal{G}|_M \to \nu(\mathcal{G}, M)$, the restrictions of the left-invariant vector fields to M span a complement to TM in $TP|_M$. In a particular, on a neighborhood of M they determine a distribution of rank equal to that of A. By the bracket relations, this distribution is Frobenius integrable. A similar argument applies to the vector fields $-\sigma^R$. Taking P smaller if necessary, we can assume that these foliations define surjective submersions $\mathsf{t}, \mathsf{s}\colon P \to M$, with σ^L, σ^R tangent to the respective fibers, and with $\mathsf{t} \circ i = \mathsf{s} \circ i = \mathrm{id}$. Define vector fields X^i_σ in $P \times P \times P$ as above, and embed $M \hookrightarrow P \times P \times P$ by $m \mapsto (m, m, m)$. Along M, hence also on some neighborhood of M inside $P \times P \times P$, the vector fields X^0_σ, X^2_τ span a distribution of rank equal to twice the rank of A. The bracket relations guarantee that this distribution is integrable, hence they define a foliation. Since the intersection of this distribution with the tangent bundle of $M \subseteq P \times P \times P$ is trivial, we conclude that the flow-out of M under these vector fields defines a (germ of a) submanifold $\Lambda \subseteq P \times P \times P$, of dimension $2\,\mathrm{rank}(A) + \dim M = 2\dim(A) - \dim M$ This is our candidate for the graph of the multiplication map.

By construction, $\Lambda \subseteq P \times P \times P$ contains M, and is invariant under the local flow of all vector fields X^0_σ, X^2_σ. In fact, it is also invariant under the local flow of X^1_τ for $\tau \in \Gamma(A)$, since these vector fields are tangent to Λ along M, and hence everywhere since the commute with all X^0_σ, X^2_σ.

Under projection $P \times P \times P \to P \times P$, $(p, p_1, p_2) \mapsto (p_1, p_2)$, the vector fields X^0_σ, X^2_σ are related to $(-\sigma^R, 0)$ and $(0, \sigma^L)$, respectively. Hence, this projection restricts to a diffeomorphism from Λ onto a neighborhood of the diagonal embedding of M in $P^{(2)} = P\ {}_{\mathsf{s}}\!\times_{\mathsf{t}} P \subseteq P \times P$. Taking the inverse map, followed by projection to the first P-factor, defines a multiplication map $\mathrm{Mult}_P\colon P^{(2)} \to P$; strictly speaking it is defined only on some neighborhood of M in $P^{(2)}$. By construction, $\Lambda = \mathrm{Gr}(\mathrm{Mult}_P)$.

Letting $\mathrm{id}_P\colon P \to P$ be the identity relation (given by the diagonal in $P \times \overline{P}$, the associativity of the groupoid multiplication means that

$$\Lambda \circ (\Lambda \times \mathrm{id}_P) = \Lambda \circ (\mathrm{id}_P \times \Lambda)$$

as relations $P \times P \times P \dashrightarrow P$, where the circle means composition of relations. In fact, we can see that both sides are given by

$$\Lambda^{[2]} \subseteq P \times (P \times P \times P),$$

the submanifold generated from the diagonal $M^{[3]}_\Delta$ (consisting of elements (m, m, m, m)) by the action of vector fields of the form

$$(-\sigma^R, -\sigma^R, 0, 0),\ (0, \sigma^L, -\sigma^R, 0),\ (0, 0, \sigma^L, -\sigma^R). \qquad \square$$

10. Integrability of Lie algebroids: The transitive case

A Lie groupoid $\mathcal{G} \rightrightarrows M$ is said to *integrate* a Lie algebroid $A \to M$ if there exists an isomorphism of Lie algebroids

$$A \cong \mathrm{Lie}(\mathcal{G}).$$

Lie's third theorem states that every finite-dimensional Lie algebra integrates to a Lie group G. In one of the early articles on the subject, Pradines [34] announced that a similar statement holds true for Lie algebroids. That this is not so was observed several years later by Almeida and Molino [1], in the context of their work on 'developability of foliations'. This then opened the question of specifying the obstructions to integrability, and after work by a number of authors (e.g., [14, 16, 17, 29, 32]), and building on ideas of Duistermaat-Kolk [19], Cattaneo-Felder [8], and Weinstein, this question was finally settled by Crainic and Fernandes in [11].

We will begin our discussion of the integration problem with the case of transitive Lie algebroids. Here, the integration problem had been solved by Mackenzie [29], but our approach will be somewhat different from his.

10.1. **The Almeida-Molino counter-example.** Let M be a connected, simply connected manifold. Given a closed 2-form

$$\omega \in \Omega^2(M), \quad \mathrm{d}\omega = 0$$

consider the Lie algebroid $A = TM \times \mathbb{R}$, with the bracket on sections given as

$$[X + f, Y + g] = [X, Y] + \nabla_X(g) - \nabla_Y(f) + \omega(X, Y)$$

for vector fields $X, Y \in \mathfrak{X}(M)$. Let us *assume* that A is integrable to a Lie groupoid $\mathcal{G} \rightrightarrows M$. Since the Lie groupoid is transitive, it the Atiyah groupoid $\mathcal{G}(P)$ of a principal G-bundle $P \to M$, where G is a 1-dimensional structure group. Since M was assumed to be 1-connected, we can take the fibers of P, and hence the structure group G, to be connected; thus $G = \mathbb{R}/\pi_1(G)$ is either a circle (if $\pi_1(G)$ is non-trivial) or all of $\mathbb{R}$ (if $\pi_1(G)$ is trivial). The canonical splitting $TM \to A,\ v \mapsto v + 0$ of the Lie algebroid A corresponds to a principal connection on P, with ω as its curvature 2-form.

Note that even if $\pi_1(G)$ is non-trivial, the corresponding lattice in $\mathbb{R}$ need not be the standard $\mathbb{Z}$ – that is, the 'size' of the circle G may depend on the Lie algebroid A. The main observation is that we can detect the 'size' via holonomy of the connection. The holonomy associates to every loop $\lambda\colon S^1 \to M$ an element $\mathrm{Hol}(\lambda) \in G = \mathbb{R}/\pi_1(G)$. This holonomy is defined by integration of the connection, but since λ is contractible it may also be computed in terms of the curvature. Indeed, if $\phi\colon D^2 \to M$ is a smooth map from the disk, extending λ, then

$$\mathrm{Hol}(\lambda) = \int_{D^2} \phi^*\omega \mod \pi_1(G)$$

For this to be consistent, the right hand side cannot depend on how we extend λ to a map from the disk. Given two extensions $\phi_\pm\colon D^2 \to M$, the difference is the integral over the map from the 2-sphere

$$\phi\colon S^2 \to M$$

whose restrictions to the upper and lower hemisphere are $\phi_\pm$. Hence, the consistency requirement is that for all maps $\phi\colon S^2 \to M$, the integral

$$\int_{S^2} \phi^*\omega$$

takes values in $\pi_1(G) \subseteq \mathbb{R}$. Hence, letting $\Lambda \subseteq \mathbb{R}$ be the subgroup given as image of the group homomorphism

$$\pi_2(M) \to \mathbb{R},\ [\phi] \mapsto \int_{S^2} \phi^*\omega,$$

we have that $\Lambda \subseteq \pi_1(G) \subseteq \mathbb{R}$.In particular, Λ is a discrete subgroup of $\mathbb{R}$. To summarize we arrive at the following condition:

> **Criterion:** If the Lie algebroid $A = TM \times \mathbb{R}$, with bracket defined by the closed 2-form ω, is integrable, then the subgroup Λ must be *discrete*.

It is easy to give an example where this criterion fails:

Example 10.1. Take $M = S^2 \times S^2$, let $\sigma \in \Omega^2(S^2)$ be the standard area form, and put with $\omega = \mathrm{pr}_1^*\,\sigma + \sqrt{2}\,\mathrm{pr}_2^*\,\sigma$. Here the set of integrals of ω over 2-spheres is $\mathbb{Z} + \sqrt{2}\mathbb{Z}$, which is dense in $\mathbb{R}$.

On the other hand, if the criterion is satisfied then A is indeed integrable, using a standard construction (see Pressley-Segal [35]) of reconstructing a principal circle bundle with connection from its curvature 2-form. The Lie groupoid is explicitly given as

$$\mathcal{G} = \{[(\gamma, u)]|\ \gamma \in C^\infty([0,1], M),\ u \in \mathbb{R}/\Lambda\}$$

with the equivalence relation

$$(\gamma_1, u_1) \sim (\gamma_0, u_0) \Leftrightarrow \gamma_1(0) = \gamma_0(0),\ \gamma_1(1) = \gamma_0(1),\ u_1 = u_0\ \ \mathrm{Hol}(\gamma_1^{-1} * \gamma_0)$$

where the holonomy may be *defined* using the integral of ω over a disk bounded by $\gamma_1^{-1} * \gamma_0$. The groupoid structure is induced by the concatenation of paths,

$$[(\gamma', u')] \circ [(\gamma, u)] = [(\gamma' * \gamma, u' + u)].$$

We leave it as an exercise to show that this Lie groupoid has 1-connected source fibers.

10.2. **Transitive Lie algebroids.** For any transitive Lie algebroid $A \to M$, we have an exact sequence of Lie algebroids,

$$0 \to L \to A \to TM \to 0, \tag{28}$$

where $L = \ker(\mathsf{a}) \subseteq A$. Its fibers are the isotropy Lie algebras, $\mathfrak{g}_m = \ker(\mathsf{a}_m)$.

The splitting theorem for Lie algebroids shows that *locally*, on a neighborhood U of any given point $m \in M$, $A|_U = TU \times \mathfrak{h}$ for a fixed Lie algebra $\mathfrak{h}$. In particular,

$$L \to M$$

is a (locally trivial) Lie algebra bundle with fibers $L_m \equiv \mathfrak{g}_m \cong \mathfrak{h}$.

Given a Lie algebra $\mathfrak{h}$, let $\mathrm{Tran}_{\mathfrak{h}}(M)$ be the set of isomorphism classes of transitive Lie algebroids over M, with structure Lie algebra $\mathfrak{h}$. Similarly, for a given Lie group H, let

$\mathrm{Prin}_H(M)$ be the isomorphism classes of principal H-bundles with structure group H. If $\mathfrak{h} = \mathrm{Lie}(H)$, we have the map

$$\mathrm{Prin}_H(M) \to \mathrm{Tran}_{\mathfrak{h}}(M) \tag{29}$$

taking a principal bundle to its Atiyah algebroid. If a transitive Lie algebroid $A \to M$ is integrable, then the Lie groupoid $\mathcal{G}$ integrating it is the Atiyah groupoid of a principal H-bundle $P \to M$. That is, A is integrable if and only if its class in $\mathrm{Trans}_{\mathfrak{h}}(M)$ lies in the image of (29) for *some* Lie group H integrating $\mathfrak{h}$. We stress that this Lie group H need not be connected.

Let $\mathfrak{z}_m$ be the center of $\mathfrak{g}_m$, and Z_m the center of the simply connected Lie group integrating $\mathfrak{g}_m$.

Let $\mathrm{Cent}(L) \subseteq L$ be the subbundle with fibers $\mathfrak{z}_m$.

Lemma 10.2. *The bundle* $\mathrm{Cent}(L)$ *has a canonical flat connection.*

Proof. Given $X \in \mathfrak{X}(M)$ and $\tau \in \Gamma(\mathrm{Cent}(L))$, define a covariant derivative in terms of the Lie algebroid bracket by

$$\nabla_X \tau = [\sigma, \tau], \quad X \in \mathfrak{X}(M),\ \tau \in \Gamma(L)$$

where $\sigma \in \Gamma(A)$ is any section with $\mathsf{a}(\sigma) = X$. This is well-defined, because σ is unique up to a section of L, and sections of L have trivial bracket with all sections of $\mathrm{Cent}(L)$. The Jacobi identity for the Lie bracket on $\Gamma(A)$ shows that this connection is flat. □

As a consequence, the bundle $\bigcup_{m \in M} Z_m$ also has a flat connection. Hence, for any smooth path $\gamma\colon [0,1] \to M$ from $m = \gamma(0)$ to $m' = \gamma(1)$, we obtain a parallel transport homomorphisms

$$\mathcal{P}(\gamma)\colon \mathfrak{z}_m \to \mathfrak{z}_{m'}, \quad \mathcal{P}(\gamma)\colon Z_m \to Z_{m'} \tag{30}$$

depending only on the homotopy class of γ.

10.3. **Splittings.** Any choice of a splitting $j\colon TM \to A$ of the exact sequence (28) defines a connection on L, by $\nabla_X = [j(X), \cdot]$. On sections of $\mathrm{Cent}(L) \subseteq L$, this coincides with the canonical flat connection. Since

$$\nabla_X[\tau_1, \tau_2] = [\nabla_X \tau_1, \tau_2] + [\tau_1, \nabla_X \tau_2]$$

by the Jacobi identity, this connection is compatible with the Lie algebra structure; in particular, the parallel transport on fibers is by Lie algebra isomorphisms. The curvature of this connection $F_\nabla \in \Omega^2(M, \mathrm{End}(L))$ is given by $F_\nabla = [\omega, \cdot]$, where $\omega \in \Omega^2(M, L)$ is the L-valued 2-form

$$\omega(X_1, X_2) = [j(X_1), j(X_2)] - j([X_1, X_2]).$$

In terms of the identification $A = TM \oplus L$ defined by j, the Lie algebroid bracket on sections is given by a formula

$$[X_1 + \tau_1, X_2 + \tau_2] = [X_1, X_2] + \nabla_{X_1}\tau_2 - \nabla_{X_2}\tau_1 + [\tau_1, \tau_2] + \omega(X_1, X_2)$$

The local trivializations $A_U = TU \times \mathfrak{h}$ considered above define splittings of $A|_U$ for which $\omega = 0$, and the connection ∇ on L is flat.

Remark 10.3. If A is the Atiyah algebroid of a principal bundle $P \to M$, then the bundle L is the gauge bundle $\mathfrak{gau}(P)$, the splitting j is equivalent to a principal connection on P, and ω becomes the curvature 2-form of the principal connection.

10.4. **Gauge transformations of transitive Lie algebroids.** Locally, any transitive Lie algebroid is of the form $TM \times \mathfrak{h}$ for a fixed Lie algebra $\mathfrak{h}$. We are interested in *gauge transformations* of such Lie algebroids

Proposition 10.4. *Let $\mathfrak{h}$ be a Lie algebra, and $\mathrm{Aut}(\mathfrak{h})$ its Lie algebra automorphisms. Given a manifold M, the group of automorphisms of the trivial Atiyah algebroid $TM \times \mathfrak{h}$, inducing the identity on the base, is the subgroup*

$$\mathrm{Gau}_{\mathcal{LA}}(TM \times \mathfrak{h}) \subseteq \Omega^1(M, \mathfrak{h}) \rtimes C^\infty(M, \mathrm{GL}(\mathfrak{h}))$$

consisting of all pairs (θ, Φ) such that θ is a solution of the Maurer-Cartan equation

$$\mathrm{d}\theta + \frac{1}{2}[\theta, \theta] = 0,$$

and Φ is an $\mathrm{Aut}(\mathfrak{h}) \subseteq \mathrm{GL}(\mathfrak{h})$-valued function such that

$$\mathrm{d}\Phi + \mathrm{ad}_\theta \circ \Phi = 0.$$

Proof. The group of vector bundle automorphism of $TM \times \mathfrak{h}$, preserving the anchor map and inducing the identity on the base, is

$$\mathrm{Gau}_{AV}(TM \times \mathfrak{h}) = \Omega^1(M, \mathfrak{h}) \rtimes C^\infty(M, \mathrm{GL}(\mathfrak{h})),$$

where an element (θ, Φ) of this group acts on sections by

$$(\theta, \Phi) \cdot (X + \xi) = X + \iota_X \theta + \Phi(\xi).$$

The bracket on sections of $TM \times \mathfrak{h}$ is

$$[X + \xi, Y + \zeta] = [X, Y] + [\xi, \zeta] + \mathcal{L}_X \zeta - \mathcal{L}_Y \xi.$$

Hence, (θ, Φ) preserves this bracket if and only if the following three equations are satisfied, for all X, Y, ξ, ζ:

$$\begin{aligned} \Phi([\xi, \zeta]) &= [\Phi(\xi), \Phi(\zeta)] \\ \theta([X, Y]) &= \mathcal{L}_X \iota_Y \theta - \mathcal{L}_Y \iota_X \theta + [\iota_X \theta, \iota_Y \theta] \\ \Phi(\mathcal{L}_X \zeta) &= \mathcal{L}_X(\Phi(\zeta)) + [\iota_X \theta, \Phi(\zeta)]. \end{aligned}$$

The first condition says that Φ takes values in Lie algebra automorphisms, the second condition means that θ is a Maurer-Cartan element. The third equation means $\mathcal{L}_X \Phi + \mathrm{ad}_{\theta(X)} \circ \Phi = 0$, hence $\mathrm{d}\Phi + \mathrm{ad}_\theta \circ \Phi = 0$. □

Remark 10.5. We may obtain general transitive Lie algebroids by taking trivial bundles $TU \times \mathfrak{h}$ over charts, and gluing with the help of transition automorphisms. This approach, and a discussion of the integration problem from this perspective, is detailed in [29].

We will now construct a canonical group homomorphism, for any given $m \in M$,

$$\mathrm{Gau}_{\mathcal{LA}}(TM \times \mathfrak{h}) \to \mathrm{Hom}(\pi_1(M, m), \mathrm{Cent}(H)).$$

Let $\widetilde{M}$ be the universal cover of M with respect to m. Given transition data (θ, Φ) as above, the Maurer-Cartan element θ, regarded as a flat connection on the trivial $\widetilde{H}$-bundle, determines a unique map $g \in C^\infty(\widetilde{M}, H)$ with

$$g|_{\widetilde{m}} = e, \quad \widetilde{\theta} = g^{-1}\mathrm{d}g;$$

here $\widetilde{m} \in \widetilde{M}$ is the base point corresponding to m. (More precisely, $g^{-1}\mathrm{d}g$ is the pull-back of the left-invariant Maurer-Cartan form $\theta^L \in \Omega^1(\widetilde{H}, \mathfrak{h})$ under the map $g\colon \widetilde{M} \to H$). Letting $\widetilde{\Phi}$ be the pull-back of Φ, we have

$$\mathrm{d}\Phi + \mathrm{ad}_\theta \circ \Phi = 0 \Leftrightarrow \mathrm{d}(\mathrm{Ad}_{\widetilde{g}} \circ \widetilde{\Phi}) = 0 \Leftrightarrow \mathrm{Ad}_{\widetilde{g}} \circ \widetilde{\Phi} = \mathrm{const}\,.$$

That is,

$$\widetilde{\theta} = \widetilde{g}^{-1}\mathrm{d}\widetilde{g}, \quad \widetilde{\Phi} = \mathrm{Ad}_{\widetilde{g}^{-1}} \circ \Psi$$

for some *fixed* automorphism $\Psi \in \mathrm{Aut}(\mathfrak{h})$. Conversely, given $g \in C^\infty(\widetilde{M}, H)$, the condition that the Lie algebra valued 1-form $g^{-1}\mathrm{d}g$ and the $\mathrm{Aut}(\mathfrak{h})$-valued function $\mathrm{Ad}_{g^{-1}}$ descend to M mean precisely that g is quasi-periodic with respect to the center. That is, the restriction of g to $\pi_1(M, m) \subseteq \widetilde{M}$ defines a group homomorphism

$$\kappa\colon \pi_1(M, m) \to \mathrm{Cent}(H),$$

and the pull-back of g under the deck transformation given by $[\lambda]$ changes g by multiplication by $\kappa([\lambda])$.

10.5. **Classification of Lie algebroids over 2-spheres.** Recall that if H is a connected Lie group, then

$$\mathrm{Prin}_H(S^2) \cong \pi_1(H). \tag{31}$$

The element of $\pi_1(H)$ associated to a principal bundle $P \to S^1$ is the homotopy class of the transition map $S^1 \to H$ of the principal bundle, after choice of trivializations over the upper and lower hemispheres of S^2, and with S^1 viewed as the equator. The following gives a similar description for transitive Lie algebroids over S^2.

Theorem 10.6. *Let $\mathfrak{h}$ be a Lie algebra, let $\widetilde{H}$ be the connected, simply connected Lie group integrating $\mathfrak{h}$, and let $\mathrm{Cent}(\widetilde{H})$ be its center. There is a canonical isomorphism,*

$$\mathrm{Tran}_{\mathfrak{h}}(S^2) \cong \mathrm{Cent}(\widetilde{H}).$$

Proof. Identify $\mathbb{R}/\mathbb{Z} = S^1 \subseteq S^2$ with the equator, and let $m \in S^1 \subseteq S^2$ be the base point (corresponding to $0 \in S^1$). We consider Lie algebroids A with a given identification $\mathfrak{g}_m \cong \mathfrak{h}$. Consider the open covering of S^2 given by the sets

$$U_\pm \subseteq S^2$$

defined by removing the south pole (for U_+) or north pole (for U_-). The intersection

$$C := U_+ \cap U_-$$

is an annulus around the equator. Given a transitive Lie algebroid $A \to S^2$, we may choose trivializations

$$A|_{U_\pm} = TU_\pm \times \mathfrak{h}$$

extending the given trivialization of L at m. Over C, these are related by gluing data (θ, Φ), where $\Phi|_m = \mathrm{Id}$ (since the identification of $\mathfrak{g}_m$ with $\mathfrak{h}$ is fixed). Let $(\widetilde{\theta}, \widetilde{\Phi})$ be their pull-back to the universal cover $\widetilde{C}$. There exists a unique function

$$g \in C^\infty(\widetilde{C}, \widetilde{H})$$

with $g(0) = e$ (where 0 is the origin of $\mathbb{R} \subseteq \widetilde{C}$) such that $\widetilde{\theta} = g^{-1}\mathrm{d}g$. As discussed above, $\widetilde{\Phi}$ must be of the form

$$\widetilde{\Phi} = \mathrm{Ad}_{g^{-1}} \circ \Psi,$$

where $\Psi \in \mathrm{Aut}(\mathfrak{h})$ is constant, and since $\Phi|_m = \mathrm{Id}$ we have $\Psi = \mathrm{Id}$, thus $\widetilde{\Phi} = \mathrm{Ad}_{g^{-1}}$.

The fact that $\widetilde{\Phi}$ descends to $C = \widetilde{C}/\mathbb{Z}$ means that Ad_g must descend to C. That is, denoting the action of the generator of $\mathbb{Z}$ by $x \mapsto x+1$, we have that $x \mapsto g(x+1)g(x)^{-1}$ takes values in $\mathrm{Cent}(\widetilde{H})$. Furthermore, since $g^{-1}\mathrm{d}g$ descends means that this function is constant. That is, $g(x+1) = cg(x)$ where

$$c := g(1) \in \mathrm{Cent}(\widetilde{H}).$$

We claim that this element does not depend on the choice of trivialization. A change of trivialization over U_+ (preserving the identification $\mathfrak{g}_m \cong \mathfrak{h}$) is an automorphism given by $h_+ \in C^\infty(U_+, \widetilde{H})$, with $h_+|_m = e$. It amounts to replacing g with $\widetilde{h}_+ g$, where $\widetilde{h}_+$ is the pull-back of h. But then $(\widetilde{h}_+ g)(1) = \widetilde{h}_+(1)g(1) = g(1)$. Similarly, the change of trivialization over U_- does not change $g(1)$. We have thus constructed a map

$$\mathrm{Trans}_{\mathfrak{h}}(S^2) \to \mathrm{Cent}(\widetilde{H}).$$

The map is surjective: given $c \in \mathrm{Cent}(\widetilde{H})$ we obtain a transitive Lie algebroid by choosing $g \in C^\infty(\widetilde{C}, \widetilde{H})$ such that $g(0) = e$, $g(1) = c$, and such that $\mathrm{Ad}_{g^{-1}}$ descends to C, defining $\Phi \in C^\infty(C, \mathrm{Aut}(\mathfrak{h}))$. The $\mathfrak{h}$-valued 1-form $g^{-1}\mathrm{d}g$ descends to a Maurer-Cartan element θ, and (θ, Φ) are the desired gluing data. The map is injective, because g is determined by these properties up to change of trivialization. (We leave it as an exercise to spell out the details of the latter step.) $\square$

Example 10.7. If $\mathfrak{h} = \mathbb{R}$, so that $\widetilde{H} = \mathrm{Cent}(\widetilde{H}) = \mathbb{R}$, we recover the identification with $H^2(S^2, \mathbb{R}) = \mathbb{R}$. If $\mathfrak{h} = \mathfrak{su}(2)$, we have $\widetilde{H} = \mathrm{SU}(2)$, hence $\mathrm{Cent}(\widetilde{H}) = \{e, c\}$ (a trivial and a non-trivial element). These correspond to the Atiyah algebroids of the trivial and non-trivial SO(3)-bundle over S^2.

The group structure on $\mathrm{Prin}_H(S^2)$, defined by the isomorphism $\mathrm{Prin}_H(S^2) \cong \pi_1(H)$, is realized by a 'connected sum' construction: Given two principal bundles $P_i \to S^2$, one obtains $P \to S^2 \# S^2$ by choosing local trivializations near the base points, and identifying the principal bundles using those identifications. It corresponds to the product on $\pi_1(H)$. In the same way, $\mathrm{Tran}_{\mathfrak{h}}(S^2)$ has a group structure given by a connected sum construction, and it amounts to multiplying the elements in $\mathrm{Cent}(\tilde{G})$.

Let us now consider the map (29) for the special case $M = S^2$.

Lemma 10.8. *Suppose H is a Lie group with* $\mathrm{Lie}(H)=\mathfrak{h}$*. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Prin}_H(S^2) \arrow[r] \arrow[d] & \mathrm{Tran}_{\mathfrak{h}}(S^2) \arrow[d] \\ \pi_1(H) \arrow[r] & \mathrm{Cent}(\widetilde{H}). \end{tikzcd}$$

Here, the bottom map is the inclusion of $\pi_1(H)$ into the kernel of the group homomorphism $\widetilde{H}\to H$; since $\widetilde{H}$ is connected, the latter is contained in the center of $\widetilde{H}$.

Proof. We may assume H is connected. Then any principal H-bundle P is obtained by gluing two copies of trivial bundles $U_\pm\times H$ by some transition function $g'\colon C\to H$, representing an element of $\pi_1(H)$. Accordingly, the associated Lie algebroid $A(P)$ is obtained by gluing two copies of $U_\pm\times\mathfrak{h}$ by the same a transition function. Lifted to $\widetilde{C}$, this becomes the transition function defining c; in particular c must be in the kernel of the map $\widetilde{H}\to H$, that is, $c\in\pi_1(H)\subseteq\widetilde{H}$. □

Remark 10.9. We will also need the following 'homotopy invariance' of this classification: For any $s\in[0,1]$ the Lie algebroid pull-back under the inclusion $S^2\to[0,1]\times S^2,\ x\mapsto(s,x)$ induces an isomorphism

$$\mathrm{Tran}_{\mathfrak{h}}([0,1]\times S^2)\to\mathrm{Tran}_{\mathfrak{h}}(S^2).$$

To see this, repeat the argument for the classification of transitive Lie algebroids over S^2 with the covering by $[0,1]\times U_\pm$. Choosing trivializations over these open sets, the transition function is given by a smooth map $g\colon[0,1]\times\widetilde{C}\to\widetilde{H}$, and the same argument as before shows that this function is quasi-periodic with respect to the translation action of $\mathbb{Z}$.

10.6. **The monodromy groups.** Let us now turn to arbitrary transitive Lie algebroids $A\to M$ over connected manifolds M. As before, we will use the notation $L=\ker(\mathsf{a})$, a bundle of Lie algebras.

Definition 10.10. The monodromy group of the transitive Lie algebroid A at $m\in M$ is the image $\Lambda_m\subseteq Z_m$ of the group homomorphism

$$\delta_A\colon\pi_2(M,m)\to Z_m$$

taking $[f]$ to the class of $f^!A\in\mathrm{Prin}_{\mathfrak{g}_m}(S^2)$.

Given a smooth path $\gamma\colon[0,1]\to M$ from $\gamma(0)=m$ to $\gamma(1)=m'$, we obtain a commutative diagram

$$\begin{tikzcd} \pi_2(M,m) \arrow[r, "\delta_A"] \arrow[d, "{[\gamma]}"'] & Z_m \arrow[d, "\mathcal{P}(\gamma)"] \\ \pi_2(M,m') \arrow[r, "\delta_A"'] & Z_{m'} \end{tikzcd}$$

In particular, $\mathcal{P}(\gamma)$ takes Λ_m to $\Lambda_{m'}$.

Proposition 10.11. *A necessary condition for the integrability of the transitive Lie algebroid A is that the monodromy groups Λ_m are discrete.*

Proof. Suppose A is integrable. Then $A = A(P)$ for some principal H-bundle, where H is a Lie group with Lie algebra $\mathfrak{h} = \mathfrak{g}_m$. Let $\pi_1(H) \subseteq \widetilde{H}$ be its fundamental group. Given a smooth map $f\colon S^2 \to M$, taking the base point of S^2 to m, it follows that $f^! A = A(f^*P)$. By Lemma 10.8,

$$\delta_A([f]) \in \pi_1(H)$$

for any such f. That is,

$$\Lambda_m \subseteq \pi_1(H),$$

and in particular Λ_m must be discrete. □

10.7. **Construction of an integration.** We will now show how to construct an integration of transitive Lie algebroids, provided that the monodromy groups are discrete. The construction will use the choice of a splitting $j\colon TM \to A$ of the Lie algebroid. As we saw, such a splitting defines a connection $\nabla_X = [j(X), \cdot]$ on the Lie algebra bundle $L \to M$. The resulting parallel transport along paths

$$\mathcal{P}(\gamma)\colon \mathfrak{g}_m \to \mathfrak{g}_{m'} \tag{32}$$

is by Lie algebra isomorphisms; it agrees with the canonical parallel transport (30) on $\mathrm{Cent}(L)$. The parallel transport of a concatenation of paths is the composition of parallel transports:

$$\mathcal{P}(\gamma' * \gamma) = \mathcal{P}(\gamma') \circ \mathcal{P}(\gamma). \tag{33}$$

A small technical problem is that the concatenation of smooth paths need no longer be smooth. But there are various simple ways around this issue: For example, one can restrict attention to paths γ with 'sitting end points' [**?**], i.e., paths that are constant near $t = 0$ and $t = 1$. (There is an associated notion of homotopies of paths with sitting end points.)

The parallel transport exponentiates to group isomorphisms $\widetilde{G_m} \to \widetilde{G_{m'}}$, and the induced map $Z_m \to Z_{m'}$ coincides with that given by the flat connection on Z. In particular, it takes Λ_m to $\Lambda_{m'}$. Letting

$$U_m = \widetilde{G_m}/\Lambda_m$$

the parallel transport also gives group isomorphisms

$$\mathcal{P}(\gamma)\colon U_m \to U_{m'},$$

again with the property (33) under concatenation. In particular, every loop based at m defines a group automorphism of U_m. On *contractible* loops $\lambda\colon [0,1] \to M$, we can do better:

Proposition 10.12. *For contractible smooth loops $\lambda\colon [0,1]\to M$ based at $m\in M$, there are canonically defined holonomies*

$$\mathrm{Hol}_m(\lambda)\in U_m \tag{34}$$

with $\mathrm{Hol}_m(\lambda'*\lambda)=\mathrm{Hol}_m(\lambda')\,\mathrm{Hol}_m(\lambda)$*, and such that for any smooth path γ from m to m',*

$$\mathrm{Hol}_{m'}(\gamma*\lambda*\gamma^{-1})=P(\gamma)\big(\mathrm{Hol}_m(\lambda)\big).$$

In particular, $\mathcal{P}(\lambda)\colon U_m\to U_m$ is conjugation by $\mathrm{Hol}_m(\lambda)$.

Proof. To define (34), let us think of λ as a smooth map $\lambda\colon S^1\to M$, taking $[0]\in S^1$ to M. After a choice of trivialization

$$\lambda^! A\cong T[0,1]\times\mathfrak{g}_m,$$

compatible with the given inclusion of $\mathfrak{g}_m$ at the base point, the splitting of $j^!A$ becomes an $\mathfrak{g}_m$-valued 1-form on S^1, and we can take its holonomy

$$\widehat{\mathrm{Hol}}_m(\lambda)\in\widetilde{G_m}.$$

This element depends on the choice of trivialization of $\lambda^!A$. However, since our loop is contractible, we may extend it to a smooth map $\psi\colon D^2\to M$, and we may restrict attention to trivializations of $\lambda^!A$ that are induced from trivializations of $\psi^!A$. Two such maps $\psi_\pm\colon D^2\to M$ combine to a map from the sphere, $\phi\colon S^2\to M$, and the two holonomies $\widehat{\mathrm{Hol}}^{\pm}_m(\lambda)$ are related by

$$\widehat{\mathrm{Hol}}^{+}_m(\lambda)=c\,\widehat{\mathrm{Hol}}^{-}_m(\lambda)$$

where $c\in\Lambda_m$ is the element corresponding to $\phi^!A\in\mathrm{Prin}_{\mathfrak{g}_m}(S^2)$. It follows that the image

$$\mathrm{Hol}_m(\lambda)\in U_m$$

is independent of the choice of extension ψ. $\square$

Given two based paths $\gamma_0,\gamma_1\colon[0,1]\to M$ from m to m', with $\gamma_1\simeq\gamma_0$, we now define

$$c(\gamma_1,\gamma_0)=\mathrm{Hol}_m(\gamma_1^{-1}*\gamma_0)\in U_m.$$

Then, if $\gamma_2\simeq\gamma_1\simeq\gamma_0$,

$$c(\gamma_2,\gamma_1)c(\gamma_1,\gamma_0)=c(\gamma_2,\gamma_0)$$

Furthermore, under concatenation of paths,

$$c(\gamma_1'*\gamma,\gamma_0'*\gamma)=\mathcal{P}(\gamma^{-1})c(\gamma_1',\gamma_0'),$$

and more generally

$$c(\gamma_1'*\gamma_1,\gamma_0'*\gamma_0)=\mathcal{P}(\gamma_1^{-1})(c(\gamma_1',\gamma_0'))\;c(\gamma_1,\gamma_0).$$

Also,

$$\mathcal{P}(\gamma_1^{-1})\mathcal{P}(\gamma_0)(u)=\mathrm{Ad}(c(\gamma_1,\gamma_0))u.$$

Theorem 10.13. *A transitive Lie algebroid $A \to M$ is integrable to a Lie groupoid (equivalently, it is the Atiyah algebroid of a principal bundle) if and only if the monodromy groups Λ_m are discrete. In this case, the source-simply connected Lie groupoid $\mathcal{G} \rightrightarrows M$ integrating A is the set of equivalence classes $[(\gamma, u)]$, where $\gamma\colon [0,1] \to M$ is a path from m to m' and $u \in U_m$, with the equivalence relation*

$$(\gamma_1, u_1) \sim (\gamma_0, u_0) \Leftrightarrow \gamma_1 \simeq \gamma_0, \quad u_1 = c(\gamma_1, \gamma_0)\ u_0.$$

The groupoid multiplication is given by

$$[(\gamma', u')] \circ [(\gamma, u)] = \big[\big(\gamma' * \gamma,\ (\mathcal{P}(\gamma^{-1}).u')u\big)\big].$$

The isotropy groups $\mathcal{G}_m$ fit into an exact sequence

$$1 \to U_m \to \mathcal{G}_m \to \pi_1(M, m) \to 1$$

Proof. Observe first that the relation $\sim$ is indeed an equivalence relation, and that the groupoid multiplication is well-defined: For example, if $(\gamma_1, u_1) \sim (\gamma_0, u_0)$ then

$$\big(\gamma' * \gamma_1,\ (\mathcal{P}(\gamma_1^{-1}).u')u_1\big) \sim \big(\gamma' * \gamma_0,\ (\mathcal{P}(\gamma_0^{-1}).u')u_0\big).$$

But

$$(\mathcal{P}(\gamma_1^{-1}).u')u_1 = (\mathcal{P}(\gamma_1^{-1}).u')c(\gamma_1, \gamma_0)u_0 = c(\gamma_1, \gamma_0)(\mathcal{P}(\gamma_0^{-1}).u')u_0$$

as required since $c(\gamma' * \gamma_1,\ \gamma' * \gamma_0) = c(\gamma_1, \gamma_0)$. It is also straightforward to check that the proposed groupoid multiplication is associative, and that its units are the elements of M, embedded as classes $[(m, 1)]$.

To see that $\mathcal{G}$ is souce simply connected, consider the source fiber of $m \in M$. It consists of all $[(\gamma, u)]$ such that γ is a path based at m. Consider a family (γ_s, u_s) for $s \in [0,1]$, with $\gamma_0 = \gamma_1$ the constant path at m and $u_1 = u_0 = 1$. The u_s form a loop in U_m, representing an element $c \in \pi_1(U_m) = \Lambda_m$. By definition of Λ_m, this element is realized by some map $\phi\colon S^2 \to M$ based at m. Equivalently, there is a family of contractible loops $\lambda_s \in \mathrm{Loop}_m(M)$ starting and ending at m, so that $s \mapsto \mathrm{Hol}_m(\lambda_s)$ represents the same element c. (See the Lemma below). Hence we have

$$u_s = \mathrm{Hol}_m(\lambda_s)\widetilde{v}_s,$$

where v_s is a contractible loop, and so

$$[(\gamma_s, u_s)] = [(\widetilde{\gamma}_s, \widetilde{u}_s)].$$

This loop in $\mathcal{G}$ is contractible, using any homotopy u_s^r with the constant loop and putting $\widetilde{\gamma}_s^r(t) = \widetilde{\gamma}_s(rt)$.

The isotropy groups $\mathcal{G}_m$ are equivalence classes $[(\lambda, u)]$ such that λ is a loop based at m, and $u \in U_m$, subject to the relation $(\lambda_1, u_1) \sim (\lambda_0, u_0)$ if and only if $\lambda_1^{-1} * \lambda_0$ is contractible and $u_1 = \mathrm{Hol}_m(\lambda_1^{-1} * \lambda_0)u_0$. The kernel of the natural map $\mathcal{G}_m \to \pi_1(M, m)$ is thus U_m. □

Lemma 10.14. *Any element of $\pi_1(U_m) = \Lambda_m$ may be realized as a loop $s \mapsto \mathrm{Hol}_m(\lambda_s) \in U_m$, where $\lambda_s \in \mathrm{Loop}_m(M)$ is a family of loops based at m, starting and ending at m.*

Proof. Given an element $c \in \Lambda_m$, choose a smooth map $\phi\colon S^2 \to M$ such that $c = \delta_A(\phi)$. We may thing of ϕ as a family λ_s of (contractible) loops $t \mapsto \lambda_s(t)$, with $\lambda_0 = \lambda_1$ being trvial loops. (Think of S^2 as being sliced by hyperplances passing through the base point.) Since λ_s is contractible, $\mathrm{Hol}_m(\lambda_s) = u_s$ is defined, and by construction of c the loop $s \mapsto u_s$ represents c. `more details` □

Corollary 10.15. *If M is contractible, then every transitive Lie algebroid $A \to M$ is isomorphic to $TM \times \mathfrak{h}$ for some Lie algebra $\mathfrak{h}$.*

This follows because the integrability assumptions are trivially satisfied, hence A corresponds to some principal H-bundle. But every principal H-bundles over a contractible base is trivial.

Remarks 10.16.
(a) By construction, we have an exact sequence of Lie groupoids
$$1 \to \mathcal{U} \to \mathcal{G} \to \Pi(M) \to 1$$
where $\mathcal{U} = \bigcup_{m \in M} U_m$.
(b) The exact sequence for $\mathcal{G}_m$ shows in particular that $\mathcal{G}_m$ is connected if and only if M is 1-connected, and $\mathcal{G}_m$ is simply connected if and only if the monodromy groups are trivial.

Remark 10.17. As already mentioned, the integrability of transitive Lie algebroids was worked out by Mackenzie [29, Theorem 8.3.6], although formulated and proved differently (using a Čech theory approach).

11. Integrability of non-transitive Lie algebroids

A general Lie algebroid $A \to M$ defines a generalized foliation of M, and is a union of its restrictions $A_{\mathcal{O}}$ to the leaves $\mathcal{O} \subseteq M$ of this foliation. If A is integrable to a source-simply connected Lie groupoid $\mathcal{G} \rightrightarrows M$, then the orbits of $\mathcal{G}$ are the leaves of A, and the restrictions $\mathcal{G}_{\mathcal{O}}$ to the orbits is a source-simply connected Lie groupoid integrating $A_{\mathcal{O}}$. Hence, integrability of the restrictions $A_{\mathcal{O}}$ is necessary for the integrability of A. It is, however, not sufficient as the following example shows.

Example 11.1. Consider the foliation $\mathcal{F}$ of $M = S^2 \times \mathbb{R}$ with leaves $M_t = S^2 \times \{t\}$. Let ω be the standard area form on S^2, and let
$$A_t = TS^2 \times \mathbb{R} \to S^2$$
be the family of Lie algebroids defined by the 2-forms $t\omega$. That is, the bracket on sections of A_t is given by
$$[X_1 + f_1, X_2 + f_2] = \mathcal{L}_{X_1} f_2 - \mathcal{L}_{X_2} f_1 + t\omega(X_1, X_2).$$
The union
$$A = \bigcup_t A_t$$
is a Lie algebroid with $\mathsf{a}(A) = T_{\mathcal{F}}M$. We claim that this Lie algebroid is not integrable.

Each A_t has a source-simply connected integration given by $\mathcal{G}_t = \mathcal{G}(P_t) \rightrightarrows S^2$ for principal $\mathbb{R}/\Lambda_t$-bundles $P_t \to M_t$, where $\Lambda_t = t\mathbb{Z}$. Note that for $t = 0$, the structure group is all of $\mathbb{R}$, while for small non-zero t, the structure groups are circle groups, where the size of the circle goes to zero as $t \to 0$. It seems 'plausible' that these cannot fit together into a global, smooth Lie groupoid.

To see this clearly, suppose that $\mathcal{G} \to M$ is a source-simply connected Lie groupoid integrating A. The map $(\mathsf{t}, \mathsf{s})\colon \mathcal{G} \to M \times M$ is a groupoid homomorphism of constant rank. The pre-image $\mathcal{H} = (\mathsf{t}, \mathsf{s})^{-1}(M)$ is hence an embedded subgroupoid of $\mathcal{G}$. This subgroupoid is the union of isotropy groups, $\mathcal{G}_m$:

$$\mathcal{H} = \bigcup_{m \in M} \mathcal{G}_m.$$

We have an exact sequence of Lie groupoids

$$1 \to \mathcal{H} \to \mathcal{G} \to \mathrm{Pair}(M) \to 1$$

integrates the exact sequence of Lie algebroids

Fix a base point $x \in S^2$, and let

$$i\colon \mathbb{R} \to M = S^2 \times \mathbb{R}, \quad t \mapsto (x, t)$$

be the corresponding inclusion. This is transverse to the anchor, hence $i^!\mathcal{G}$ is a Lie groupoid integrating $i^!A$. But $i^!A$ is just the *trivial* Lie algebroid $\mathbb{R}\times\mathbb{R}$ (a trivial vector bundle, regarded as a Lie algebroid with zero anchor), hence its source-simply connected integration is $\mathbb{R}\times\mathbb{R} \rightrightarrows \mathbb{R}$ (a trivial vector bundle, regarded as a Lie groupoid). It follows that $i^!\mathcal{G}$ is a group bundle, which is a quotient of $\mathbb{R} \times \mathbb{R}$ by some subbundle $\Lambda \subseteq \mathbb{R} \times \mathbb{R}$. But we know that the fiber Λ_t at $t \in \mathbb{R}$ is $\mathbb{R}/t\mathbb{Z}$. This is a contradiction, since $\Lambda = \{(tn, t)|\ n \in \mathbb{Z}, t \in \mathbb{R}\}$ is not a submanifold of $\mathbb{R} \times \mathbb{R}$.

Given a Lie algebroid $A \to M$, and any $m \in M$, let $\mathfrak{g}_m = \ker(\mathsf{a}_m)$. We define the monodromy group

$$\Lambda_m \subseteq \mathrm{Cent}(\widetilde{G_m})$$

to be the monodromy group of the restrictions of A to the leaf $\mathcal{O}$ through m. As we saw, discreteness of Λ_m is necessary and sufficient for the integrability of $A_{\mathcal{O}}$, where $\mathcal{O}$ is the leaf containing m. To ensure integrability of A itself, we need to compare with the monodromy groups of nearby leaves. Let

$$\Lambda_m^0 \subseteq \mathrm{Cent}(\mathfrak{g}_m)$$

be the pre-image of Λ_m under the exponential map $\mathrm{Cent}(\mathfrak{g}_m) \to \mathrm{Cent}(\widetilde{G_m})$. Clearly, Λ_m is discrete if and only if Λ_m^0 is discrete. An advantage of passing to Λ_m^0 is that these may be regarded as subgroups of A_m. Let

$$\Lambda^0 = \bigcup_{m \in M} \Lambda_m^0 \subseteq A$$

be their union.

Theorem 11.2 (Crainic-Fernandes [11])**.** *The Lie algebroid $A \to M$ is integrable to a (possibly non-Hausdorff) Lie groupoid $\mathcal{G} \rightrightarrows M$ if and only if the monodromy groups are* uniformly discrete*, in the sense that there exists an open neighborhood of the zero section $M \subseteq A$ that does not contain any non-zero elements of Λ^0.*

Crainic-Fernandes give an explicit construction of the groupoid $\mathcal{G} \rightrightarrows M$ integrating A, as the space of $\mathcal{LA}$-paths in A modulo $\mathcal{LA}$-homotopy. The groupoid itself was described also by

Cattaneo-Felder [8], as well as Severa [37], but the key question of smoothness was answered by Crainic-Fernandes.

Remark 11.3. An $\mathcal{LA}$-path in A is a Lie algebroid morphism $T[0,1] \to A$. Such an $\mathcal{LA}$-path is given by the base path $\gamma\colon [0,1] \to M$, together with a lift of γ to a path $\widehat{\gamma}\colon [0,1] \to A$ such that

$$\mathsf{a} \circ \widehat{\gamma} = \frac{d\gamma}{dt}.$$

Recall from Section 7.5.1 the notion of homotopy of $\mathcal{LA}$ morphisms. [...]

Following Severa [37], one can define a homotopy between two such paths to be an $\mathcal{LA}$-morphism $T[0,1]^2 \to A$ such that its restrictions to the tangent bundle of the horizontal sides of the square $[0,1]^2$ are constant maps, while the restrictions to the vertical sides are the given maps γ_0, γ_1. This defines an equivalence relation, and $\mathcal{G}$ is the set of equivalence classes. The groupoid structure is such that $[\phi'] \circ [\phi] = [\psi]$ whenever there is a morphism from the tangent bundle of a triangle, restricting to ϕ, ϕ', ψ on the three boundaries.

Example 11.4. Recall that a Lie algebroid with trivial anchor is the same as a family of Lie algebras $\mathfrak{g}_m$ depending smoothly on $m \in M$. The source-simply connected integration of such a Lie algebroid is a family of simply connected Lie groups G_m depending smoothly on m. The existence of such an integration (namely, that the G_m fit together into a smooth manifold) was proved in 1966 by Douady-Lazard [17]. It now follows directly from the Crainic-Fernandes theorem since the leaves $\mathcal{O}$ are points. More generally, whenever the leaves of a Lie algebroid A are 2-connected, the monodromy groups are trivial, hence the criterion is trivially satisfied.

Example 11.5. Another situation where the monodromy groups Λ^0_m are trivial occurs if the isotropy Lie algebras $\mathfrak{g}_m$ have trivial center (for example, if they are semisimple).

Example 11.6. [11] If $A \to M$ is integrable, then every Lie subalgebroid of A is again integrable. This follows since the monodromy groups of the Lie subalgebroid are contained in those of A.

Example 11.7. Let $\mathfrak{g}$ be a Lie algebra, and $A = \mathfrak{g} \times M$ the actions Lie algebroid for a $\mathfrak{g}$-action on M. By a result of Dazord [15] (see also Palais [33]), such an action is always integrable. This follows from the Crainic-Fernandes theorem, since the monodromy group at m is contained in $\pi_1(G_m) \subseteq \widetilde{G_m}$, where G_m is the Lie subgroup of the simply connected Lie group G integrating $\mathfrak{g}$. `explain why this is the case.` Hence, Λ^0_m is contained in the set of all $\xi \in \mathfrak{g} \cong A_m$ such that $\exp(\xi) = e$. Since the exponential map is a diffeomorphism near 0, this implies that the condition of the theorem is satisfied.

12. Lie algebroid cohomology, Lie groupoid cohomology

12.1. **The de Rham complex of a Lie algebroid.** Given a manifold M, one can define the Lie derivatives $\mathcal{L}_X$ on the space $\Omega(M) = \Gamma(\wedge^\bullet T^*M)$ of differential forms in terms of the contraction operators ι_X and the given Lie derivative on functions $f \in \Omega^0(M) = C^\infty(M)$. One takes $\mathcal{L}_X$ to be the unique linear operator of degree 0 such that its commutator with contraction operators is given by

$$\iota_Y \circ \mathcal{L}_X = \mathcal{L}_X \circ \iota_Y - \iota_{[X,Y]}, \quad \mathcal{L}_X(f) = X(f) \text{ for } f \in C^\infty(M) = \Omega_0(M).$$

Similarly, one defines the exterior differential to be the linear operator of degree 1 such that

$$\iota_X \circ \mathrm{d} = \mathcal{L}_X - \mathrm{d} \circ \iota_X.$$

These definitions generalize from TM to arbitrary Lie algebroids $A \to M$: On the graded space

$$\Gamma(\wedge^\bullet A^*)$$

we have the operators ι_σ of contraction with $\sigma \in \Gamma(A)$, we define Lie derivatives $\mathcal{L}_\sigma$ to be the degree 0 operators satisfying

$$\iota_\tau \circ \mathcal{L}_\sigma = \mathcal{L}_\sigma \circ \iota_\tau - \iota_{[\sigma,\tau]}, \quad \mathcal{L}_\sigma(f) = \mathsf{a}(\sigma)(f),$$

and we define an exterior differential by

$$\iota_\sigma \circ \mathrm{d}_A = \mathcal{L}_\sigma - \mathrm{d}_A \circ \iota_\sigma.$$

Explicitly, for $\omega \in \Gamma(\wedge^k A^*)$,

$$(\mathrm{d}_A\omega)(\sigma_1, \ldots, \sigma_{k+1}) = \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \widehat{\sigma_j}, \ldots, \sigma_{k+1}) + \sum_{i=1}^{k+1} (-1)^{i+1} \mathcal{L}_{\mathsf{a}(\sigma_i)} \omega(\sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_{k+1}) \tag{35}$$

where entries with a hat are to be omitted. The usual properties for the de Rham complex of differential forms generalize:

Proposition 12.1 (Cartan calculus)**.** *One has the formulas*

$$[\mathcal{L}_\sigma, \iota_\tau] = \iota_{[\sigma,\tau]}, \quad [\mathcal{L}_\sigma, \mathcal{L}_\tau] = \mathcal{L}_{[\sigma,\tau]}, \quad [\mathcal{L}_\sigma, \mathrm{d}_A] = 0, \quad [\iota_\sigma, \iota_\tau] = 0, \quad [\iota_\sigma, \mathrm{d}_A] = \mathcal{L}_\sigma, \quad [\mathrm{d}_A, \mathrm{d}_A] = 0$$

where the brackets indicate graded commutators*.*

Proof. The first, fourth, and fifth formula are immediate (from the definitions). The second formula holds true on functions; to prove it in general one has to show that the commutator of the two sides with any contraction ι_κ is equal. For the left hand side we find

$$[[\mathcal{L}_\sigma, \mathcal{L}_\tau], \iota_\kappa] = [[\mathcal{L}_\sigma, \iota_\kappa], \mathcal{L}_\tau] + [\mathcal{L}_\sigma, [\mathcal{L}_\tau, \iota_\kappa]] = [\iota_{[\sigma,\kappa]}, \mathcal{L}_\tau] + [\mathcal{L}_\sigma, \iota_{[\tau,\kappa]}] = \iota_{[[\sigma,\kappa],\tau]} + \iota_{[\sigma,[\tau,\kappa]]}$$

which agrees with the right hand

$$[\mathcal{L}_{[\sigma,\tau]}, \iota_\kappa] = \iota_{[[\sigma,\tau],\kappa]}$$

by the Jacobi identity. The third formula is verified similarly: Since the operator $[\mathcal{L}_\sigma, \mathrm{d}_A]$ has degree 1, it suffices to show that its commutator with all ι_τ is zero:

$$[[\mathcal{L}_\sigma, \mathrm{d}_A], \iota_\tau] = -[[\mathcal{L}_\sigma, \iota_\tau], \mathrm{d}_A] + [\mathcal{L}_\sigma, [\mathrm{d}_A, \iota_\tau]] = -[\iota_{[\sigma,\tau]}, \mathrm{d}_A] + [\mathcal{L}_\sigma, \mathcal{L}_\tau] = -\mathcal{L}_{[\sigma,\tau]} + [\mathcal{L}_\sigma, \mathcal{L}_\tau] = 0.$$

Likewise, the last formula follows from

$$[[\mathrm{d}_A, \mathrm{d}_A], \iota_\sigma] = 2[[\mathrm{d}_A, \iota_\sigma], \mathrm{d}_A] = 2[\mathcal{L}_\sigma, \mathrm{d}_A] = 0.$$

∎

Similar arguments may be used to show,

Lemma 12.2. *The operators $\iota_\sigma, \mathcal{L}_\sigma, \mathrm{d}_A$ on $\Gamma(\wedge A^*)$ are graded derivations of degrees $-1, 0, 1$, respectively.*

Since d_A is an odd operator, the identity $[\mathrm{d}_A, \mathrm{d}_A] = 0$ means that $\mathrm{d}_A \circ \mathrm{d}_A = 0$. One may thus define the cohomology groups:

Definition 12.3. The de Rham cohomology groups $H^\bullet(A)$ of the Lie algebroid A are the cohomology groups of the complex $(\Gamma(\wedge A^*), \mathrm{d}_A)$.

Note that $H^\bullet(A)$ has a natural graded algebra structure, coming from the wedge product on $\Gamma(\wedge A^*)$.

For $A = TM$, one recovers the usual de Rham complex of differential forms. For $A = \mathfrak{g}$ a Lie algebra, one obtains the *Chevalley-Eilenberg complex* $\wedge\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Recall that the latter may also be regarded as the complex of left-invariant differential forms on G.

A bit more generally, one can also consider the de Rham cohomology of A with coefficients in an A-module $V \to M$. Recall that this means that A comes with a flat A-connection. One defines contractions, Lie derivatives, and a differential on

$$\Gamma(\wedge^\bullet A^* \otimes V)$$

by the same inductive formulas as before, with the understanding that $\mathcal{L}_\sigma \tau$ for

$$\tau \in \Gamma(\wedge^0 A^* \otimes V) = \Gamma(V)$$

is the given A-representation $\nabla_\sigma \tau$.

12.2. **The Lie algebroid structure.** As we saw, a Lie algebroid structure on A makes the sections of $\wedge A^*$ into a differential graded algebra $(\Gamma(\wedge A^*), \mathrm{d}_A)$, where the Jacobi identity of the bracket on A is ultimately responsible for $\mathrm{d}_A \circ \mathrm{d}_A = 0$. That the converse is true as well was first observed by Vaintrob [?] (generalizing a well-known-fact for Lie algebras):

Proposition 12.4. *Let $A \to M$ be a vector bundle, and let d_A be a differential on $\Gamma(\wedge A^*)$, that is, d_A is a derivation of degree 1 with $\mathrm{d}_A \circ \mathrm{d}_A = 0$. Then d_A determines a unique Lie algebroid structure on A for which d_A is the de Rham differential.*

Proof. Given d_A, define 'Lie derivatives' $\mathcal{L}_\sigma$ on $\Gamma(\wedge A^*)$ by Cartan's identity $\mathcal{L}_\sigma = \mathrm{d}_A \circ \iota_\sigma + \iota_\sigma \circ \mathrm{d}_A$, and define a bracket on sections by the formula

$$\omega([\sigma_1, \sigma_2]) = [\mathcal{L}_{\sigma_1}, \iota_{\sigma_2}]\omega, \quad \omega \in \Gamma(\wedge^1 A^*).$$

Then $\iota_{[\sigma_1,\sigma_2]} = [\mathcal{L}_{\sigma_1}, \iota_{\sigma_2}]$. (A priori, this formula holds on $\Gamma(\wedge^1 A^*)$, but since $\mathcal{L}_\sigma, \iota_\sigma$ are graded derivations, it holds in fact on all of $\Gamma(\wedge A^*)$. Then, the first, fourth, and fifth identity from Cartan's calculus (Lemma 12.1) hold by definition, and the sixth is the assumption $\mathrm{d}_A^2 = 0$. The identity $[\mathcal{L}_\sigma, \mathrm{d}_A] = 0$ follows from

$$[\mathcal{L}_\sigma, \mathrm{d}_A] = [[\iota_\sigma, \mathrm{d}_A], \mathrm{d}_A] = \frac{1}{2}([[\iota_\sigma, \mathrm{d}_A], \mathrm{d}_A] + [[\mathrm{d}_A, \iota_\sigma], \mathrm{d}_A]) = -\frac{1}{2}[[\mathrm{d}_A, \mathrm{d}_A], \iota_\sigma] = 0,$$

while $[\mathcal{L}_{\sigma_1}, \mathcal{L}_{\sigma_2}] = \mathcal{L}_{[\sigma_1,\sigma_2]}$ follows from

$$[\mathcal{L}_{\sigma_1}, \mathcal{L}_{\sigma_2}] = [[\mathrm{d}_A, \iota_{\sigma_1}], \mathcal{L}_{\sigma_2}] = [[\iota_{\sigma_1}, \mathcal{L}_{\sigma_2}], \mathrm{d}_A] = [\iota_{[\sigma_1,\sigma_2]}, \mathrm{d}_A] = \mathcal{L}_{[\sigma_1,\sigma_2]}.$$

By the calculation as in the proof of Proposition 12.1, this last identity implies the Jacobi identity for the bracket $[\cdot,\cdot]$, so that the latter is indeed a Lie bracket. It remains to show that this bracket admits an anchor map. Define

$$\mathsf{a}\colon A \to TM$$

on the level of sections $\sigma \in \Gamma(A)$ by

$$\mathcal{L}_{\mathsf{a}(\sigma)}(f) = \mathcal{L}_\sigma(f)$$

for $f \in C^\infty(M)$ (viewed on the right hand side as an element of $\Gamma(\wedge^0 A^*)$). It defines a bundle map since the right hand side is C^∞-linear in σ. Then

$$\iota_{[\sigma_1,f\sigma_2]} = [\mathcal{L}_{\sigma_1}, f\iota_{\sigma_2}] = f[\mathcal{L}_{\sigma_1},\iota_{\sigma_2}] + \mathcal{L}_{\sigma_1}(f)\iota_{\sigma_2} = f\iota_{[\sigma_1,\sigma_2]} + \mathcal{L}_{\mathsf{a}(\sigma_1)}(f)\iota_{\sigma_2},$$

which shows the Leibnitz rule for the bracket, $[\sigma_1, f\sigma_2] = f[\sigma_1,\sigma_2] + \mathsf{a}(\sigma_1)(f)\sigma_2$. □

It is easily seen that for a direct product of Lie algebroids, $A = A_1 \times A_2$ over $M = M_1 \times M_2$, we have that

$$\Gamma(\wedge A^*) = \Gamma(\wedge A_1^*) \otimes \Gamma(\wedge A_2^*)$$

as differential graded algebras – in turn, one may take this to be the definition of the Lie algebroid structure on the direct product.

Proposition 12.5. *Let $A \to M$ be a Lie algebroid, and $B \subseteq A$ is a vector subbundle along a submanifold $N \subseteq M$. Then B is a Lie subalgebroid of A if and only if the* B-horizontal space

$$\{\omega \in \Gamma(\wedge A^*)|\ \iota_\sigma\omega|_N = 0 \textit{ for all } \sigma \in \Gamma(A,B)\}$$

is a subcomplex for the differential d_A. (Equivalently, the projection $\phi^*\colon \Gamma(\wedge A^*) \to \Gamma(\wedge B^*)$ is a cochain map for a (unique) differential d_B on $\Gamma(\wedge B^*)$.)

Proof. Let $\phi\colon B \hookrightarrow A$ be the inclusion map, with base map $i\colon N \to M$, and $\phi^*\colon \Gamma(\wedge B^*) \to \Gamma(\wedge A^*)$ be the induced maps on sections of the dual bundle. Suppose $\ker(\phi^*)$ is a subcomplex. Then $\Gamma(\wedge B^*)$ inherits a differential such that

$$\phi^* \circ \mathrm{d}_A = \mathrm{d}_B \circ \phi^*.$$

In particular, B inherits the structure of a Lie algebroid. If $\sigma \in \Gamma(A,B)$, with restriction τ, we have that

$$\phi^* \circ \iota_\sigma = \iota_\tau \circ \phi^*, \quad \phi^* \circ \mathcal{L}_\sigma = \mathcal{L}_\tau \circ \phi^*.$$

Consequently, if $\sigma_1, \sigma_2 \in \Gamma(A,B)$, restricting to sections $\tau_1, \tau_2 \in \Gamma(B)$, then

$$\phi^* \circ \iota_{[\sigma_1,\sigma_2]} = \phi^* \circ [\mathcal{L}_{\sigma_1}, \iota_{\sigma_2}] = [\mathcal{L}_{\tau_1}, \iota_{\tau_2}] \circ \phi^* = \iota_{[\tau_1,\tau_2]} \circ \phi^*.$$

Applying this to arbitrary $\omega \in \ker(\phi^*)$, this shows in particular that $[\sigma_1,\sigma_2] \in \Gamma(A,B)$, so that $\Gamma(A,B)$ is closed under the Lie bracket. Similarly, if $f \in C^\infty(M)$ vanishes along N, and $\sigma \in \Gamma(A,B)$ restricts to $\tau \in \Gamma(B)$, then (since ϕ^* coincides with i^* on functions)

$$\phi^*\mathcal{L}_{\mathsf{a}(\sigma)}f = \phi^*\iota_\sigma \mathrm{d}_A f = \iota_\tau\phi^*\mathrm{d}_A f = \iota_\tau \mathrm{d}_B \phi^* f = 0,$$

which shows that the vector field $\mathsf{a}(\sigma)$ is tangent to N. Hence, $\mathsf{a}(B) \subseteq TN$. This shows that B is a Lie subalgebroid.

Conversely, let B be a Lie subalgebroid of A. To show $\phi^*\mathrm{d}_A\omega = \mathrm{d}_B\phi^*\omega$ for all $\omega \in \Gamma(\wedge^k A^*)$, apply both sides to sections $\tau_1, \ldots, \tau_k \in \Gamma(B)$, which we may take to be restrictions of sections $\sigma_1, \ldots, \sigma_k \in \Gamma(A, B)$. Since B is a Lie subalgebroid, the bracket $[\sigma_i, \sigma_j]$ restricts to $[\tau_i, \tau_j]$, and $\mathsf{a}(\tau_i)$ are i-related to $\mathsf{a}(\sigma_i)$. Hence, the desired identity follows from the formula (35) for the differential. This proves that ϕ^* is a cochain map, and in particular its kernel is a subcomplex. $\square$

Example 12.6. For any Lie algebroid A, the diagonal in $A \times A$ is a Lie subalgebroid. The corresponding map on de Rham complexes

$$\Gamma(\wedge(A \times A)^*) = \Gamma(\wedge A^*) \otimes \Gamma(\wedge A^*) \to \Gamma(\wedge A^*)$$

is just the algebra multiplication, hence it is a cochain map.

More generally, we obtain:

Proposition 12.7. *Let $A \to M$, $B \to N$ be Lie algebroids, and let $\varphi\colon B \to A$ be a bundle map with base map $F\colon N \to M$. Then φ is a morphism of Lie algebroids if and only if the pull-back map on sections, $\Gamma(\wedge A^*) \to \Gamma(\wedge B^*)$, is a cochain map.*

Proof. Suppose $\varphi^*\colon \Gamma(\wedge B^*) \to \Gamma(\wedge A^*)$ is a cochain map. The inclusion $B \cong \mathrm{Gr}(\varphi) \to A \times B$ factors as a composition of maps

$$B \to B \times B \to A \times B,$$

where the first map is the diagonal inclusion, and the the second map is $\varphi \times \mathrm{id}_B$. The map on de Rham complexes is thus $\phi^* \otimes \mathrm{id}_{\Gamma(\wedge B^*)}\colon \Gamma(\wedge A^*) \otimes \Gamma(\wedge B^*) \to \Gamma(\wedge B^*) \to \Gamma(\wedge B^*)$, followed by multiplication. It is thus a cochain map, proving that $\mathrm{Gr}(\phi)$ is a Lie subalgebroid.

Conversely, if $\mathrm{Gr}(\phi)$ is a Lie subalgebroid, we see that the map $\Gamma(\wedge A^*) \otimes \Gamma(\wedge B^*) \to \Gamma(\wedge B^*)$, $x \otimes y \mapsto \phi^*(x)y$ is a cochain map:

$$\mathrm{d}_B(\phi^*(x)y) = \phi^*(\mathrm{d}_A x)y + \phi^*(x)\mathrm{d}_B y.$$

Applying this to $y = 1$, it follows that ϕ^* is a cochain map. $\square$

12.3. Examples.

12.3.1. *Foliations.* By Frobenius' theorem, a vector subbundle $B \subseteq TM$ is the tangent bundle $T_{\mathcal{F}}M$ of a foliation $\mathcal{F}$ if and only if B is a Lie subalgebroid of TM. In terms of the differential complexes, this is equivalent to requiring that the space of B-horizontal forms (i.e. $\iota_X\omega = 0$ whenever X takes values in B) is a subcomplex of $\Omega(M)$. The quotient complex $(\wedge(T^*_{\mathcal{F}}M), \mathrm{d}_A)$ is called the *tangential de Rham complex.* See e.g. [**?**, Chapter III].

12.3.2. $\mathfrak{g}$-*actions.* If $A = M \times \mathfrak{g}$ is the action Lie algebroid for a $\mathfrak{g}$-action on M, then $\Gamma(\wedge A^*) \cong C^\infty(M) \otimes \wedge\mathfrak{g}^*$ is the Chevalley-Eilenberg complex with coefficients in the $\mathfrak{g}$-module $C^\infty(M)$.

12.3.3. *Poisson manifolds.* Let M be a manifold with a Poisson bracket. Hence, the corresponding Poisson tensor $\pi \in \Gamma(\wedge^2 TM)$ satisfies $[\pi, \pi] = 0$. One obtains a differential $\mathrm{d}_\pi = [\pi, \cdot]$ on $\Gamma(\wedge TM)$, making the latter into a differential graded algebra. Hence, T^*M inherits a Lie algebroid structure.

12.3.4. *b-forms.* Given an embedded hypersurface $N \subseteq M$, one has a Lie algebroid $A = T^bM$ whose sections are the vector fields tangent to N. The corresponding complex $\Gamma(\wedge A^*) = \Omega^b(M)$ can be regarded as differential forms on $M\backslash N$ developing a first order pole along N. This is the starting point for Melrose's b-calculus (in his work, M is a manifold with boundary $N = \partial M$).

12.4. **The Lie groupoid complex.** Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, recall that $\mathcal{G}^{(p)}$ is the space of p-arrows in $\mathcal{G}$. In this section, we will use the 'simplicial notation' and write $B_p\mathcal{G}$ for the space of p-arrows: Thus

$$B_p\mathcal{G} = \{(g_1, \ldots, g_p)|\ g_i \in \mathcal{G},\ \mathsf{s}(g_i) = \mathsf{t}(g_{i+1})\}.$$

It will be convenient to regard g_i as an arrow from a base point $\mathsf{s}(g_i) = m_i$ to a base point $\mathsf{t}(g_i) = m_{i-1}$; hence a p-arrow comes with $p+1$ base points

$$m_0, \ldots, m_p.$$

There are $p+1$ *face maps*, where

$$\partial_i \colon B_p\mathcal{G} \to B_{p-1}\mathcal{G},\ i = 0, \ldots, p$$

amounts to 'omitting the i-th base point'. Explicitly,

$$\partial_i(g_1, \ldots, g_p) = \begin{cases} (g_2, \ldots, g_p) & i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_p) & 0 < i < p, \\ (g_1, \ldots, g_{p-1}) & i = p. \end{cases}$$

For $p = 1$ we have $\partial_0(g) = \mathsf{s}(g)$, $\partial_1(g) = \mathsf{t}(g)$. There are also degeneracy maps $\epsilon_i \colon B_p\mathcal{G} \to B_{p+1}\mathcal{G}$, 'repeating the i-th base point' by inserting a trivial arrow. That is,

$$\epsilon_i(g_1, \ldots, g_p) = (g_1, \ldots, g_i, m_i, g_{i+1}, \ldots, g_p).$$

Definition 12.8. The complex $(\mathsf{C}^\bullet(\mathcal{G}), \delta)$ of differentiable groupoid cochains is given by

$$\mathsf{C}^p(\mathcal{G}) = C^\infty(B_p\mathcal{G}), \quad \delta = \sum_{i=0}^{p+1} (-1)^i \partial_i^*.$$

The *normalized subcomplex* $\widetilde{\mathsf{C}}^\bullet(\mathcal{G})$ is the subcomplex consisting of $f \in C^\infty(B_p\mathcal{G})$ such that $\epsilon_i^* f = 0$ for all i.

That is, the normalized subcomplex consists of functions on $B_p\mathcal{G}$ which vanish on $(g_1, \ldots, g_p) \in B_p\mathcal{G}$ whenever one of the entries g_i is a unit. Using the relations between face maps, it is not hard to check to verify that δ does indeed square to zero, and that $\widetilde{\mathsf{C}}^\bullet(\mathcal{G})$ is indeed a subcomplex.

Remark 12.9. Also of interest is a localized version of these complexes, denoted $\mathsf{C}^\bullet_M(\mathcal{G})$ and $\widetilde{\mathsf{C}}^\bullet_M(\mathcal{G})$, whose cochains are the germs of functions on $B_p\mathcal{G}$ along $M \subseteq B_p\mathcal{G}$.

Example 12.10. A 0-cochain $f \in \mathsf{C}^0(\mathcal{G})$ is a function on M. It is a cocycle if and only if $(\delta f)(g) = (\mathsf{t}^* f)(g) = (\mathsf{s}^* f)(g) = 0$, i.e. f is constant along the orbits of $\mathcal{G}$. (One might say: f is a $\mathcal{G}$-invariant function on M.) A 1-cochain $f \in \mathsf{C}^1(\mathcal{G})$ is a cocycle if and only if

$$(\delta f)(g_0, g_1) = f(g_1) - f(g_0 g_1) + f(g_0) = 0.$$

These are the *multiplicative functions* on $\mathcal{G}$, i.e., groupoid homomorphisms $\mathcal{G} \to \mathbb{R}$.

Example 12.11. Let $\mathcal{G} = \mathrm{Pair}(M)$ be the pair groupoid. Then $B_p\mathcal{G}$ consists of $p+1$-tuples $(m_0, \ldots, m_p)$, with the i-th face map forgetting the i-th entry. (The normalized subcomplex consists of functions that vanish whenever two of the entries coincide.)

Consider the cocycles in low degree: For $f \in \mathsf{C}^0(\mathcal{G})$, we have that $(\delta f)(m_0, m_1) = f(m_1) - f(m_0)$, which vanishes if and only if f is constant. For $f \in \mathsf{C}^1(\mathcal{G})$, we have that $(\delta f)(m_0, m_1, m_2) = f(m_1, m_2) - f(m_0, m_2) + f(m_0, m_1)$, which vanishes if and only if $f(m_0, m_1) + f(m_1, m_2) = f(m_0, m_2)$ for all triples of points. Any 1-cocycle is a coboundary: Fixing a base point m_*, we have that

$$f(m_1, m_2) = -f(m_*, m_1) + f(m_*, m_2) = (\delta g)(m_1, m_2)$$

with $g(m) = f(m_*, m)$.

More generally, the formula

$$(\mathsf{h}f)\,(m_0, \ldots, m_{k-1}) = f(m_*, m_0, \ldots, m_{k-1}),$$

defines a *homotopy operator* for δ:

$$h \circ \delta + \delta \circ h = \mathrm{id} - i \circ \pi$$

where i and π are inclusion of and projection to $\mathbb{R} \subseteq \mathsf{C}^0(\mathcal{G})$. It follows that, $H^\bullet(\mathcal{G})$ is trivial. Notice that the homotopy operator no longer works for the localized complex $\mathsf{C}^\bullet_M(\mathcal{G})$. In fact, the latter has more interesting cohomology $H^\bullet_M(\mathcal{G})$: One can show that this *Alexander-Spanier complex* computes the de Rham cohomology of M.

Some comments and generalizations:

(a) Just as for the Lie algebroid complex, one can consider groupoid cochains with coefficients in a $\mathcal{G}$-representation $p\colon V \to M$. Recall that such a $\mathcal{G}$-representation is given by a vector bundle together with an action $\mathcal{G}\,{}_{\mathsf{s}}\!\times_p V \to V$ such that the maps $g\colon V_{\mathsf{s}(g)} \to V_{\mathsf{t}(g)}$ are linear. Equivalently, the action groupoid $\mathcal{G} \ltimes V \rightrightarrows V$ is a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$. Applying the functor $B_\bullet$, we obtain vector bundles

$$B_p(\mathcal{G} \ltimes V) \to B_p(\mathcal{G}).$$

The fiber of $B_p(\mathcal{G} \times V)$ over $(g_1, \ldots, g_p) \in B_p\mathcal{G}$, with base points $m_0, \ldots, m_p$, may be regarded as sequences $v_0, \ldots, v_p$ with $v_i \in V_{m_i}$, such that $v_{i-1} = g_i \circ v_i$. (Of course, such a sequence is uniquely determines by any one of the v_i, for example by v_p.) The sections of this bundle define the cochain groups complex

$$\mathsf{C}^p(\mathcal{G}, V) = \Gamma(B_p(\mathcal{G} \ltimes V));$$

the differential is as before the alternating sum of pullbacks.

(b) Another generalization is to consider differential forms on $B_p\mathcal{G}$. This defines a double complex,

$$\mathsf{C}^{p,q}(\mathcal{G}) = \Omega^q(B_p\mathcal{G})$$

A q-form ω on $\mathcal{G}$ is a 1-cocycle if it is multiplicative: Thus,

$$\mathrm{Mult}_{\mathcal{G}}^*\,\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega$$

where $\mathrm{Mult}_{\mathcal{G}}\colon \mathcal{G}^{(2)} \to \mathcal{G}$ is the groupoid multiplication and $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathcal{G}^{(2)} \to \mathcal{G}$ are the two projections.

(c) The complex $\mathsf{C}(\mathcal{G})$ has a (non-commutative) ring structure given by *cup product*, as follows. Given $f \in \mathsf{C}^p(\mathcal{G}) = C^\infty(B_p\mathcal{G})$ and $f' \in \mathsf{C}^{p'}(\mathcal{G}) = C^\infty(B_{p'}\mathcal{G})$, we define $f \cup f'$ by

$$(f \cup f')(g_1, \ldots, g_{p+p'}) = f(g_1, \ldots, g_p) + f(g_{p+1}, \ldots, g_{p+p'}).$$

The cup product satisfies

$$\delta(f \cup f') = \delta(f) \cup f' + (-1)^p f \cup \delta(f'),$$

so it defines a cup product in cohomology $H^\bullet(\mathcal{G})$.

12.5. **Weinstein-Xu's van Est map.** Suppose $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, with Lie algebroid $A \to M$. We then have two cochain complexes

$$\widetilde{\mathsf{C}}^\bullet(\mathcal{G}) \subseteq \mathsf{C}^\bullet(\mathcal{G}) = C^\infty(B_\bullet\mathcal{G}), \qquad \mathsf{C}^\bullet(A) = \Gamma(\wedge^\bullet A^*).$$

We would like to describe a cochain map

$$\mathrm{VE}^\bullet \colon \widetilde{\mathsf{C}}^\bullet(\mathcal{G}) \to \mathsf{C}^\bullet(A)$$

amounting to a 'differentiation procedure'. In the case that $\mathcal{G}$ is a Lie group, with A its Lie algebra, this will be the classical *van Est map* [**?**].

To describe this map, consider the map

$$\Phi \colon B_p\mathcal{G} \to M^{p+1}$$

taking a p-arrow to its $p+1$ base points $(m_0, \ldots, m_p)$. Its i-th component is the anchor map for a $\mathcal{G}$-action on $B_p\mathcal{G}$, given by

$$h \cdot (g_1, \ldots, g_p) = (g_1, \ldots, g_i h^{-1}, h g_{i+1}, \ldots, g_p).$$

For $i = 0$, the right hand side is to be interpreted as $(hg_1, g_2, \ldots, g_p)$, and for $i = p$ as $(g_1, g_2, \ldots, g_p h^{-1})$. Accordingly, every section $\sigma \in \Gamma(A)$ gives rise to $p+1$ vector fields defining the corresponding infinitesimal actions: Regarding $B_p\mathcal{G}$ as a submanifold of $\mathcal{G}^{p+1}$, these are the restrictions of the vector fields

$$\sigma^{(i)} = (0, \ldots, 0, \underbrace{\sigma^L}_{i}, \underbrace{-\sigma^R}_{i+1}, 0, \ldots, 0),$$

For $i = 0$, this is to be interpreted as $(-\sigma^R, 0, \ldots, 0)$ and for $i = p$ as $(0, \ldots, 0, \sigma^L)$.

Example 12.12. Let $\mathcal{G} = \mathrm{Pair}(M)$, $A = TM$. Then the map Φ is a diffeomorphism $B_p\mathcal{G} = M^{p+1}$. In terms of this identification, the $p+1$ vector fields defined by $X \in \Gamma(A) = \mathfrak{X}(M)$ are simply

$$X^{(i)} = (0, \ldots, \underbrace{X}_{i}, \ldots, 0) \in \mathfrak{X}(M^{p+1})$$

The formula for the Van Est map is then as follows: For $f \in \widetilde{\mathsf{C}}^p(\mathcal{G}) = C^\infty(B_p\mathcal{G})$,

$$\mathrm{VE}(f)(\sigma_1, \ldots, \sigma_p) = \sum_{s \in \mathfrak{S}_p} \mathrm{sign}(s)\ \mathcal{L}(\sigma^{(1)}_{s(1)}) \cdots \mathcal{L}(\sigma^{(p)}_{s(p)})(f)\big|_M \tag{36}$$

where the sum is over the permutation group $\mathfrak{S}_p$. Note that the 0-th action does not enter this formula.

Lemma 12.13. *The right hand side of* (36) *is C^∞-linear in $\sigma_1, \ldots, \sigma_p$, hence it gives a well-defined section* $\mathrm{VE}(f) \in \Gamma(\wedge^p A^*)$.

Proof. Using that the vector fields $\sigma^{(i)}$ are all defined on all of $\mathcal{G}^p$, we may compute (36) by using any extension of f to $\mathcal{G}^p$, still with the property that f vanishes whenever one of the entries lies in M. (We will use the same notation f for the extension.) The difference $\mathrm{VE}(f)(g\sigma_1, \ldots, \sigma_p) - g\,\mathrm{VE}(f)(\sigma_1, \ldots, \sigma_p)$, for a given function f, is a linear combination of terms of the form $\mathcal{L}(\tau_1^{(i_1)}) \cdots \mathcal{L}(\tau_r^{(i_r)})(f)|_M$ with $1 \le i_1 < \ldots < i_r \le p$ where r is strictly less than p. Using the formula for the vector fields $\sigma^{(i)}$, and since f is normalized, it is not hard to see that such terms are all zero. `to be polished. A cleaner way is to pull everything back to` $E_p\mathcal{G}$ $\square$

Theorem 12.14 (Weinstein-Xu [41])**.** *The van Est map* (36) *is a cochain map* $\mathrm{VE}\colon \widetilde{\mathsf{C}}(\mathcal{G}) \to \mathsf{C}(A)$, *intertwining the products.*

The van Est map is also well-defined on the localized complex $\widetilde{\mathsf{C}}_M(\mathcal{G})$. We will see that the resulting map

$$\mathrm{VE}\colon \widetilde{\mathsf{C}}_M(\mathcal{G}) \to \mathsf{C}(A)$$

induces an isomorphism in cohomology. In the case of $\mathcal{G} = \mathrm{Pair}(M)$, this gives the isomorphism between Alexander-Spanier cohomology and de Rham cohomology.

12.6. **The Crainic double complex.** Our approach to the Weinstein-Xu theorem will relate the two complexes using a double complex, due to Crainic [10]. The argument is similar to A. Weil's Čech-de Rham double complex (see e.g. [4]), used to prove the isomorphism between Čech cohomology and de Rham cohomology. The sequence of manifolds

$$E_p\mathcal{G} = \{(a_0, \ldots, a_p) \in \mathcal{G}^{p+1} |\ \mathsf{s}(a_0) = \ldots = \mathsf{s}(a_p)\}$$

is a simplicial manifold, for the face maps

$$\partial_i\colon E_p\mathcal{G} \to E_{p-1}\mathcal{G}, \quad i = 0, \ldots, p$$

omitting the i-th entry, and degeneracy maps

$$\epsilon_i\colon E_p\mathcal{G} \to E_{p+1}\mathcal{G}, \quad i = 0, \ldots, p$$

repeating the i-th entry. There is a morphism of simplicial manifolds $\kappa_\bullet\colon E_\bullet\mathcal{G} \to B_\bullet\mathcal{G}$, where

$$\kappa_p\colon E_p\mathcal{G} \to B_p\mathcal{G}, \ \ (a_0, \ldots, a_p) \mapsto (a_0a_1^{-1}, a_1a_2^{-1}, \ldots, a_{p-1}a_p^{-1}).$$

(This being a morphism means that $\kappa_\bullet$ intertwines the respective face and degeneracy maps.) The map κ_p is the quotient map for the principal $\mathcal{G}$-action on $E_p\mathcal{G}$, with anchor map

$$\pi_p\colon E_p\mathcal{G} \to M$$

taking $(a_0, \ldots, a_p)$ to the common source, and given by

$$h \cdot (a_0, \ldots, a_p) = (a_0h^{-1}, \ldots, a_ph^{-1}).$$

On the other hand, the $p+1$ $\mathcal{G}$-actions on $E_p\mathcal{G}$ given by left multiplication descend to the $p+1$ actions on $B_p\mathcal{G}$ described above. In particular, the vector field $(0,0,\ldots,-\sigma^R,0,\ldots)$ (with $-\sigma^R$ as the i-th entry) descends to $\sigma^{(i)}$. Note that π_p is also a simplicial map (where M is a simplicial manifold in a trivial way: it is given by M itself for all p, and all face and degeneracy maps are the identity maps). It has a right inverse $i_p\colon M \to E_p\mathcal{G}$ which also defines a simplicial map.

$$\begin{array}{ccc} E_p\mathcal{G} & \xrightarrow{\kappa_p} & B_p\mathcal{G} \\ {\scriptstyle \pi_p}\big\downarrow\big\uparrow{\scriptstyle i_p} & & \\ M & & \end{array}$$

The Crainic double complex is given by

$$\mathsf{C}^{r,s} = \Gamma(\wedge^s \pi_r^* A)$$

with the two differentials defines as follows. For any fixed s, the collection bundles $\wedge^s\pi_r^*A \to E_r\mathcal{G}$ define a *simplicial vector bundle* $\wedge^s\pi_\bullet^*A \to E_\bullet\mathcal{G}$, i.e., there are face and degeneracy maps

$$\partial_i\colon\ \wedge^s\, \pi_r^*A \to \wedge^s\pi_{r-1}^*A, \qquad \epsilon_i\colon\ \wedge^s\, \pi_r^*A \to \wedge^s\pi_{r+1}^*A$$

covering the face and degeneracy maps on the base. Put differently, we may regard $\wedge^s A \to M$ as a simplicial vector bundle given by $A \to M$ in all degrees, and with all face and degeneracy maps being the identity maps; then $\wedge^s\pi_\bullet^*A$ is just its pullback in the category of simplicial manifolds. This defines the simplicial differential

$$\delta = \sum_{i=0}^{r+1}(-1)^i\partial_i^*\colon :\ \mathsf{C}^{r,s} \to \mathsf{C}^{r+1,s}.$$

On the other hand, for fixed r, we may regard $\pi_r^*A \to E_r\mathcal{G}$ as the tangent bundle to the fibers of $E_r\mathcal{G} \to B_r\mathcal{G}$:

$$\pi_r^*A \cong \ker(T\kappa_r) \subseteq TE_r\mathcal{G}.$$

(This is just the usual isomorphism for the vertical bundle of any groupoid principal bundle; in case $\mathcal{G} = G$ is a group, with Lie algebra $A = \mathfrak{g}$, it is the isomorphism $\ker(T\kappa_r) = E_rG \times \mathfrak{g}$.) Since this is the tangent bundle to a foliation, it is in particular a Lie algebroid, hence there is a Chevalley-Eilenberg differential

$$\mathrm{d}_{CE}\colon \mathsf{C}^{r,s} \to \mathsf{C}^{r,s+1}.$$

This Chevalley Eilenberg and simplicial differentials commute in the ungraded sense (since each ∂_i is a Lie algebroid morphism):

$$\mathrm{d}_{CE}\circ\delta = \delta\circ\mathrm{d}_{CE}.$$

For a double complex, we prefer if the two differentials commute in the graded sense. We hence put

$$\mathrm{d} = (-1)^r\mathrm{d}_{CE}\colon \mathsf{C}^{r,s} \to \mathsf{C}^{r,s+1};$$

then

$$(C^{\bullet,\bullet}, \mathrm{d}, \delta)$$

is the desired double complex, with horizontal differential δ and vertical differential d.

This double complex comes with two 'augmentation' maps. First, there is the map

$$\kappa_r^*\colon \widetilde{\mathsf{C}}^r(\mathcal{G}) \to \mathsf{C}^{r,0}.$$

Its image are the $\mathcal{G}$-invariant functions on $E_r\mathcal{G}$, regarded as sections of $\wedge^0\pi_r^*A^*$. Second, regarding $A \to M$ as a simplicial Lie algebroid $A_\bullet \to M_\bullet$ as above, it too defines a double complex $(\mathsf{D}^{\bullet,\bullet}, \mathrm{d}, \delta)$, where

$$\mathsf{D}^{r,s} = \Gamma(\wedge^s A_r^*),$$

and with the simplicial differential δ and with $\mathrm{d} = (-1)^r \mathrm{d}_{CE}$. The map

$$\pi_r^* A \cong \ker(T\kappa_r) \to A_r$$

is a morphism of Lie algebroids, inducing a morphism of complexes, for all fixed r,

$$\pi_r^* \colon \mathsf{C}^\bullet(A_r) \to \mathsf{C}^{r,\bullet}$$

which is indeed a map of double complexes $\mathsf{D}^{\bullet,\bullet} \to \mathsf{C}^{\bullet,\bullet}$. Note that this map is injective, hence we may think of $\mathsf{D}^{\bullet,\bullet}$ as a sub-double complex.

Remark 12.15. Warning: the map ι_r^* is *not* a cochain map for d_{CE}, since this does not correspond to a Lie algebroid morphism. It is still a cochain map for δ, though.

The idea of the construction is now as follows. Consider the double complex $\mathrm{Tot}^\bullet(\mathsf{C})$ with the total differential $\mathrm{d} + \delta$. We have a cochain map

$$\mathsf{C}^\bullet(\mathcal{G}) \to \mathrm{Tot}^\bullet(\mathsf{C}).$$

We would like to define a cochain map from $\mathrm{Tot}^\bullet(\mathsf{C})$, by essentially 'inverting' the map π^*.

12.7. **Perturbation lemma.** We will use the following Lemma from homological algebra, called the *basic perturbation lemma*. Suppose $(\mathsf{C}^{\bullet,\bullet}, \mathrm{d}, \delta)$ is a double complex, concentrated in non-negative degrees, with the corresponding total complex $(\mathrm{Tot}^\bullet(\mathsf{C}), \mathrm{d}+\delta)$. Let $i \colon \mathsf{D}^{\bullet,\bullet} \hookrightarrow \mathsf{C}^{\bullet,\bullet}$ be a sub-double complex. Suppose the horizontal differential δ admits a homotopy operator

$$\mathsf{h} \colon \mathsf{C}^{\bullet,\bullet} \to \mathsf{C}^{\bullet-1,\bullet}$$

with $\mathsf{h}|_{\mathsf{D}} = 0$, so that

$$[\mathsf{h}, \delta] = 1 - i \circ p$$

for some projection $p \colon \mathsf{C}^{\bullet,\bullet} \to \mathsf{D}^{\bullet,\bullet}$. The perturbation lemma modifies this homotopy for δ into a homotopy operator for the total differential.

Lemma 12.16 (Brown, Gugenheim)**.** *Let*

$$\mathsf{h}' = \mathsf{h}(1 + \mathrm{d}\mathsf{h})^{-1} = (1 + \mathsf{h}\mathrm{d})^{-1}\mathsf{h}.$$

Then

$$[\mathsf{h}', \mathrm{d} + \delta] = \mathrm{id} - i \circ p'$$

where $p' = p(1 + \mathrm{d}\mathsf{h})^{-1}$ is a new projection to D.

Proof. We calculate:

$$\begin{aligned}
(1 + \mathsf{h}\mathrm{d})\ [\mathsf{h}', \mathrm{d} + \delta]\ (1 + \mathrm{d}\mathsf{h}) &= (1 + \mathsf{h}\mathrm{d})\mathsf{h}'(\mathrm{d} + \delta)(1 + \mathrm{d}\mathsf{h}) + (1 + \mathsf{h}\mathrm{d})(\mathrm{d} + \delta)\mathsf{h}'(1 + \mathrm{d}\mathsf{h}) \\
&= \mathsf{h}(\mathrm{d} + \delta)(1 + \mathrm{d}\mathsf{h}) + (1 + \mathsf{h}\mathrm{d})(\mathrm{d} + \delta)\mathsf{h} \\
&= [\mathsf{h}, \mathrm{d} + \delta] \\
&= 1 - i \circ p + [\mathsf{h}, \mathrm{d}],
\end{aligned}$$

to be compared with

$$\begin{aligned}(1+\mathsf{h}\mathrm{d})\,(1-i\circ p')\,(1+\mathrm{d}\mathsf{h}) &= (1+\mathsf{h}\mathrm{d})(1+\mathrm{d}\mathsf{h})-(1+\mathsf{h}\mathrm{d})\,i\circ p\\ &= 1-i\circ p+[\mathsf{h},\mathrm{d}]-\mathsf{h}\,\mathrm{d}\,i\circ p.\end{aligned}$$

Bit $\mathsf{h}\,\mathrm{d}\,i\,p=\mathsf{h}\,i\,\mathrm{d}\,p=0$, since i is a cochain map for d, and since h vanishes on the range of i, by assumption. □

In short, if the inclusion $\mathsf{D}\hookrightarrow\mathsf{C}$ is a homotopy equivalence for the differential δ, and the homotopy operator h has the property above, then we can turn it into a homotopy equivalence for the total ('perturbed') differential.

12.8. **Construction of the van Est map.** We return to the double complex $\mathsf{C}^{\bullet,\bullet}$ for a Lie groupoid $\mathcal{G}\rightrightarrows M$, and its subcomplex $\mathsf{D}^{\bullet,\bullet}$.

Lemma 12.17. *There exists a homotopy operator* h *for the differential* δ *on* $\mathsf{C}^{\bullet,\bullet}$ *with*

$$[\mathsf{h},\delta]=1-\pi^*\circ\iota^*.$$

This homotopy operator vanishes on the range of π^*.

Proof. Define maps

$$h_j\colon E_r\mathcal{G}\to E_{r+1}\mathcal{G},\quad j=0,\ldots,r$$

by

$$h_j(a_0,\ldots,a_p)=(a_0,\ldots,a_j,\underbrace{m,\ldots,m}_{r+1-j}).$$

These maps lift to bundle maps $\pi_r^*A\to\pi_{r+1}^*A$, denoted by the same letters. One can verify that

$$\mathsf{h}=\sum_{j=0}^{r}(-1)^{j+1}h_j^*\colon\Gamma(\wedge^s\pi_r^*A)\to\Gamma(\wedge^s\pi_{r+1}^*A)$$

is a homotopy operator with the desired property. □

Remark 12.18. It may be useful to explain this calculation on the 'homology side'. Consider $\mathbb{Z}E_p\mathcal{G}$, the formal linear combinations with coefficients in $\mathbb{Z}$ (to be precise, we only take linear conbinations of elements with the same source in M). Then δ is pullback under the map

$$\partial\colon\mathbb{Z}E_p\mathcal{G}\to\mathbb{Z}E_{p-1}\mathcal{G},\quad(a_0,\ldots,a_p)=\sum_i(-1)^i(a_0,\ldots,\widehat{a_i},\ldots).$$

while h is pullback under

$$h\colon\mathbb{Z}E_p\mathcal{G}\to\mathbb{Z}E_{p+1}\mathcal{G},\quad(a_0,\ldots,a_p)\mapsto\sum_{j=0}^{p}(-1)^{j+1}(a_0,\ldots,a_j,m,\ldots,m).$$

Let us demonstrate the calculation of $h\delta+\delta h$ for $p=2$:

$$\begin{aligned}h\delta(a_0,a_1,a_2) &= h\big((a_1,a_2)-(a_0,a_2)+(a_0,a_1)\big)\\ &= -(a_1,m,m)+(a_1,a_2,m)+(a_0,m,m)-(a_0,a_2,m)-(a_0,m,m)+(a_0,a_1,m)\\ &= -(a_1,m,m)+(a_1,a_2,m)-(a_0,a_2,m)+(a_0,a_1,m)\end{aligned}$$

$$\begin{aligned}\delta h(a_0,a_1,a_2) =& h\big(-(a_0,m,m,m)+(a_0,a_1,m,m)-(a_0,a_1,a_2,m)\big)\\ =& -(m,m,m)+(a_0,m,m)-(a_0,m,m)+(a_0,m,m)\\ &+(a_1,m,m)-(a_0,m,m)+(a_0,a_1,m)-(a_0,a_1,m)\\ &-(a_1,a_2,m)+(a_0,a_2,m)-(a_0,a_1,m)+(a_0,a_1,a_2)\\ =& -(m,m,m)+(a_1,m,m)-(a_1,a_2,m)+(a_0,a_2,m)-(a_0,a_1,m)+(a_0,a_1,a_2)\end{aligned}$$

So, $(h\delta+\delta h)(a_0,a_1,a_2)=-(m,m,m)+(a_0,a_1,a_2)$.

We are now in position to construct a cochain map from the complex $\widetilde{\mathsf{C}}^\bullet(\mathcal{G})$ to the complex $\mathsf{C}^\bullet(A)$. We begin with the pullback map $\kappa^*\colon \widetilde{\mathsf{C}}^\bullet(\mathcal{G})\to \mathsf{C}^{\bullet,\bullet}$. It takes values in $\mathsf{C}^{\bullet,0}$, hence can be regarded as a cochain map to the total complex.

$$\kappa^*\colon \widetilde{\mathsf{C}}^\bullet(\mathcal{G})\to \mathrm{Tot}^\bullet(\mathsf{C}^{\bullet,\bullet}).$$

The map $\iota^*\colon \mathsf{D}^{\bullet,\bullet}$ is homotopy equivalence with respect to δ, with homotopy operator h. By the basic perturbation lemma, we can modify it into a homotopy equivalence $\iota^*\circ(1+\mathrm{d}h)^{-1}$ with respect to $\mathrm{d}+\delta$. Hence, the composition

$$\iota^*\circ(1+\mathrm{d}h)^{-1}\circ\kappa^*\colon \widetilde{\mathsf{C}}^\bullet(\mathcal{G})\to \mathrm{Tot}^\bullet(\mathsf{D}^{\bullet,\bullet})$$

is a cochain map. This is almost what we want, except that we don't want to work with $\mathsf{D}^{\bullet,\bullet}$ which contains an infinite number of copies of the complex $\mathsf{C}^\bullet(A)$. Note however that the differential

$$\delta=\sum_{i=0}^{r+1}(-1)^i\partial_i^*\colon \Gamma(\wedge^s A_r^*)\to\Gamma(\wedge^s A_r^*)$$

(with $A_r=A$ for all r alternates between 0 and the identity, and is trivial in degree $r=0$. It's easy to see that the inclusion of the $r=0$ column

$$\mathsf{C}^s(A)\hookrightarrow \mathsf{D}^{\bullet,s}$$

as $\mathsf{D}^{0,s}$ is a homotopy equivalence. Hence, we conclude that

$$\iota_0^*\circ(1+\mathrm{d}h)^{-1}\circ\kappa^*\colon \widetilde{\mathsf{C}}^\bullet(\mathcal{G})\to \mathsf{C}^s(A)$$

is a cochain map.

Definition 12.19. The composition

$$\iota_0^*\circ(1+\mathrm{d}h)^{-1}\circ\kappa^*\colon \widetilde{\mathsf{C}}^\bullet(\mathcal{G})\to \mathsf{C}^s(A)$$

is called the *van Est map* for the groupoid $\mathcal{G}\rightrightarrows M$.

In this formula, $(1+\mathrm{d}h)^{-1}=\sum_{n=0}^\infty(-1)^n(\mathrm{d}\circ h)^n$ is well-defined on any element of finite degree. We can also write

$$\mathrm{VE}=\iota_0^*\circ(1+[\mathrm{d},h])^{-1}\circ\kappa^*$$

because $\mathrm{d} \circ \kappa^* = 0$ and $(1 + [\mathrm{d}, h])^{-1} = (1 + \mathrm{d}h)^{-1} + \sum_{j=1}^{\infty} (h\mathrm{d})^j$. This formula is somewhat easier to work with since $[\mathrm{d}, h]$ is closer to being a derivation. On elements of degree p the formula simplifies to

$$\mathrm{VE} = (-1)^p \iota_0^* \circ (\mathrm{d} \circ h)^p \circ \kappa_p^* \colon \mathsf{C}^p(\mathcal{G}) \to \mathsf{C}^p(A);$$

again, we can replace $\mathrm{d} \circ h$ with $[\mathrm{d}, h]$ if desired.

To get a better understanding of this formula, consider the zig-zag on elements of the form

$$f_0 \bar{\otimes} \cdots \bar{\otimes} f_r \bar{\otimes} \alpha \in \mathsf{C}^{r,s}$$

where $f_i \in C^\infty(\mathcal{G})$ $\alpha \in \Gamma(\wedge^s A^*)$, and the tensor products $\bar{\otimes}$ are over $C^\infty(M)$ (viewing $C^\infty(\mathcal{G})$ as a module over $C^\infty(M)$ via the source map). We have

$$\mathsf{h}(f_0 \bar{\otimes} \cdots \bar{\otimes} f_r \bar{\otimes} \alpha) = \sum_{j=0}^{r-1} (-1)^{j+1} \; f_0 \bar{\otimes} \cdots \bar{\otimes} f_j \bar{\otimes} \underbrace{1 \bar{\otimes} \cdots \bar{\otimes} 1}_{r-j-1} \bar{\otimes} \; \iota^*(f_{j+1} \cdots f_r)\alpha.$$

$$\pi_r^* \alpha = \underbrace{1 \bar{\otimes} \cdots \bar{\otimes} 1}_{r+1} \bar{\otimes} \alpha.$$

$$\iota_r^*(f_0 \bar{\otimes} \cdots \bar{\otimes} f_r \bar{\otimes} \alpha) = \iota^*(f_0 \cdots f_r)\alpha \in \wedge^s A^*.$$

Finally, the differential is given by the following formula (taking into account that d is a derivation relative to the wedge product)

$$\mathrm{d}\,(f_0 \bar{\otimes} \cdots \bar{\otimes} f_r \bar{\otimes} \alpha) = \sum_{j=0}^{r} \sum_k f_0 \bar{\otimes} \cdots \bar{\otimes} \mathcal{L}(X_k^L) f_j \bar{\otimes} \cdots \bar{\otimes} f_r \bar{\otimes} (\beta^k \wedge \alpha) + (-1)^r f_0 \bar{\otimes} \cdots \bar{\otimes} f_r \bar{\otimes} \mathrm{d}_{CE} \alpha.$$

In the first term, we take X_k be a local frame of sections of A, with dual frame β^k of A^*.

Remark 12.20. It is instructive to consider the case of a pair groupoid $\mathcal{G} = \mathrm{Pair}(M)$. Here

$$B_p\mathcal{G} = M^{p+1}, \quad E_p\mathcal{G} = M^{p+1} \times M.$$

The quotient map κ_p is just the obvious projection to the first factor, $\pi_p \colon E_p\mathcal{G} \to M_p$ is projection to the last factor. The space

$$C^\infty(B_p\mathcal{G}) = C^\infty(M^{p+1})$$

has a dense subspace $C^\infty(M)^{\otimes(p+1)}$, consisting of functions $f_0 \otimes f_1 \otimes \cdots \otimes f_p$. Likewise, $\mathsf{C}^{r,s}$ has a dense subspace consisting of all

$$(f_0 \otimes f_1 \otimes \cdots \otimes f_p) \otimes \alpha$$

with $\alpha \in \Omega^s(M)$. Consider the zig-zag $(\mathrm{d} \circ h)^p$ for $p = 2$. We have

$$\begin{aligned} f_0 \otimes f_1 \otimes f_2 &\xrightarrow{\kappa} (f_0 \otimes f_1 \otimes f_2) \otimes 1 \\ &\xrightarrow{h} -(f_0 \otimes 1) \otimes f_1 f_2 + (f_0 \otimes f_1) \otimes f_2 \\ &\xrightarrow{\mathrm{d}} (f_0 \otimes 1) \otimes \mathrm{d}(f_1 f_2) - (f_0 \otimes f_1) \otimes \mathrm{d}f_2 \\ &\xrightarrow{h} -f_0 \otimes \mathrm{d}(f_1 f_2) + f_0 \otimes f_1 \mathrm{d}f_2 \\ &\xrightarrow{\mathrm{d}} f_0 \otimes \mathrm{d}f_1 \mathrm{d}f_2 \\ &\xrightarrow{\iota^*} f_0\, \mathrm{d}f_1 \mathrm{d}f_2 \end{aligned}$$

More generally, one finds that

$$\mathrm{VE}(f_0 \otimes \cdots \otimes f_p) = f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_p,$$

using a 'completed' tensor product. In any case, products

$$\in C^\infty(M^{p+1})$$

are dense. Likewise

$$C^{r,s} = C^\infty(M)^{\otimes(p+1)} \otimes \Omega^s(M)$$

The map κ_p^* is given by

$$f_0 \otimes f_1 \otimes \cdots \otimes f_p \otimes 1.$$

The map h takes this to

$$-f_0 \otimes f_1 \otimes \cdots \otimes f_p \otimes 1$$

Appendix A. Deformation to the normal cone

A.1. **Basic properties.** Let (M, N) be a pair consisting of a manifold M and a submanifold N. Similar to the normal bundle functor, the *deformation to the normal cone* is a covariant functor from manifold pairs to manifolds, taking a pair (M, N) to a manifold $\mathcal{D}(M, N)$ and a smooth map of pairs $\varphi\colon (M_1, N_1) \to (M_2, N_2)$ to a smooth map

$$\mathcal{D}(\varphi)\colon \mathcal{D}(M_1, N_1) \to \mathcal{D}(M_2, N_2). \tag{37}$$

The manifold $\mathcal{D}(M, N)$ is a set-theoretic union of two submanifolds

$$\mathcal{D}(M, N) = \nu(M, N) \sqcup (M \times \mathbb{R}^\times),$$

and given $\varphi\colon (M_1, N_1) \to (M_2, N_2)$ the map $\mathcal{D}(\varphi)$ preserves this decomposition, and is given by $\nu(\varphi)\colon \nu(M_1, N_1) \to \nu(M_2, N_2)$ on the first piece and by $(m, t) \mapsto (\varphi(m), t)$ on the second piece. The deformation functor is uniquely determined by these properties, and the 'normalization' that

$$\mathcal{D}(\mathbb{R}, 0) = \mathbb{R} \times \mathbb{R}$$

as a manifold, where the diffeomorphism is given by $s \mapsto (s, 0)$ on $\nu(\mathbb{R}, 0) = \mathbb{R}$ and by $(s, t) \mapsto (\frac{1}{t} s, t)$ on $\mathbb{R} \times \mathbb{R}^\times$.

Before describing the construction of $\mathcal{D}(M, N)$ in more detail, we list some of the basic properties.

(a) The multiplicative group $\mathbb{R}^\times$ acts smoothly on $\mathcal{D}(M,N)$. Let λ_a denote the action of $a \in \mathbb{R}^\times$. On $\mathcal{D}(M,N)_0 = \nu(M,N)$ this is the action by scalar multiplication,

$$\lambda_a(\xi) = a\xi$$

while on the complement $M \times \mathbb{R}^\times$ the action is $\lambda_a(m,t) = (m, a^{-1}t)$. Given a map φ of pairs, the induced map $\mathcal{D}(\varphi)$ is $\mathbb{R}^\times$-equivariant.

The vector field with flow $s \mapsto \lambda_{\exp(s)}$ is given by $t\frac{\partial}{\partial t}$ on $M \times \mathbb{R}^\times$, and by minus the Euler vector field on $\nu(M,N)$.

(b) If $V \to M$ is a vector bundle, we have a canonical diffeomorphism

$$\mathcal{D}(V,M) \to V \times \mathbb{R}$$

given by the family of maps

$$\mathcal{D}(V,M)_t \to V, \quad \begin{cases} v \mapsto t^{-1}v & t \neq 0 \\ v \mapsto v & t = 0 \end{cases}$$

(with the standard isomorphism $\nu(V,M) = V$ for $t = 0$). In terms of this diffeomorphism, the $\mathbb{R}^\times$-action on $V \times \mathbb{R}$ is

$$\lambda_a(v,t) = (av, a^{-1}t).$$

(c) As a special case (taking V to be the zero vector bundle), we have that

$$\mathcal{D}(M;M) = M \times \mathbb{R}$$

with its standard manifold structure. Hence $(M,N) \to (M;M)$ defines a smooth map

$$\mathcal{D}(M,N) \to M \times \mathbb{R}.$$

On $\mathcal{D}(M,N)_0 = \nu(M,N)$ this is the bundle projection to $N \subseteq M$, and on $M \times \mathbb{R}^\times$ it is the obvious inclusion.

Similarly, $(N,N) \to (M,N)$ defines a smooth map

$$, \quad N \times \mathbb{R} \to \mathcal{D}(M,N).$$

For all t, this is the inclusion of $N \subseteq \mathcal{D}(M,N)_t$.

(d) Let $f\colon (M,N) \to (\mathbb{R};0)$, a smooth real-valued function vanishing on N. It induces a map

$$\mathcal{D}(f)\colon \mathcal{D}(M,N) \to \mathcal{D}(\mathbb{R};0) \cong \mathbb{R} \times \mathbb{R}.$$

This function take $[v] \in \nu(M,N)$, represented by $v \in TM|_N$, to $(v(f), 0)$, and $(m,t) \in M \times \mathbb{R}^\times$ to $(\frac{1}{t}f(m), t)$. We conclude that the function

$$\tilde{f}\colon \mathcal{D}(M,N) \to \mathbb{R}$$

given by $[v] \mapsto v(f)$ on $\nu(M,N)$, and by $(m,t) \mapsto \frac{1}{t}f(m)$ on the complement, is smooth. In terms of the $\mathbb{R}^\times$-action, the functions $\widetilde{f}$ are characterized by the property $\widetilde{f}(m,1) = f(m)$ together with homogeneity of degree 1:

$$\lambda_a^*\widetilde{f} = a\widetilde{f}.$$

(e) More generally, if $E \to M$ is a vector bundle, and σ a section vanishing along $N \subseteq M$, then the map $\sigma\colon (M,N) \to (E;M)$ induces a map
$$\mathcal{D}(\sigma)\colon \mathcal{D}(M,N) \to \mathcal{D}(E;M) = E \times \mathbb{R};$$
here $\mathcal{D}(\sigma)_0$ is the normal derivative $\mathrm{d}^N\sigma\colon \nu(M,N) \to E|_N \subseteq E$, while $\mathcal{D}(\sigma)_t$ for $t \neq 0$ is given by σ.

(f) A tubular neighborhood embedding $\psi\colon \nu(M,N) \to M$ determines, by functoriality, an embedding
$$\mathcal{D}(\psi)\colon \nu(M,N) \times \mathbb{R} \to \mathcal{D}(M,N).$$

A.2. **Charts on $\mathcal{D}(M,N)$.** To describe in more detail the manifold structure on $\mathcal{D}(M,N)$, we should specify an atlas. For $k \le n$ consider the pair $(\mathbb{R}^n, \mathbb{R}^k)$ as the local model for the pair (M,N). Write the coordinates on $\mathbb{R}^n$ as $(x,y) = (x_1,\ldots,x_k,y_1,\ldots,y_l)$ where $k+l=n$. We declare
$$\mathcal{D}(\mathbb{R}^n,\mathbb{R}^k) = \mathbb{R}^n \times \mathbb{R},$$
by the map given on $\nu(\mathbb{R}^n,\mathbb{R}^k) = \mathbb{R}^k \times \mathbb{R}^l$ by the obvious identification with $\mathbb{R}^n \times \{0\}$, and on $\mathbb{R}^n \times \mathbb{R}^\times$ by the map
$$(x,y,t) \mapsto (x, \frac{1}{t}y, t).$$

Recall that a submanifold chart for (M,N) is given by a chart $\phi\colon U \to \mathbb{R}^n$ for M such that
$$\phi\colon (U, U \cap N) \to (\mathbb{R}^n, \mathbb{R}^k).$$
Given any such submanifold chart, we obtain a chart
$$\mathcal{D}(\phi)\colon \mathcal{D}(U, U\cap N) \to \mathcal{D}(\mathbb{R}^n,\mathbb{R}^k) \cong \mathbb{R}^n \times \mathbb{R}. \tag{38}$$
for the deformation space.

Proposition A.1. *The charts* (38) *define an atlas on* $\mathcal{D}(M,N)$.

Proof. It is clear that these charts cover all of $\mathcal{D}(M,N)$ (note that a submanifold chart need not actually meet N). We have to check the compatibility of charts. Write $\widetilde{U} = \mathcal{D}(U, N\cap U) \subseteq \mathcal{D}(M,N)$ so that
$$\widetilde{U} = \nu_N|_{N\cap U} \cup (U \times \mathbb{R}^\times).$$
Let $x^1,\ldots,x^k,y^1,\ldots,y^l$ be the local coordinates defined by a submanifold chart near $m \in N$, and denote the resulting local coordinates on $\mathcal{D}(M,N)$ by
$$x_1,\ldots,x_k,\widetilde{y}_1,\ldots,\widetilde{y}_l,t.$$
Thus $\widetilde{y}_i|_{U\times\mathbb{R}^\times} = \frac{1}{t}y_i$ for $t \neq 0$. Suppose $X_1,\ldots,X_k,Y_1,\ldots,Y_l$ are a new set of submanifold coordinates. Then
$$X_i = F_i(x,y),\ Y_j = G_j(x,y),$$
with $G_j(x,0) = 0$. The resulting change of local coordinates on the deformation space is then given for $t \neq 0$ by
$$(x,\widetilde{y},t) \mapsto (X,\widetilde{Y},t) = \Big(F(x,t\widetilde{y}), \frac{1}{t}G(x,t\widetilde{y}), t\Big)$$

It extends smoothly to $t = 0$, and for $t = 0$ is given exactly by the normal map for the change of coordinates. This shows that the change of coordinates is smooth. $\square$

A.3. **Euler-like vector fields.** Let $X \in \mathfrak{X}(M, N) = \Gamma(TM, TN)$, a vector field on M that is tangent to N. Then X defines a vector field $\mathcal{D}(X)$ on the deformation space. This vector field is tangent to all fibers $\mathcal{D}(M, N)_t$, and is given by X for $t \neq 0$ and by the linear appoximation $\nu(X)$ for $t = 0$.

Theorem A.2. *A vector field $X \in \mathfrak{X}(M)$ is Euler-like along N if and only if the vector field*

$$W = \frac{1}{t} X + \frac{\partial}{\partial t}$$

on $M \times \mathbb{R}^\times$ extends smoothly to a vector field on $\mathcal{D}(M, N)$.

Proof. Recall from a above that the vector field $t\frac{\partial}{\partial t}$ on $M \times \mathbb{R}^\times$ extends smoothly to the deformation space $\mathcal{D}(M, N)$, and is given on $\nu(M, N)$ by *minus* the Euler vector field on $\nu(M, N)$. On the other hand, the vector field X, regarded as a vector field on $M \times \mathbb{R}^\times$, extends to the vector field $\mathcal{D}(X)$, given on $\nu(M, X)$ by the Euler vector field. Hence, $X + t\frac{\partial}{\partial t}$ extends to a vector field on $\mathcal{D}(M, N)$ that vanishes along the submanifold $\nu(M, N)$. This implies that $\frac{1}{t}$ times this vector field extends smoothly to $\mathcal{D}(M, N)$. $\square$

Remark A.3. It may be instructive to repeat this proof in local coordinates. Let $x_1, \ldots, x_k, y_1, \ldots, y_l$ be coordinates of a submanifold chart, so that N is given by $y = 0$. Recall from (23) that X is Euler-like along N if and only if

$$X = \sum_i a^i(x, y) \frac{\partial}{\partial x^i} + \sum_j (y^j + b^j(x, y)) \frac{\partial}{\partial y^j}$$

where the a^i vanish for $y = 0$, and b^j vanishes to second order for $y = 0$. The vector field $\mathcal{D}(X)$ is given by the same formula in the 'product' coordinates of $\mathcal{D}(M, N)$. A change to the standard $\mathcal{D}(M, N)$ coordinates $\widetilde{t} = t,\ \widetilde{x} = x,\ \widetilde{y} = \frac{1}{t} y$ gives

$$\mathcal{D}(X) = \sum_i a^i(x, t\widetilde{y}) \frac{\partial}{\partial x^i} + \sum_j \widetilde{y}^j \frac{\partial}{\partial \widetilde{y}^j} + \sum_j \frac{1}{t} b^j(x, t\widetilde{y})) \frac{\partial}{\partial \widetilde{y}^j}$$

On the other hand, the vector field $\frac{\partial}{\partial t}$ becomes, after coordinate change,

$$\frac{\partial}{\partial t} \rightsquigarrow \frac{\partial}{\partial t} - t \sum_j \widetilde{y}_j \frac{\partial}{\partial \widetilde{y}_j}.$$

So,

$$W = \frac{\partial}{\partial t} + \frac{1}{t} \sum_i a^i(x, t\widetilde{y}) \frac{\partial}{\partial x^i} + \sum_j \frac{1}{t^2} b^j(x, t\widetilde{y})) \frac{\partial}{\partial \widetilde{y}^j}$$

which extends smoothly to $t = 0$.

Since $W \sim \frac{\partial}{\partial t}$, its flow is given by a family of diffeomorphisms $\mathcal{D}(M,N)_t \to \mathcal{D}(M,N)_{t-s}$. Since $[W, \mathcal{D}(X)] = 0$, this flow intertwines the vector fields $\mathcal{D}(X)_t$. In particular, we obtain a diffeomorphism

$$\nu(M,N) = \mathcal{D}(M,N)_0 \to M = \mathcal{D}(M,N)_1,$$

taking $\nu(X)$ to X. Since the restriction of W to $\mathcal{D}(N,N) = N \times \mathbb{R}$ is the vector field $\frac{\partial}{\partial t}$, the family of diffeomorphisms considered above induce the identity on $\nu(\mathcal{D}(M,N)_t, N) \cong \nu(M,N)$, hence it gives tubular neighborhood embeddings according to our definition.

`should explain more clearly the completeness issues`

A.4. **Vector bundles.** For any vector bundle $A \to M$ of rank k and subbundle $B \to N$, the deformation space defines a vector bundle $\mathcal{D}(A,B) \to \mathcal{D}(M,N)$, again of rank k. Its restriction to the zero fiber is the normal bundle $\nu(A,B) \to \nu(M,N)$. In fact, $\nu(A,B)$ is a double vector bundle

$$\begin{array}{ccc} \nu(A,B) & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ \nu(M,N) & \longrightarrow & N \end{array}$$

with core $A|_N/B$. The sections of $\nu(A,B)$ over $\nu(M,N)$ are generated by the linear and core sections. Regarding sections of a vector bundle as sections that are homogeneous of degree -1, the core sections are vector fields on $\nu(A,B)$ of homogeneity $(-1,-1)$, while the linear sections are those of homogeneity $(0,-1)$.

In terms of the deformation space, every σ of A whose restriction to N takes values in B, one obtains a section $\mathcal{D}(\sigma)$ if $\mathcal{D}(A)$. Its restriction to $\nu(M,N)$ is the linear section defined by σ. (Note that $\mathcal{D}(\sigma)$ is invariant under natural $\mathbb{R}^\times$-action on the deformation space, hence $\nu(\sigma)$ is invariant under the resulting $\mathbb{R}^\times$-action on $\nu(A,B) \to B$. The latter is the horizontal scalar multiplication). For a general section σ, the section $t(\sigma \times 0) \in \Gamma(A \times \mathbb{R}^\times)$ again extends to a section of $\mathcal{D}(A,B)$. Relative to the $\mathbb{R}^\times$-action, this section is homogeneous of degree -1; hence we conclude that the resulting section of $\mathcal{D}(A,B) \to \mathcal{D}(M,N)$ is homogeneous of degree -1 for the horizontal scalar multiplication; that is, it is a core section.

If A is an anchored vector bundle, with anchor map $\mathsf{a}\colon A \to TM$, and B is an anchored subbundle along N (that is, $\mathsf{a}(B) \subseteq TN$), then the anchor map

$$\mathsf{a}\colon (A,B) \to (TM,TN)$$

defines a map of deformation spaces

$$\mathcal{D}(\mathsf{a})\colon \mathcal{D}(A,B) \to \mathcal{D}(TM,TN) \subseteq T\mathcal{D}(M,N),$$

where we are identifying $\mathcal{D}(TM,TN)$ with the subbundle $\ker(T\pi)$. In this way, $\mathcal{D}(A,B) \to \mathcal{D}(M,N)$ is an anchored vector bundle, with the fibers $\mathcal{D}(A,B)|_{\pi^{-1}(t)}$ as anchored subbundles. For $t = 0$, we obtain the anchored vector bundle $\nu(A,B) \to \nu(M,N)$.

A.5. **Lie groupoids.** Let us apply the normal cone construction to a Lie groupoid and Lie subgroupoid.

Theorem A.4. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{H} \rightrightarrows N$ a Lie subgroupoid. Then $\mathcal{D}(\mathcal{G}, \mathcal{H})$ is a Lie groupoid, containing $\nu(\mathcal{G}, \mathcal{H}) \rightrightarrows \nu(M, N)$ and $\mathcal{G} \times \mathbb{R} \rightrightarrows M \times \mathbb{R}^\times$ as Lie subgroupoids.*

Proof. The structure maps are all obtained 'by functoriality', for example

$$\mathsf{s}\colon (\mathcal{G}, \mathcal{H}) \to (M, N)$$

induces

$$\mathcal{D}(\mathsf{s})\colon \mathcal{D}(\mathcal{G}, \mathcal{H}) \to \mathcal{D}(M, N),$$

and similarly for t and for the inclusion of units. The multiplication map is obtained by applying the functor $\mathcal{D}$ to

$$\mathrm{Mult}_{\mathcal{G}}\colon (\mathcal{G}^{(2)}, \mathcal{H}^{(2)}) \to (\mathcal{G}, \mathcal{H}),$$

and observing that

$$\mathcal{D}(\mathcal{G}^{(k)}, \mathcal{H}^{(k)}) = \mathcal{D}(\mathcal{G}, \mathcal{H})^{(k)}$$

for all k. For the latter, note that each of the projections $(\mathcal{G}^{(k)}, \mathcal{H}^{(k)}) \to (\mathcal{G}, \mathcal{H})$, given by $(g_1, \ldots, g_k) \mapsto g_i$, give rise to maps $\mathcal{D}(\mathcal{G}^{(k)}, \mathcal{H}^{(k)}) \to \mathcal{D}(\mathcal{G}, \mathcal{H})$, and that the resulting map

$$\mathcal{D}(\mathcal{G}^{(k)}, \mathcal{H}^{(k)}) \to \mathcal{D}(\mathcal{G}, \mathcal{H})^k,$$

is a diffeomorphism onto $\mathcal{D}(\mathcal{G}, \mathcal{H})^{(k)}$. ∎

Example A.5. If G is a Lie group, and H is a Lie subgroup, then the group $\nu(G, H)$ is the semi-direct product $H \ltimes (\mathfrak{g}/\mathfrak{h})$ of H with the vector space $\mathfrak{g}/\mathfrak{h}$. The normal cone construction exhibits the semi-direct product as a 'limit' of G. For $\mathfrak{h} = \{0\}$, one obtains $\mathfrak{g}$ as a 'limit' of G.

One can see it more explicitly if G is given as a matrix Lie group, $G \subseteq \mathrm{GL}(n, \mathbb{R})$. The inclusion map takes $\{e\}$ to $\{0\}$, hence it induces an embedding

$$\mathcal{D}(G, \{e\}) \to \mathcal{D}((\mathrm{Mat}(n, \mathbb{R}), \{0\}) \cong \mathrm{Mat}(n, \mathbb{R}) \times \mathbb{R}.$$

The image of this embedding is

$$\big\{(x, 0)|\ x \in \mathfrak{g}\big\} \sqcup \Big\{\big(\frac{1}{t}A, t\big)|\ A \in G,\ t \in \mathbb{R}\Big\} \subseteq \mathrm{Mat}(n, \mathbb{R}) \times \mathbb{R}.$$

To see that this is indeed a submanifold, observe that a neighborhood of $\mathfrak{g} \times \{0\}$ is given as the image of the map $\mathfrak{g} \times \mathbb{R} \to \mathrm{Mat}(n, \mathbb{R}) \times \mathbb{R}$, $\quad (x, t) \mapsto (\frac{1}{t}\exp(tx), t)$.

Example A.6. Given any Lie groupoid, the proposition exhibits $\nu(\mathcal{G}, M) = A(\mathcal{G})$ (viewed as a vector bundle, hence as a Lie groupoid $A(\mathcal{G}) \rightrightarrows M$) as a Lie subgroupoid of $\mathcal{D}(\mathcal{G}, M) \rightrightarrows M \times \mathbb{R}$, with complement $\mathcal{G} \times \mathbb{R}^\times$. This applies, in particular, to the pair groupoid $\mathrm{Pair}(M) = M \times M$. The groupoid

$$\mathcal{D}(M \times M, M) \rightrightarrows M \times \mathbb{R}$$

is known as *Connes's tangent groupoid*, since it plays a key role in Connes' approach [9] to the Atiyah-Singer index theorem.

A.6. **Lie algebroids.** The Lie algebroid analogue to Theorem A.4 reads as follows:

Theorem A.7. *Let $A \to M$ be a Lie algebroid, and $B \to N$ a Lie subalgebroid. Then the Lie algebroid structure on $A \times 0_{\mathbb{R}^\times} \to M \times \mathbb{R}^\times$ extends uniquely to a Lie algebroid structure on $\mathcal{D}(A,B)$. In particular, this induces a Lie algebroid structure on $\nu(A,B) \to \nu(M,N)$.*

Proof. As explained above the anchor map for $A \to M$ induces an anchor map for $\mathcal{D}(A,B) \to \mathcal{D}(M,N)$. Any section $\sigma \in \Gamma(A,B)$ (that is, a section of A whose restriction to N takes values in B) determines a section $\mathcal{D}(\sigma)$ of the deformation space; arbitrary sections $\tau \in \Gamma(A)$ determine sections $\widehat{\tau} \in \Gamma(\mathcal{D}(A,B))$, given as the extension of $t(\tau \times 0) \in \Gamma(A \times 0_{\mathbb{R}^\times})$. These sections generate $\Gamma(\mathcal{D}(A,B))$ as a module over smooth functions on $\mathcal{D}(M,N)$. Hence, to show that the Lie algebroid structure on $A \times 0_{\mathbb{R}^\times}$ extends, it suffices to show that the Lie algebroid bracket of sections of these two types, which a priori is given only as a section of $A \times 0_{\mathbb{R}^\times}$, extends to a section of $\mathcal{D}(A,B)$. But this is straightforward; the desired extensions are given by the formulas

$$[\,\widehat{\tau_1}\,,\,\widehat{\tau_2}\,] = t\widehat{[\tau_1,\tau_2]},\quad [\mathcal{D}(\sigma),\,\widehat{\tau}\,] = \widehat{[\sigma,\tau]},\quad [\mathcal{D}(\sigma_1),\mathcal{D}(\sigma_2)] = \mathcal{D}([\sigma_1,\sigma_2]).$$

□

Restricting these brackets to $t = 0$, we obtain the Lie algebroid structure of $\nu(A,B)$. For $\tau \in \Gamma(A)$, let $c(\tau) \in \Gamma(\nu(M,N))$ be the core section defined by $\tau|_N \mod B \in \Gamma(A|_N/B)$. Then

$$[c(\tau_1), c(\tau_2)] = 0,\ \ [\nu(\sigma), c(\tau)] = c([\sigma,\tau]),\quad [\nu(\sigma_1),\nu(\sigma_2)] = \nu([\sigma_1,\sigma_2]).$$

The anchor map for linear and core sections is

$$\mathsf{a}(\nu(\sigma)) = \nu(\mathsf{a}(\sigma)),\qquad \mathsf{a}(c(\tau)) = 0.$$

Remark A.8. If $M = \mathrm{pt}$, so that $A = \mathfrak{g}$ and $B = \mathfrak{h}$ are just Lie algebras, we see that the Lie algebroid structure on $\nu(\mathfrak{g},\mathfrak{h}) \to \mathrm{pt}$ is that of the semi-direct product $\mathfrak{g}/\mathfrak{h} \rtimes \mathfrak{h}$.

Let us now specialize to the case that $B = i^!A$, where $i\colon N \hookrightarrow M$ is the inclusion of a submanifold transverse to the anchor.

Lemma A.9. *The bundle $\nu(A, i^!A) \to \nu(M,N)$ has a distinguished section $\epsilon\colon \nu(M,N) \to \nu(A, i^!A)$ such that $\mathsf{a}(\epsilon)$ is the Euler vector field. There exists a section $\sigma \in \Gamma(A, i^!A)$ such that $\nu(\sigma) = \epsilon$.*

Proof. Recall that $\nu(A, i^!A)$ is canonically isomorphic, as a double vector bundle, to $p^!i^!A$. For any anchored vector bundle $B \to N$ (here $i^!A$), and any vector bundle $p\colon V \to N$ (here $\nu(M,N) \to N$), the bundle $p^!B \to V$ is given by $TV \times_{TN} B \to V \times_N N = V$. The section $\mathcal{E}_V \times 0$ of $TV \times B \to V \times N$ restricts over $V \times_N N$ to a section of $TV \times_{TN} B$, as required.
If $\sigma \in \Gamma(A)$ is any section such that $\mathsf{a}(\sigma)$ is Euler-like, then $\sigma \in \Gamma(A, i^!A)$ has $\nu(\sigma) = \epsilon$. □

We will call σ as in the Lemma an Euler-like section of A.

Given an Euler-like section σ, we obtain a section of $\mathcal{D}(A, i^!A)$, and a corresponding vector field $\widetilde{X}$ on the total space of $\mathcal{D}(A, i^!A)$. Note that this vector field is an infinitesimal Lie algebroid automorphism. It follows that the vector field

$$\widetilde{W} = \frac{1}{t}\widetilde{X} + \frac{\partial}{\partial t}$$

on $\mathcal{D}(A, i^!A)$ is an infinitesimal Lie algebroid automorphism, hence its flow is by Lie algebroid automorphisms. This flow gives the desired tubular neighborhood embedding

$$\nu(A, i^!A) \to A,$$

covering $\nu(M, N) \to M$.

Remark A.10. A variation:[12] Consider the canonical map

$$F \colon \mathcal{D}(M, N) \to M,$$

given on $\nu(M, N)$ be $i \circ p$ and on $M \times \mathbb{R}^\times$ by projection. This map is smooth, and transversality of N to the anchor of A guarantees the map F is transverse to the anchor. Hence, the pull-back Lie algebroid $F^!A$ is well-defined.

If σ is an Euler-like section of A, then

$$\frac{1}{t}\sigma + \frac{\partial}{\partial t} \in \Gamma(A \times T\mathbb{R}^\times)$$

extends smoothly to a section of $F^!A$. The corresponding vector field on $F^!A$ is an infinitesimal Lie algebroid automorphism, lifting the vector field W on the base. Hence, its interwines $\iota_t^!A$ with $\iota_{t-s}^!A$, where $\iota_t \colon \mathcal{D}(M, N)_t \to \mathcal{D}(M, N)$ is the inclusion of the t-fiber. In particular, we get a LA isomorphism

$$p^!i^!A = \iota_0^!A \to \iota_1^!A = A.$$

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