18.116 Riemann Surfaces, MIT Spring term, 2008

First exercise sheet. Due: March 11, during class.

Late assignments will not be accepted.

Exercise 1. Jost, section 1.3 exercise 1) and 3), section 2.1 exercise 1) and 2), and section 2.7 exercise 1), 2), and 3).

Exercise 2.

- Consider the map $\mathbb{C} \longrightarrow \mathbb{C}$ given by $z \mapsto z^k$ for $k \in \mathbb{Z}$. Show that this defines a covering $\pi : \Sigma \longrightarrow \mathbb{C}^*$. What is the domain Σ of this covering? What is the automorphism group of the covering?

- Let $z, \tilde{z}$ be coordinate charts for $\mathbb{P}^1$ such that $\tilde{z} = \frac{1}{z}$ on the overlap, as discussed in lectures. Show that the expression
$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \cdots + a_k z^k}{b_0 + b_1 z + \cdots + b_l z^l},$$
uniquely defines a holomorphic map $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. When is this a biholomorphism (i.e. holomorphic isomorphism)? Bonus: Show that every holomorphic map $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is of the above form, i.e. is a rational function.

- Show that the map f defines a covering of the complement of a finite collection of points (called branch points) in $\mathbb{P}^1$. The degree of f is the number of sheets; calculate the degree of f. Bonus: Describe the group of deck transformations of this cover.

- If Σ is a Riemann surface, show that a meromorphic function on Σ is nothing but a holomorphic map $\Sigma \longrightarrow \mathbb{P}^1$.

Exercise 3. Define an elliptic curve as the quotient $E_\tau = \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau \subset \mathbb{C} \cong \mathbb{R}^2$ is the lattice
$$\Lambda_\tau = \langle 1, \tau \rangle,$$
where $\tau \in H$ is a point in the upper half plane (and hence linearly independent with $1 \in \mathbb{C}$).

- Now let $\sigma \in \mathbb{C}^*$ and define the quotient $Q_\sigma = \mathbb{C}^*/\mathbb{Z}$ where $n \in \mathbb{Z}$ acts via $z \mapsto \sigma^n z$. Show that Q_σ is biholomorphic to E_τ for some τ. Determine the correspondence this implies between $\tau \in H$ and $\sigma \in \mathbb{C}^*$.

- With σ as above, define f via
$$f(z) = \sum_{n=-\infty}^{\infty} \frac{z}{(\sigma^{n/2} z - \sigma^{-n/2})^2}.$$
Where does this sum converge? Show that f is invariant under the above $\mathbb{Z}$-action where it is defined. Deduce that it defines a holomorphic map $f : Q_\sigma \longrightarrow \mathbb{P}^1$. What is the degree of this map? What are its branch points? What are its critical points?

Exercise 4. Let f be a non-constant meromorphic function on the Riemann surface M. Show that

- f has as many zeros as poles, counted with multiplicity (give a definition of multiplicity).

- f completely determines the complex structure on the surface Σ. (hint - describe an atlas of holomorphic charts)

Exercise 5 (Review of Hilbert spaces)**.** The typical example of an infinite-dimensional Hilbert space is the space of square-integrable functions (modulo a.e. zero functions), L^2, or the square-summable sequences, ℓ^2. These are well-known spaces, look them up anywhere.

Let us work with Hilbert spaces over $\mathbb{R}$, i.e. $\mathcal{H}$ is a real vector space equipped with a nondegenerate, real, positive-definite inner product $\langle\cdot,\cdot\rangle$ with associated norm $||v|| = \langle v, v\rangle^{1/2}$, and which is such that $\mathcal{H}$ is *complete*, i.e. Cauchy sequences converge (of course the norm induces on $\mathcal{H}$ a topology and we are using it.)

Begin by proving the Cauchy-Schwarz inequality for Hilbert spaces:

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle,$$

and conclude from it the triangle inequality for the norm.

i) **(Vector subspaces of Hilbert spaces)**

- Give examples of subspaces of ℓ^2 which a) have infinite dimension and codimension, b) which have finite dimension, and c) which have finite codimension.
- Give an example of a proper subspace of ℓ^2 which is closed.
- Give an example of a proper subspace of ℓ^2 which is not closed.
- Give an example of a proper subspace of ℓ^2 which is dense.
- if $W \subset \mathcal{H}$ is a subspace, show $\overline{W}$ is a subspace and show

$$W^\perp = (\overline{W})^\perp.$$

  Finally show that $W \cap W^\perp = \{0\}$.
- Show that if $W \subset \mathcal{H}$ is a closed subspace, then W and $\mathcal{H}/W$ naturally inherit a Hilbert space structure.
- Is it possible that $\overline{W}/W$ be nonzero but finite-dimensional?

ii) **The unit sphere**. Let $S(\mathcal{H}) \subset \mathcal{H}$ be the unit sphere in $\mathcal{H}$.

- Show that $S(\mathcal{H})$ is closed. Show it is compact iff $\dim \mathcal{H} < \infty$.
- Show that a linear map of Hilbert spaces $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is continuous if and only if $F(S(\mathcal{H}_1))$ is bounded. (This is why such maps are sometimes called "bounded operators") Show this is equivalent to the inequality

$$||Fv||_{\mathcal{H}_2} \leq C||v||_{\mathcal{H}_1} \quad \forall v \in \mathcal{H}_1, \tag{1}$$

  for some constant C (independent of v).
- Suppose that $W \subset \mathcal{H}_1$ is a dense linear subspace and $F : W \longrightarrow \mathcal{H}_2$ is a continuous linear map. Show that F has a unique extension to a continuous linear map $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$.
- A linear map of Hilbert spaces $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is called *compact* when $F(S(\mathcal{H}_1))$ is compact in $\mathcal{H}_2$ (i.e. any bounded sequence has a subsequence mapped by F to a convergent sequence). Give an example of a compact operator $\ell^2 \longrightarrow \ell^2$ which is also an injection. Can it be surjective? Can its image be dense? Give proof/examples.

iii) **The continuous dual** The operator norm of a continuous linear map $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is defined as

$$||F|| := \sup_{v \in S(\mathcal{H}_1)} ||Fv||_{\mathcal{H}_2}.$$

Show that the composition of continuous linear operators is a continuous operation in the operator norm, i.e. for A, B continous linear operators, show

$$||A \circ B|| \leq ||A||||B||.$$

Let $\mathcal{H}'$ denote the continuous dual of $\mathcal{H}$, i.e. the space of continuous linear maps $L : \mathcal{H} \longrightarrow \mathbb{R}$, equipped with operator norm, viewing $\mathbb{R}$ as a Hilbert space.

• Show that the "dualization map" $v \mapsto v^* = \langle v, \cdot \rangle$ is an injective, norm-preserving continuous linear map $\mathcal{H} \longrightarrow \mathcal{H}'$.

The Riesz representation theorem states that the dualization map is an isomorphism of Hilbert spaces.

• Show that if $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a continous linear operator, then $F^* : \mathcal{H}_2' \longrightarrow \mathcal{H}_1'$ defined by

$$F^* \mu = \mu \circ F$$

is a continuous linear map. If F is injective, under what conditions is F^* surjective? Show that if $\mathrm{Im} F$ is dense, then F^* is injective.

iv) **Weak convergence** A sequence $(x_n)_{n \in \mathbb{N}}$ is said to be *weakly convergent* to $x \in \mathcal{H}$ (we write $x_n \rightharpoonup x$) if

$$\lim_{n \to \infty} f(x_n - x) = 0 \quad \forall f \in \mathcal{H}'.$$

• Show that a convergent sequence in $\mathcal{H}$ is automatically weakly convergent. For this reason, sometimes usual convergence in $\mathcal{H}$ is called *strong convergence.*

• Give an example of a weakly convergent sequence in ℓ^2 which is not strongly convergent.

• Let $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a continuous linear map of Hilbert spaces. Show that if $x_n \rightharpoonup x$ then $F x_n \rightharpoonup F x$. If F is compact, show $F x_n$ converges strongly to $F x$.

• Prove that any bounded sequence $(x_n)_{n \in \mathbb{N}}$ in a Hilbert space $\mathcal{H}$ has a weakly convergent subsequence. [Hint: Show that it suffices to show convergence $(x_{n_k}, y) \longrightarrow (x, y)$ for y in the closure $\overline{S}$ of the span S of (x_n).]

• If $x_n \rightharpoonup x$, show

$$||x|| \leq \liminf_{n \to \infty} ||x_n||.$$

Also show that $x_n \to x$ iff both $x_n \rightharpoonup x$ and

$$||x|| = \lim_{n \to \infty} ||x_n||.$$