Geometry and Topology, 1300Y

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Contents

1 An introduction to homotopy theory **2**
- 1.1 Cell complexes . . . 3
- 1.2 The fundamental group(oid) . . . 4
- 1.3 $\pi_1(S^1) = \mathbb{Z}$ . . . 6
- 1.4 Further computations of π_1 . . . 8
- 1.5 The Van Kampen theorem . . . 9
- 1.6 Covering spaces . . . 12
- 1.7 Group actions and Deck transformations . . . 17

2 Homology **19**
- 2.1 Simplicial homology . . . 19
- 2.2 Singular homology . . . 21
- 2.3 H_0 and H_1 . . . 23
- 2.4 Relative homology and the excision property . . . 24
- 2.5 Excision implies Simplicial = Singular homology . . . 29
- 2.6 Axioms for homology . . . 31
- 2.7 Mayer-Vietoris sequence . . . 32
- 2.8 Degree . . . 34
- 2.9 Cellular homology . . . 35

3 Cohomology **39**
- 3.1 Cup product . . . 40
- 3.2 Contravariance of cohomology . . . 41
- 3.3 Cap product . . . 41

1 An introduction to homotopy theory

This semester, we will continue to study the topological properties of manifolds, but we will also consider more general topological spaces. For much of what will follow, we will deal with arbitrary topological spaces, which may, for example, not be Hausdorff (recall the quotient space $R_0 = \mathbb{R} \sqcup \mathbb{R}/(a \sim b \text{ iff } a = b \neq 0)$, or locally Euclidean (for example, the Greek letter θ), or even locally contractible (for example, the Hawaiian earring, given by the union of circles at $(1/n, 0)$ of radius $1/n$ for all positive $n \in \mathbb{Z}$ in $\mathbb{R}^2$ with the induced topology).

While we relax the type of space under consideration, we suitably relax the notion of equivalence which we are interested in: we will often be concerned not with homeomorphism (topological equivalence), but rather *homotopy equivalence*, which we recall now.

Definition 1. Continuous maps $f_0, f_1 : X \longrightarrow Y$ are *homotopic*, i.e. $f_0 \simeq f_1$, when there is a continuous map $F : X \times I \longrightarrow Y$, called a homotopy, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We sometimes write $F : f_0 \Rightarrow f_1$ to denote the homotopy.

The homotopy relation $\simeq$ is an equivalence relation: if $F_{01} : f_0 \Rightarrow f_1$ and $F_{12} : f_1 \Rightarrow f_2$ for maps $f_i : X \longrightarrow Y$, then

$$F_{02}(t,x) = \begin{cases} F_{01}(2t,x) & : \ 0 \leq t \leq 1/2 \\ F_{12}(2t-1,x) & : \ 1/2 \leq t \leq 1 \end{cases}$$

gives a homotopy $F_{02} : f_0 \Rightarrow f_2$. Check reflexivity and identity yourself!

The homotopy relation is also compatible with the natural category structure on continuous functions: If $F : f_0 \Rightarrow f_1$ for $f_i : X \longrightarrow Y$, and $G : g_0 \Rightarrow g_1$ for $g_i : Y \longrightarrow Z$, then the composition

$$X \times I \xrightarrow{(F, \pi_I)} Y \times I \xrightarrow{G} Z$$

defines a homotopy $g_0 \circ f_0 \Rightarrow g_1 \circ f_1$. As a result of this, we may consider a new category, where the objects are topological spaces and the morphisms are *homotopy classes* of continuous maps.

Definition 2. Topological spaces, and homotopy classes of maps between them, form a category, **HTop**, called the *homotopy category of spaces.*

Because the notion of morphism is different in **HTop**, this changes the meaning of isomorphism – we are no longer dealing with homeomorphism.

Definition 3. Topological spaces X, Y are said to be *homotopy equivalent*(or *homotopic* or have the same homotopy type $X \simeq Y$) when they are isomorphic in the homotopy category. This means that there are maps $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$.

Example 1.1. *(Homotopy equivalences)*

- *The one-point space $\{*\}$ is homotopic to $\mathbb{R}$, since $* \mapsto 0$ and $x \mapsto * \ \forall x \in \mathbb{R}$ define continuous maps f, g which are homotopy inverses of each other. Similarly $\{*\} \simeq B^n \simeq \mathbb{R}^n \ \forall n$. Any space $\simeq *$ we call* contractible.
- *The solid torus $B^2 \times S^1$ is homotopic to S^1.*
- *Any vector bundle $E \longrightarrow X$ is homotopic to X itself.*
- *The "pair of pants" surface with boundary $S^1 \sqcup S^1 \sqcup S^1$ is homotopic to the letter θ.*

When considering maps $f : X \longrightarrow Y$, we may choose to consider equivalence classes of maps which are homotopic *only away from a distinguished subset $A \subset X$*. These are called homotopies *relative to A*.

Definition 4. For $f_i : X \longrightarrow Y$ and $A \subset X$, we say $F : f_0 \Rightarrow f_1$ is a homotopy rel A when $F(x,t) = f_0(x)$ for all $x \in A$ and for all t.

This is useful in the case that a space X can be "continuously contracted" onto a subspace $A \subset X$: we formalize this as follows:

Definition 5. A retraction of X onto the subspace $A \subset X$ is a continuous map $r : X \longrightarrow A$ such that $r|_A = \mathrm{Id}_A$. In other words, a retraction is a self-map $r : X \longrightarrow X$ such that $r^2 = r$, where we take $A = \mathrm{im}(r)$.

We then say that $A \subset X$ is a (strong) deformation retract of X when $\mathrm{Id}_X : X \longrightarrow X$ is homotopic (rel A) to a retract $r : X \longrightarrow A$.

Proposition 1.2. *If $A \subset X$ is a deformation retract of X, then $A \simeq X$.*

Proof. Take $r : X \longrightarrow A$ and the inclusion $\iota : A \longrightarrow X$. Then $\iota \circ r = r \simeq \mathrm{Id}_X$ by assumption, and $r \circ \iota = \mathrm{Id}_A$. Hence we have $X \simeq A$. □

Deformation retracts are quite intuitive and easy to visualize - they also can be used to understand any homotopy equivalence:

Proposition 1.3. *(See Hatcher, Cor. 0.21) X, Y are homotopy equivalent iff there exists a space Z containing X, Y and deformation retracting onto each.*

1.1 Cell complexes

The construction of a mapping cylinder M_f of a continuous map $f : X \longrightarrow Y$ is an example of the coarse type of gluing and pasting constructions we are allowed to do once we go beyond manifolds. In this section we will introduce more such constructions, and introduce a class of spaces which is very convenient for algebraic topology.

A *cell complex*, otherwise known as a CW complex, is a topological space constructed from disks (called *cells*), step by step increasing in dimension. The basic procedure in the construction is called "attaching an n-cell". An n-cell is the interior e^n of a closed disk D^n of dimension n. How to attach it to a space X? Simply glue D^n to X with a continuous map $\varphi : S^{n-1} \longrightarrow X$, forming:

$$X \sqcup D^n / \{x \sim \varphi(x) \ : \ x \in \partial D^n\}.$$

The result is a topological space (with the quotient topology), but as a set, is the disjoint union $X \sqcup e^n$.
Building a cell complex X

- Start with a discrete set X^0, whose points we view as 0-cells.
- Inductively form the n-skeleton X^n from X^{n-1} by attaching a set of n-cells $\{e^n_\alpha\}$ to X^{n-1}.
- Either set $X = X^n$ for some $n < \infty$, or set $X = \bigcup_n X^n$, where in the infinite case we use the weak topology: $A \subset X$ is open if it is open in X^n $\forall n$.

While cell complexes are not locally Euclidean, they do have very good properties, for example they are Hausdorff and locally contractible. Any manifold is homotopy equivalent to a cell complex.

Example 1.4. *The 1-skeleton of a cell complex is a graph, and may have loops.*

Example 1.5. *The classical representation of the orientable genus g surface as a $4g$-gon with sides identified cyclically according to $\ldots aba^{-1}b^{-1}\ldots$ is manifestly a cell complex with a single 0-cell, $2g$ 1-cells and a single 2-cell. One sees immediately from this representation that to puncture such a surface at a single point would render it homotopy equivalent to a "wedge" of $2g$ circles, i.e. the disjoint union of $2g$ circles where $2g$ points, one from each circle, are identified.*

Example 1.6. *The n-sphere may be expressed as a cell complex with a single 0-cell and a single n-cell. So $S^n = e^0 \sqcup e^n$.*

Example 1.7. *The real projective space $\mathbb{R}P^n$ is the quotient of S^n by the antipodal involution. Hence it can be expressed as the upper hemisphere with boundary points antipodally identified. Hence it is a n-cell attached to $\mathbb{R}P^{n-1}$ via the antipodal identification map. since $S^0 = \{-1, 1\}$, we see $\mathbb{R}P^0 = e^0$ is a single 0-cell, and $\mathbb{R}P^n = e^0 \sqcup e^1 \sqcup \cdots \sqcup e^n$.*

Note that in the case of $\mathbb{R}P^2$, the attaching map for the 2-cell sends opposite points of S^1 to the same point in $\mathbb{R}P^1 = S^1$. Hence the attaching map $S^1 \longrightarrow S^1$ is simply $\theta \mapsto 2\theta$. Compare this with the attaching map $\theta \mapsto \theta$, which produces the 2-disc instead of $\mathbb{R}P^2$.

Example 1.8. *The complex projective space, $\mathbb{C}P^n$, can be expressed as $\mathbb{C}^n$ adjoin the $n-1$-plane at infinity, where the attaching map $S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$ is precisely the defining projection of $\mathbb{C}P^{n-1}$, i.e. the generalized Hopf map. As a result, as a cell complex we have*

$$\mathbb{C}P^n = e^0 \sqcup e^2 \sqcup \cdots \sqcup e^{2n}.$$

1.2 The fundamental group(oid)

We are all familiar with the idea of connectedness of a space, and the stronger notion of path-connectedness: that any two points x, y may be joined by a continuous path. In this section we will try to understand the fact that there may be *different homotopy classes of paths connecting* x, y, or in other words[1], that the space of paths joining x, y may be disconnected.

To understand the behaviour of paths joining points in a topological space X, we first observe that these paths actually form a category: Define a category $\mathcal{P}(X)$, whose objects are the points in X, and for which the morphisms from $p \in X$ to $q \in X$ are the finite length paths joining them, i.e. define

$$\mathrm{Hom}(p, q) := \{\gamma : [0, \infty) \longrightarrow X \quad : \quad \exists R > 0 \text{ with } \gamma(0) = p, \quad \gamma(t) = q \quad \forall t \geq R\}.$$

We may then define the length of the path to be $T_\gamma = \mathrm{Inf}\{T \ : \ \gamma(t) = q \ \forall \ t \geq T\}$.

The composition

$$\mathrm{Hom}(p, q) \times \mathrm{Hom}(q, r) \longrightarrow \mathrm{Hom}(p, r)$$

is known as "concatenation of paths", which simply means that

$$(\gamma_2\gamma_1)(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_{\gamma_1} \\ \gamma_2(t - T_{\gamma_1}) & t \geq T_{\gamma_1} \end{cases}$$

The path category $\mathcal{P}(X)$ has a space of objects, X, and a space of arrows (morphisms), which is a subspace of $C^0([0, \infty), X)$. As a result, it is equipped with a natural topology: Take the given topology on X, and take the topology on arrows induced by the "compact-open" topology on $C^0([0, \infty), X)$. You can verify that the category structure is compatible with this topology.

[Recall: open sets in the compact-open topology are arbitrary unions of finite intersections of sets of the form $C^0((X, K), (Y, U))$, for $K \subset X$ compact and $U \subset Y$ open.]

Just as we simplified the category **Top** to form **HTop**, we can simplify our path category $\mathcal{P}(X)$ by keeping the objects, but considering two paths $\gamma, \gamma' \in \mathrm{Hom}(p, q)$ to be equivalent when they are homotopic rel boundary in the following sense[2]

Definition 6. Paths γ_0, γ_1 are *homotopic paths* (and we write $\gamma_0 \simeq \gamma_1$) when there is a homotopy

$$H : I \times [0, \infty) \longrightarrow X$$

such that $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t)$ for all t, and there exists an $R > 0$ for which $H(s, 0) = \gamma_i(0)$ and $H(s, t) = \gamma_i(1)$ $\forall t > R$, for all s.

[1]Consider this a heuristic statement - it is a delicate matter to compare $C^0(I \times I, X)$ and $C^0(I, C^0(I, X))$.

[2]We showed that homotopy is an equivalence relation; for the same reason, homotopy rel $A \subset X$ is, too.

Exercise 1. If γ_0, γ_1 are paths $[0,1] \longrightarrow X$, then they are homotopic paths if and only if there is a homotopy rel endpoints

$$H : I \times I \longrightarrow X$$

with $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$ for all t. (rel endpoints means that $H(s,0) = \gamma_i(0)$ and $H(s,1) = \gamma_i(1)$ for all s.)

Finally, note that if $\gamma_{pq} \in \mathrm{Hom}(p,q)$ and $\gamma_{qr} \in \mathrm{Hom}(q,r)$, and if we homotopically deform these paths $\gamma_{pq} \overset{h}{\Rightarrow} \gamma'_{pq}$ and$\gamma_{qr} \overset{k}{\Rightarrow} \gamma'_{qr}$, then the concatenation (and rescaling) of h and k gives a homotopy from $\gamma_{qr}\gamma_{pq}$ to $\gamma'_{qr}\gamma'_{pq}$. This shows that the category structure descends to homotopy classes of paths.

Modding out paths by homotopies, we obtain a new category, which we could call $H\mathcal{P}(X)$, but it is actually called $\Pi_1(X)$, the fundamental groupoid of X. Note that since $\mathcal{P}(X)$ has a topology from compact-open, then so does its quotient $\Pi_1(X)$. The reason it is called a groupoid is that it is a special kind of category: every morphism is *invertible*: given any homotopy class of path $[\gamma] : p \longrightarrow q$, we can form $[\gamma]^{-1} = [\gamma^{-1}]$, where $\gamma^{-1}(t) = \gamma(T_\gamma - t)$. Draw a diagram illustrating a homotopy from $\gamma^{-1}\gamma$ to the constant path p, proving that any path class is invertible in the fundamental groupoid.

Definition 7. The fundamental groupoid $\Pi_1(X)$ is the category whose objects are points in X, and whose morphisms are homotopy classes of paths[3] between points. It is equipped with the quotient of the compact-open topology.

From any groupoid, we can form a bunch of groups: pick any object $x_0 \in X$ in the category, and consider the space of all self-morphisms $\mathrm{Hom}(x_0, x_0)$ in the category. Since all morphisms in a groupoid are invertible, it follows that $\mathrm{Hom}(x_0, x_0)$ is a group – it is called the *isotropy group* of x_0. Since the fundamental groupoid has a natural topology for which the category structure is continuous, it follows that $\mathrm{Hom}(x_0, x_0)$ is a *topological group*.

Definition 8. The fundamental group of the pointed[4]space (X, x_0) is the topological group $\pi_1(X, x_0) := \mathrm{Hom}(x_0, x_0)$ of homotopy classes of paths beginning and ending at x_0.

In any groupoid, the isotropy groups of objects x, y are always isomorphic if $\mathrm{Hom}(x,y)$ contains at least one element γ, since the map $g \mapsto \gamma g \gamma^{-1}$ defines an isomorphism $\mathrm{Hom}(x,x) \longrightarrow \mathrm{Hom}(y,y)$ Therefore we obtain:

Proposition 1.9. *If $x, y \in X$ are connected by a path σ, then $\gamma \mapsto [\sigma]\gamma[\sigma]^{-1}$ defines an isomorphism $\pi_1(X,x) \longrightarrow \pi_1(X,y)$.*

Example 1.10. *Let X be a convex set in $\mathbb{R}^n$ (this means that the linear segment joining $p, q \in X$ is contained in X) and pick $p, q \in X$. Given any paths $\gamma_0, \gamma_1 \in \mathcal{P}(p,q)$, the linear interpolation $s\gamma_0 + (1-s)\gamma_1$ defines a homotopy of paths $\gamma_0 \Rightarrow \gamma_1$. Hence there is a single homotopy class of paths joining p, q, and so $\Pi_1(X)$ maps homeomorphically via the source and target maps (s,t) to $X \times X$, and the groupoid law is $(x,y) \circ (y,z) = (x,z)$. This is called the* pair groupoid *over X. The fundamental group $\pi_1(X, x_0)$ is simply $s^{-1}(x_0) \cap t^{-1}(x_0) = \{(x_0, x_0)\}$, the trivial group.*

The final remark to make concerning the category $\mathcal{P}(X)$ of paths on a space X, and its homotopy descendant $\Pi_1(X)$, the fundamental groupoid, is that they depend functorially on the space X.

Proposition 1.11. *$\mathcal{P} : X \mapsto \mathcal{P}(X)$ is a functor from* **Top** *to the category of categories, taking morphisms (continuous maps) $f : X \longrightarrow Y$ to morphisms (functors) $f \circ - : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$. Furthermore, a homotopy $H : f \Rightarrow g$ defines a natural transformation $\mathcal{P}(f) \Rightarrow \mathcal{P}(g)$, and hence a homotopy equivalence $X \simeq Y$ gives rise to an equivalence of categories $\mathcal{P}(X) \simeq \mathcal{P}(Y)$.*

[3]Note also that any path $\tilde{\gamma}$ may be reparamatrized via $\gamma(t) := \tilde{\gamma}(T_\gamma t)$ to a path of unit length, and that $\gamma' \simeq \gamma$, so that up to homotopy, only unit length paths need be considered (this is the usual convention when defining the fundamental group(oid)).

[4]A pointed space is just a pair $(X, A \subset X)$ where A happens to consist of a single point. Recall that pairs form a category, with $\mathrm{Hom}((X,A),(Y,B)) = \{f \in C^0(X,Y) \ : \ f(A) \subset B\}$.

These properties descend to the fundamental groupoid, as well as to the fundamental group, implying that for any continuous map of pointed spaces $f : (X, x_0) \longrightarrow (Y, y_0)$, we obtain a homomorphism of groups $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$, given simply by composition $[\gamma] \mapsto [f \circ \gamma]$. This last fact is usually proven directly, since it is so simple.

1.3 $\pi_1(S^1) = \mathbb{Z}$

In this section we will compute the fundamental group of S^1. The method we use will help us develop the theory of covering spaces. We essentially follow Hatcher, Chapter 1.

Theorem 1.12. *The map $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$ given by $n \mapsto [\omega_n]$, for $\omega_n(s) = e^{2\pi i n s}$, is an isomorphism.*

Proof. Consider the map $p : \mathbb{R} \longrightarrow S^1$ defined by $p(s) = e^{2\pi i s}$. It can be viewed as a projection of a single helix down to a circle. The loop ω_n may be factored as a linear path $\tilde{\omega}_n(s) = ns$ in $\mathbb{R}$, composed with p:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{\omega}_n"] \arrow[r, "\omega_n"'] & S^1 \end{tikzcd}$$

We say that $\tilde{\omega}_n$ is a "lift" of ω_n to the "covering space" $\mathbb{R}$. Note that $\Phi(n)$ could be defined as $[p \circ \tilde{f}]$ for any path $\tilde{f}$ in $\mathbb{R}$ joining 0 to n. This is because $\tilde{f} \simeq \tilde{\omega}_n$ via the homotopy $(1-t)\tilde{f} + t\tilde{\omega}_n$.

To check that Φ is a homomorphism, note that $\Phi(m+n)$ is represented by the loop $p \circ (\tilde{\omega}_m \cdot (\tau_m \circ \tilde{\omega}_n))$, where $\tau_m : \mathbb{R} \longrightarrow \mathbb{R}$ is the translation $\tau_m(x) = x + m$. But since[5] $p \circ \tau_m = p$, we see that the loop is equal to the concatenation $\omega_m \cdot \omega_n$. Thus $\Phi(m+n) = \Phi(m)\Phi(n)$.

To prove that Φ is surjective, we do it by taking any loop $f : I \longrightarrow S^1$ and lifting it to $\tilde{f}$ starting at 0, which then must go to an integer n. Then $\Phi(n) = [f]$ as required. For this to work, we need to prove:

a) For each path $f : I \longrightarrow S^1$ with $f(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \longrightarrow \mathbb{R}$ with $f(0) = \tilde{x}_0$.

To prove that Φ is injective, suppose that $\Phi(m) = \Phi(n)$. This means that there is a homotopy $f_t : \omega_m = f_0 \Rightarrow \omega_n = f_1$. Let us lift this homotopy to a homotopy $\tilde{f}_t$ of paths starting at 0. By uniqueness it must be that $\tilde{f}_0 = \tilde{\omega}_0$ and similarly $\tilde{f}_1 = \tilde{\omega}_1$. Since $\tilde{f}_t$ is a homotopy of paths, its endpoint is the same for all t, hence $m = n$. For this to work, we need to be able to lift the homotopy via the statement:

b) For each homotopy $f_t : I \longrightarrow S^1$ of paths starting at $x_0 \in S^1$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy $\tilde{f}_t : I \longrightarrow \mathbb{R}$ of paths starting at $\tilde{x}_0$.

Both statements $a), b)$ are lifting results and can be absorbed in the statement of the following lemma. $\square$

Lemma 1.13 (Lifting lemma)**.** *Given a map $F : Y \times I \longrightarrow S^1$ and a "initial lift" $\tilde{F}_0 : Y \times \{0\} \longrightarrow \mathbb{R}$ lifting $F|_{Y \times \{0\}}$, there is a unique "complete lift" $\tilde{F} : Y \times I \longrightarrow \mathbb{R}$ lifting F and agreeing with $\tilde{F}_0$.*

Proof. The main ingredient of the proof is to use the fact that $p : \mathbb{R} \longrightarrow S^1$ is a covering space, meaning that there is an open cover $\{U_\alpha\}$ of S^1 such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each mapped homeomorphically onto U_α by p. For example, we could take the usual cover U_0, U_1 by two open arcs.

To construct the lift $\tilde{F}$, we first lift the homotopy for small neighbourhoods $N \subset Y$, producing $\tilde{F} : N \times I \longrightarrow \mathbb{R}$. We then observe that these lifts on neighbourhoods glue together to give a complete lift.

Fix $y_0 \in Y$. By compactness of $y_0 \times I$, there is a neighbourhood N of y_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of the interval such that $F(N \times [t_i, t_{i+1}])$ is contained in some U_α for each i (call this open

[5]We see here that τ_m is a "deck transformation", an automorphism of the covering space fixing the base.

set U_i). The lift on $N \times [0, t_0]$ is given as $\tilde{F}|_{N\times\{0\}}$. Assume inductively that $\tilde{F}$ has been constructed on $N \times [0, t_i]$. For the next segment, $F(N \times [t_i, t_{i+1}]) \subset U_i$ and $\tilde{F}(y_0, t_i)$ lies inside $\tilde{U}_i$. Replacing N by a smaller neighbourhood of y_0, we may assume that $\tilde{F}(N \times \{t_i\}) \subset \tilde{U}_i$. Now we simply define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be $p|_{\tilde{U}_i}^{-1} \circ F$. In this way we get a lift $\tilde{F} : N \times I \longrightarrow \mathbb{R}$ for some neighbourhood N of y_0.

The fact that these local lifts glue to a global lift stems from the uniqueness of the lift at each point y_0 (hence two local lifts for neighbourhoods N, N' must agree on their intersection. Furthermore, the uniqueness of the complete lift is also implied by the uniqueness of the lift at each point y_0, which we now show.

Let Y be a point. Suppose $\tilde{F}, \tilde{F}'$ are two lifts of $F : I \longrightarrow S^1$ with $\tilde{F}(0) = \tilde{F}'(0)$. Choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ compatible with $\{U_i\}$ as before. Assume that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, $\tilde{F}([t_i, t_{i+1}])$ is also, and must lie in a single one of the lifts $\tilde{U}_i$ of U_i, in fact the same one which $\tilde{F}'([t_i, t_{i+1}])$ is in, since these share the same value at t_i. Since p is an isomorphism on this open set, we obtain $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$, completing the proof. ☐

Corollary 1.14. *Any nonconstant complex polynomial $f(z)$ must have a zero.*

Proof. If f has no zeros, then f must take $\mathbb{C}\backslash\{0\}$ into $\mathbb{C}\backslash\{0\}$, both homotopic to S^1. For sufficiently small R, the loop $\gamma_R(t) = f(Re^{2\pi it})$ is homotopic to a constant loop ω_0. Letting R grow sufficiently large, $f(z)$ behaves as z^n for n the degree of f, and so $\gamma_R(t)$ is homotopic to ω_n. By the theorem, $n = 0$, a contradiction. ☐

Using the same arguments you can show that f must have $n = \deg f$ zeros, counted with multiplicity.

Corollary 1.15 (Brouwer fixed point theorem)**.** *Every continuous map $h : D^2 \longrightarrow D^2$ has a fixed point.*

Proof. If h has no fixed point, then we obtain a map $r : D^2 \longrightarrow S^1$ by intersecting the ray from $h(x)$ to x with the boundary circle. This is a retraction onto the circle. But a retract $r : X \longrightarrow A$ to a subspace $A \overset{i}{\hookrightarrow} X$ satisfiees $r \circ i = \mathrm{Id}$, implying $r_* \circ i_* = \mathrm{Id}$, implying that i_* must be an injection. Contradiction. ☐

Corollary 1.16 (Borsuk-Ulam)**.** *Every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$ takes the same value on at least one pair of antipodal points.*

Proof. If not, then $\tilde{g}(x) = f(x) - f(-x)$ is an odd function $S^2 \longrightarrow \mathbb{R}^2$ with no zeros, so that $g(x) = \tilde{g}(x)/|\tilde{g}(x)|$ is well defined and still odd. Composing with the equatorial path $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$, we obtain an odd function $h : S^1 \longrightarrow S^1$. We prove that h is nontrivial in $\pi_1(S^1)$: lift h to $\tilde{h} : S^1 \longrightarrow \mathbb{R}$; since $h(s + 1/2) = -h(s)$ for $s \in [0, 1/2]$, it follows that $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$ for some odd integer q (q must be constant since it depends continuously on s but is an integer). In particular $\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q$. In other words, h is homotopic to an odd multiple of the generator of $\pi_1(S^1)$ and hence must be nontrivial. On the other hand, since η is nullhomotopic in S^2, $h = g \circ \eta$ must also be nullhomotopic, a contradiction. ☐

Borsuk-Ulam can be used to prove the famous "Ham Sandwich theorem", stating that bread, ham, and cheese, can always be cut with one slice in such a way so that all three quantities are halved. This is proved by starting with the bread: for each direction $v \in S^2$, let $P(v)$ be the plane normal to v which cuts the bread in half (the middle such plane if there is an interval of these). Then define a map $S^2 \longrightarrow \mathbb{R}^2$ via $f(v) = (c(v), h(v))$, where $c(v)$ is the volume of cheese on the side of $P(v)$ in the direction of v, and similarly for the ham $h(v)$. Borsuk-Ulam then implies that there is a plane which ensures a well-balanced meal.

Before we discuss the computation of $\pi_1(X)$ for other, more complicated examples, let's try to understand the *fundamental groupoid* of S^1.

As we saw before, any paths $\gamma, \gamma' \in \mathcal{P}(\mathbb{R})$ joining $p, q \in \mathbb{R}$ must be homotopic, i.e. there is a single homotopy class of paths joining points in $\mathbb{R}$, and so the fundamental groupoid of $\mathbb{R}$ is simply $\mathbb{R} \times \mathbb{R}$, with groupoid law $(x, y) \circ (y, z) = (x, z)$.

Now let $a, b \in S^1$ and let γ be a path from a to b. Choose $\tilde{a} \in p^{-1}(a)$, so that γ may be lifted to $\tilde{\gamma}$, starting at $\tilde{a}$ and ending at $\tilde{b} := \tilde{\gamma}(1)$. Of course $\tilde{\gamma}$ is homotopic to a unique linear path, and similarly for γ; and two such linear paths $p\circ\tilde{\gamma}, p\circ\tilde{\gamma}'$ coincide iff $\gamma' = \gamma + n,\ \ n \in \mathbb{Z}$. As a result, we see that $\Pi_1(S^1) = \mathbb{R}\times\mathbb{R}/\sim$, where $(x, y) \sim (x + n, y + n),\ \ n \in \mathbb{Z}$. Therefore we obtain that $\Pi_1(S^1)$ has a cylinder as its space of arrows, which then maps to S^1 via the source and target maps (s, t). Note also that for $p \in S^1$, $s^{-1}(p)$ is homeomorphic to $\mathbb{R}$, and t maps this to S^1 as a covering map, precisely the same one as $p : \mathbb{R} \longrightarrow S^1$ from earlier.

1.4 Further computations of π_1

The main technique for computing $\pi_1(X)$ is the Van Kampen theorem, which is an analog of the Mayer-Vietoris sequence which we learned about for de Rham cohomology. Before we get to it, we will cover some more elementary facts about computing π_1.

Proposition 1.17. *Let X, Y be path-connected. Then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Proof. Recall that a map $f : Z \longrightarrow X \times Y$ is continuous iff the projections $g : Z \longrightarrow X$, $h : Z \longrightarrow Y$ are separately continuous. Therefore if f is a loop based at (x_0, y_0), it is nothing more than a pair of loops in X and Y based at x_0 and y_0. Similarly homotopies of loops are nothing but pairs of homotopies of pairs of loops, and so $[f] \mapsto ([g], [h])$ defines the obvious isomorphism. $\square$

A natural example to consider, given that $\pi_1(S^1) \cong \mathbb{Z}$, is the torus $T = S^1 \times S^1$. Then $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

Proposition 1.18. $\pi_1(S^n) = \{0\}$ *for $n > 2$.*

Proof. Any continuous map of smooth manifolds is homotopic to a smooth map: given $f : S^1 \longrightarrow S^n$, we may find a smooth approximation $\tilde{f} : S^1 \longrightarrow \mathbb{R}^{n+1}$ which lies in a small tubular neighbourhood U of S^n. Then form $H(p, t) = r((1 - t)f(p) + t\tilde{f}(p))$, for $r : U \longrightarrow S^n$ the retraction.

By Sard's theorem, $\tilde{f}$ is not surjective for $n \geq 2$, failing to take $q \in S^n$ as a value. $S^n \backslash \{q\}$ is contractible, hence $\tilde{f}$ is homotopic to the trivial path. $\square$

Corollary 1.19. $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *for $n \neq 2$.*

1.5 The Van Kampen theorem

There are many versions of the Van Kampen theorem; all of them help us to do the following: determine the fundamental group of a space X which has been expressed as a union $\bigcup_\alpha U_\alpha$ of open sets, given the fundamental groups of each U_α and $U_\alpha \cap U_\beta$, as well as the induced maps on fundamental groups given by the inclusion (or *fibered coproduct*) diagram

$$\begin{tikzcd} U_\alpha \cap U_\beta \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & U_\alpha \arrow[d, "i_\alpha"] \\ U_\beta \arrow[r, "i_\beta"'] & U_\alpha \cup U_\beta \end{tikzcd} \tag{1}$$

Before we begin to state the theorem, we briefly review the idea of the free product of groups. Given groups G_1, G_2, we may form the *free product* $G_1 * G_2$, defined as follows: $G_1 * G_2$ is the group of equivalence classes of finite words made from letters chosen from $G_1 \sqcup G_2$, where the equivalence relation is finitely generated by $a * b \sim ab$ for a, b both in G_1 or G_2, and the identity elements $e_i \in G_i$ are equivalent to the empty word. The group operation is juxtaposition. For example, $\mathbb{Z} * \mathbb{Z}$ is the free group on two generators:

$$\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle = \{a^{i_1} b^{j_1} a^{i_2} b^{j_2} \cdots a^{i_k} b^{j_k} \;:\; i_p, j_p \in \mathbb{Z}, \quad k \geq 0\}$$

Note that from a categorical point of view[6], $G_1 * G_2$ is the *coproduct* or *sum* of G_1 and G_2 in the following sense: not only does it fit into the following diagram of groups:

$$\begin{tikzcd} & G_2 \arrow[d, "\iota_2"] \\ G_1 \arrow[r, "\iota_1"'] & G_1 * G_2 \end{tikzcd}$$

but $(\iota_1, \iota_2, G_1 * G_2)$ is the "most general" such object, i.e. any other triple (j_1, j_2, G) replacing it in the diagram must factor through it, via a unique map $G_1 * G_2 \longrightarrow G$.

The simplest version of Van Kampen is for a union $X = U_1 \cup U_2$ of two path-connected open sets such that $U_1 \cap U_2$ is path-connected and simply connected. Note that the injections ι_1, ι_2 give us induced homomorphisms $\pi_1(U_i) \longrightarrow \pi_1(X)$. By the coproduct property, this map must factor through a group homomorphism

$$\Phi : \pi_1(U_1) * \pi_1(U_2) \longrightarrow \pi_1(X).$$

Theorem 1.20 (Van Kampen, version 1)**.** *If $X = U_1 \cup U_2$ with U_i open and path-connected, and $U_1 \cap U_2$ path-connected and simply connected, then the induced homomorphism $\Phi : \pi_1(U_1) * \pi_1(U_2) \longrightarrow \pi_1(X)$ is an isomorphism.*

Proof. Choose a basepoint $x_0 \in U_1 \cap U_2$. Use $[\gamma]_U$ to denote the class of γ in $\pi_1(U, x_0)$. Use $*$ as the free group multiplication.

$\boxed{\Phi \text{ is surjective:}}$ Let $[\gamma] \in \pi_1(X, x_0)$. Then we can find a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_i, t_{i+1}])$ is contained completely in U_1 or U_2 (it might be in $U_1 \cap U_2$). Then γ factors as a product of its restrictions γ_{i+1} to $[t_i, t_{i+1}]$, i.e.

$$[\gamma]_X = [\gamma_1 \gamma_2 \cdots \gamma_n]_X$$

But the γ_i are not loops, just paths. To make them into loops we must join the subdivision points $\gamma(t_i)$ to the basepoint, and we do this as follows: if $\gamma(t_i) \in U_1 \cap U_2$ then we choose a path η_i from x_0 to $\gamma(t_i)$ lying in $U_1 \cap U_2$; otherwise we choose such a path lying in whichever of U_1, U_2 contains $\gamma(t_i)$. This is why we need $U_i, U_1 \cap U_2$ to be path-connected.

[6] Coproducts in categories are the subject of a question in Assignment 6

Then define $\tilde{\gamma}_i = \eta_{i-1}\gamma_i\eta_i^{-1}$ and we obtain a factorization of loops

$$[\gamma]_X = [\tilde{\gamma}_1]_X \cdots [\tilde{\gamma}_n]_X.$$

We chose the η_i in just such a way that each loop in the right hand side lies either in U_1 or in U_2; hence we can choose $e_i \in \{1,2\}$ so that $[\tilde{\gamma}_1]_{U_{e_1}} * \cdots * [\tilde{\gamma}_n]_{U_{e_n}}$ makes sense as a word in $\pi_1(U_1) * \pi_1(U_2)$, and hence we have $[\gamma]_X = \Phi([\tilde{\gamma}_1]_{U_{e_1}} * \cdots * [\tilde{\gamma}_n]_{U_{e_n}})$, showing surjectivity.

$\boxed{\Phi \text{ is injective:}}$ take an arbitrary element of the free product $\gamma = [a_1]_{U_{e_1}} * \cdots * [a_k]_{U_{e_k}}$ (for $e_i \in \{1,2\}$) , and suppose that $\Phi([a_1]_{U_{e_1}} * \cdots * [a_k]_{U_{e_k}}) = 1$. This means that $a_1 \cdots a_k$ is homotopically trivial in X. We wish to show that $\gamma = 1$ in the free product group.

Take the homotopy $H : I \times I \longrightarrow X$ taking $a_1 \cdots a_k$ to the constant path at x_0, and subdivide $I \times I$ into small squares $S_{ij} = [s_i, s_{i+1}] \times [t_i, t_{i+1}]$ so that each square is sent either into U_1 or U_2, and subdivide smaller if necessary to ensure that the endpoints of the domains of the loops a_i are part of the subdivision.

Set up the notation as follows: let v_{ij} be the grid point (s_i, t_i) and a_{ij} the path defined by H on the horizontal edge $v_{ij} \to v_{i+1,j}$, and b_{ij} the vertical path given by H on $v_{ij} \to v_{i,j+1}$. Then we can write $a_i = a_{p_{i-1}+1,0} \cdots a_{p_i,0}$ for some $\{p_i\}$, and we can factor each loop as a product of tiny paths:

$$\gamma = [a_1]_{U_{e_1}} * \cdots * [a_k]_{U_{e_k}} = [a_{0,0} \cdots a_{p_1,0}]_{U_{e_1}} * \cdots * [\cdots a_{p_k,0}]_{U_{e_k}}$$

Again, these paths a_{ij} (as well as the b_{ij}) are not loops, so, just as in the proof of surjectivity, choose paths h_{ij} from the basepoint to all the gridpoint images $H(v_{ij})$, staying within $U_1 \cap U_2$, U_1, or U_2 accordingly as $H(v_{ij})$. pre- and post-composing with the h_{ij}, we then obtain loops $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ lying in either U_1 or U_2.

In particular we can factor γ as a bunch of tiny loops, each remaining in U_1 or U_2:

$$\gamma = [\tilde{a}_{0,0}]_{U_{e_1}} * \cdots * [\tilde{a}_{p_1,0}]_{U_{e_1}} * \cdots * [\tilde{a}_{p_k,0}]_{U_{e_k}}$$

For each loop $\tilde{a}_{i,0}$, we may use H restricted to the square immediately above $a_{i,0}$ to define a homotopy $H_{i,0} : \tilde{a}_{i,0} \Rightarrow \tilde{b}_{i,0}\tilde{a}_{i,1}\tilde{b}_{i+1,0}^{-1}$: If $\tilde{a}_{i,0}$ is in U_1, say, and the homotopy $H_{i,0}$ occurs in U_1, then we may replace $[\tilde{a}_{i,0}]_{U_1}$ with $[\tilde{b}_{i,0}]_{U_1} * [\tilde{a}_{i,1}]_{U_1} * [\tilde{b}_{i+1,0}]_{U_1}^{-1}$ in the free product. If on the other hand $H_{i,0}$ occurs in U_2, then we observe that $\tilde{a}_{i,0}$ must lie in $U_1 \cap U_2$, and **since this is simply connected**, $[\tilde{a}_{1,0}]_{U_1} =$ empty word $= [\tilde{a}_{1,0}]_{U_2}$ in the free product, so it can be replaced with $[\tilde{b}_{i,0}]_{U_2} * [\tilde{a}_{i,1}]_{U_2} * [\tilde{b}_{i+1,0}]_{U_2}^{-1}$ in the free product. Doing this replacement for each square in the bottom row, the $[\tilde{b}_{i,0}]_{U_{e_i}}$ cancel, and we may repeat the replacement for the next row.

In this way we eventually reach the top row, which corresponds to a free product of constant paths at x_0, showing $\gamma = 1$ in the free product, as required. □

Let's give some examples of fundamental groups computed with the simple version of Van Kampen:

Example 1.21. *The wedge sum of pointed spaces (X,x), (Y,y) is $X \vee Y := X \sqcup Y / x \sim y$, and is the coproduct in the category of pointed spaces. If X, Y are topological manifolds, then let V_x, V_y be disc neighbourhoods of x, y so that $X \vee Y = U_1 \cup U_2$ with $U_1 = [X \sqcup V_y]$ and $U_2 = [V_x \sqcup Y]$. We conclude that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. For example, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$.*

At least for pointed manifolds, therefore, we can say that the π_1 functor preserves coproducts. Does this hold for all pointed spaces? No, but it does work when the point is a deformation retract of an open neighbourhood.

Example 1.22. *Let X, Y be connected manifolds of dimension n. Then their connected sum $X \sharp Y$ is naturally decomposed into two open sets $U \cup V$ with $U \cap V \cong I \times S^{n-1} \simeq S^{n-1}$. If $n > 2$ then $\pi_1(S^{n-1}) = 0$, and hence $\pi_1(X \sharp Y) = \pi_1(X) * \pi_1(Y)$.*

Example 1.23. *Using the classical $4g$-gon representation of the genus g orientable surface Σ_g, we showed that when punctured it is homotopic to $\vee_{2g} S^1$. Hence $\pi_1(\Sigma_g \backslash \{p\}) = F_{2g}$. What happens with non-orientable surfaces? What about puncturing manifolds of dimension > 2?*

The second version of Van Kampen will deal with cases where $U_1 \cap U_2$ is not simply-connected. By the inclusion diagram (1), we see that we have a canonical map from the fibered sum $\pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2)$ to $\pi_1(X)$: Van Kampen again states that this is an isomorphism. Recall that if $\iota_k : H \longrightarrow G_k$, $k = 1, 2$ are injections of groups, then the fibered product or "free product with amalgamation" may be constructed as a quotient of the free product, by additional relations generated by $(g_1\iota_1(h)) * g_2 \sim g_1 * (\iota_2(h)g_2)$ for $g_i \in G_i$ and $h \in H$. In other words,

$$G_1 *_H G_2 = (G_1 * G_2)/K,$$

where K is the normal subgroup generated by elements $\{\iota_1(h)^{-1}\iota_2(h) \ : \ h \in H\}$.

Theorem 1.24 (Van Kampen, version 2[7])**.** *If $X = U_1 \cup U_2$ with U_i open and path-connected, and $U_1 \cap U_2$ path-connected, then the induced homomorphism $\Phi : \pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2) \longrightarrow \pi_1(X)$ is an isomorphism.*

Proof. Exercise! Slight modification of the given proof, need to understand the analogous condition to the one we used to show $[\tilde{a}]_{U_1} =$ empty word $= [\tilde{a}]_{U_2}$ in the free product. □

Example 1.25. *Express the 2-sphere as a union of two discs with intersection homotopic to S^1. By Van Kampen version 2, we have $\pi_1(S^2) = (0 * 0)/K = 0$.*

Example 1.26. *Take a genus g orientable surface Σ_g. Choose a point $p \in \Sigma_g$ and let $U_0 = \Sigma_g\backslash\{p\}$. Let U_p be a disc neighbourhood of p. Then we have $\Sigma_g = U_0 \cup U_p$, with intersection $U_0 \cap U_p \simeq S^1$. The inclusion map $S^1 \longrightarrow U_p$ is trivial in homotopy while $S^1 \longrightarrow U_0$ sends $1 \in \mathbb{Z}$ to $a_1b_1a_1^{-1}b^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$. Hence the amalgamation introduces a single relation:*

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots a_g, b_g \ \ | \ \ [a_1, b_1] \cdots [a_g, b_g]\rangle$$

Example 1.27. *Do the same as above, but with $\mathbb{R}P^2 = U_0 \cup U_p$, with $\pi_1(U_0) = \mathbb{Z} = \langle a\rangle$ and the inclusion of $U_0 \cap U_p \simeq S^1$ in U_0 sends $1 \mapsto a^2$, hence we obtain*

$$\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}.$$

Example 1.28 (Perverse computation of $\pi_1(S^3)$)**.** *Express S^3 as the union of two solid tori, glued along their boundary. Visualize it by simply looking at the interior and exterior of an embedded torus in $\mathbb{R}^3 \sqcup \infty$. Fatten the tori to open sets U_0, U_1 with $U_0 \cap U_1 \simeq T^2$, so that*

$$\pi_1(S^3) = \mathbb{Z} *_{\mathbb{Z}\times\mathbb{Z}} \mathbb{Z}.$$

The notation is not enough to determine the group: we need the maps $(\iota_i)_* : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ induced by the inclusions: by looking at generating loops, we get $\iota_0(1, 0) = 1, \iota_0(0, 1) = 0$ while $\iota_1(1, 0) = 0, \iota_1(0, 1) = 1$. Hence the amalgamation kills both generators, yielding the trivial group.

The proof of Van Kampen in Hatcher is slightly more general than this, as it allows arbitrarily many open sets U_α, with only the extra hypothesis that triple intersections be path-connected (in our proof, each vertex v_{ij} is joined to the basepoint by a path: since the vertex is surrounded by 4 squares, we would need quadruple intersections to be path-connected. This can be improved by using a hexagonal decomposition, or a brick configuration, where the vertices are surrounded by only 3 2-cells). The ultimate Van Kampen theorem does not refer to basepoints or put connectivity conditions on the intersection: it states that the fundamental groupoid of $U_1 \cup U_2$ is the fibered sum of $\Pi_1(U_1)$ and $\Pi_1(U_2)$ over $\Pi_1(U_1 \cap U_2)$. Viewing the topology of X as a category (where objects are open sets and arrows are inclusions), the Van Kampen theorem can be rephrased as follows:

Theorem 1.29 (Van Kampen, version 3)**.** *Π_1 is a functor from the topology of X to groupoids, which preserves fibered sum[8].*

[7] See the proof in Hatcher

[8] See "Topology and Groupoids" by Ronald Brown.

1.6 Covering spaces

Consider the fundamental group $\pi_1(X, x_0)$ of a pointed space. It is natural to expect that the group theory of $\pi_1(X, x_0)$ might be understood geometrically. For example, subgroups may correspond to images of induced maps $\iota_* \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$ from continuous maps of pointed spaces $(Y, y_0) \longrightarrow (X, x_0)$. For this induced map to be an injection we would need to be able to lift homotopies in X to homotopies in Y. Rather than consider a huge category of possible spaces mapping to X, we restrict ourselves to a category of *covering spaces*, and we show that under some mild conditions on X, this category completely encodes the group theory of the fundamental group.

Definition 9. A covering map of topological spaces $p : \tilde{X} \longrightarrow X$ is a continuous map such that there exists an open cover $X = \bigcup_\alpha U_\alpha$ such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets (called *sheets*), each homeomorphic via p with U_α. We then refer to $(\tilde{X}, p)$ (or simply $\tilde{X}$, abusing notation) as a covering space of X.

Let $(\tilde{X}_i, p_i)$, $\quad i = 1, 2$ be covering spaces of X. A morphism of covering spaces is a covering map $\phi : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that the diagram commutes:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

We will be considering covering maps of pointed spaces $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$, and pointed morphisms between them, which are defined in the obvious fashion.

Example 1.30. *The covering space $p : \mathbb{R} \longrightarrow S^1$ has the additional property that $\tilde{X} = \mathbb{R}$ is simply connected. There are other covering spaces $p_n : S^1 \longrightarrow S^1$ given by $z \mapsto z^n$ for $n \in \mathbb{Z}$, and in fact these are the only connected ones up to isomorphism of covering spaces (there are disconnected ones, but they are unions of connected covering spaces).*

Notice that $(p_n)_* : \pi_1(S^1) \longrightarrow \pi_1(S^1)$ maps $[\omega_1] \mapsto [\omega_n] = n[\omega_1]$, hence $(p_n)_*(\pi_1(S^1)) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}$. As a result, we see that there is an isomorphism class of covering space associated to every subgroup of $\mathbb{Z}$: we associate $p : \mathbb{R} \longrightarrow S^1$ to the trivial subgroup.

Note also that we have the commutative diagram

$$\begin{tikzcd} S^1 \arrow[rr, "z^m"] \arrow[dr, "z^{mn}"'] & & \tilde{S}^1 \arrow[dl, "z^n"] \\ & S^1 & \end{tikzcd}$$

showing that we have a morphism of covering spaces corresponding to the inclusion of groups $mn\mathbb{Z} \subset n\mathbb{Z} \subset \mathbb{Z}$.

There is a natural functor from pointed covering spaces of (X, x_0) to subgroups of $\pi_1(X, x_0)$, as a consequence of the following result:

Lemma 1.31 (Homotopy lifting)**.** *Let $p : \tilde{X} \longrightarrow X$ be a covering and suppose that $\tilde{f}_0 : Y \longrightarrow \tilde{X}$ is a lifting of the map $f_0 : Y \longrightarrow X$. Then any homotopy f_t of f_0 lifts uniquely to a homotopy $\tilde{f}_t$ of $\tilde{f}_0$.*

Proof. The same proof used for the Lemma 1.13 works in this case. □

Corollary 1.32. *The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ induced by a covering space is injective, and its image $G(p, \tilde{x}_0)$ consists of loops at x_0 whose lifts to $\tilde{X}$ at $\tilde{x}_0$ are loops.*

If we choose a different basepoint $\tilde{x}_0' \in p^{-1}(x_0)$, and if $\tilde{X}$ is path-connected, we see that $G(p, \tilde{x}_0')$ is the conjugate subgroup $\gamma G(p, \tilde{x}_0)\gamma^{-1}$, for $\gamma = p_*[\tilde{\gamma}]$ for $\tilde{\gamma} : \tilde{x}_0 \to \tilde{x}_0'$.

Hence p_* defines a functor as follows:

$$\{ \text{ pointed coverings } (\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0) \} \longrightarrow \{ \text{ subgroups } G \subset \pi_1(X, x_0) \}$$

The group $G(p, \tilde{x}_0) = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is called the characteristic subgroup of the covering p.

We will prove that under some conditions on X, this is an equivalence:

Theorem 1.33 (injective)**.** *Let X be path-connected and locally path-connected. Then $G(p, \tilde{x}) = G(p', \tilde{x}')$ iff there exists a canonical isomorphism $(p, \tilde{x}) \cong (p', \tilde{x}')$.*

Theorem 1.34 (surjective)**.** *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for any subgroup $G \subset \pi_1(X, x)$, there exists a covering space $p : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ with $G = G(p, \tilde{x})$.*

The first tool is a criterion which decides whether maps to X may be lifted to $\tilde{X}$:

Lemma 1.35 (Lifting criterion)**.** *Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a covering and let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a a map with Y path-connected and locally path-connected. Then f lifts to $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, x_0))$.*

Proof. It is clear that the group inclusion must hold if f lifts, since $f_* = p_* \tilde{f}_*$. For the converse, we define $\tilde{f}$ as follows: let $y \in Y$ and let $\gamma : y_0 \to y$ be a path. Then take the path $f\gamma$ and lift it at $\tilde{x}_0$, giving $\widetilde{f\gamma}$. Define $\tilde{f}(y) = \widetilde{f\gamma}(1)$.

$\tilde{f}$ is well defined, independent of γ: if we choose $\gamma' : y_0 \to y$, then $(f\gamma')(f\gamma)^{-1}$ is a loop h_0 in the image of f_* and hence is homotopic (via h_t) to a loop h_1 which lifts to a loop $\tilde{h}_1$ at $\tilde{x}_0$. But the homotopy lifts, and hence $\tilde{h}_0$ is a loop as well. By uniqueness of lifted paths, $\tilde{h}_0$ consists of $\widetilde{f\gamma'}$ and $\widetilde{f\gamma}$ (both lifted at $\tilde{x}_0$), traversed as a loop. Since they form a loop, it must be that $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$.

$\tilde{f}$ is continuous: We show that each $y \in Y$ has a neighbourhood V small enough that $\tilde{f}|_V$ coincides with f. Take a neighbourhood U of $f(y)$ which lifts to $\tilde{f}(y) \in \tilde{U} \subset \tilde{X}$ via $p : \tilde{U} \longrightarrow U$. Then choose a path-connected neighbourhood V of y with $f(V) \subset U$. Fix a path γ from y_0 to y and then for any point $y' \in V$ choose path $\eta : y \to y'$. Then the paths $(f\gamma)(f\eta)$ have lifts $\widetilde{f\gamma}\widetilde{f\eta}$, and $\widetilde{f\eta} = p^{-1} f\eta$. Hence $\tilde{f}(V) \subset \tilde{U}$ and $\tilde{f}|_V = p^{-1} f$, hence continuous. ☐

Lemma 1.36 (uniqueness of lifts)**.** *If $\tilde{f}_1, \tilde{f}_2$ are lifts of a map $f : Y \longrightarrow X$ to a covering $p : \tilde{X} \longrightarrow X$, and if they agree at one point of Y, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. The set of points in Y where $\tilde{f}_1$ and $\tilde{f}_2$ agree is open and closed: take a neighbourhood U of $f(y)$ such that $p^{-1}(U)$ is a disjoint union of homeomorphic $\tilde{U}_\alpha$, and let $\tilde{U}_1, \tilde{U}_2$ contain $\tilde{f}_1(y), \tilde{f}_2(y)$. Then take $N = \tilde{f}_1^{-1}(\tilde{U}_1) \cap \tilde{f}_2^{-1}(\tilde{U}_2)$. If $\tilde{f}_1, \tilde{f}_2$ agree (disagree) at y, then they must agree (disagree) on all of N. ☐

Proof of injectivity. If there is an isomorphism $f : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$, then taking induced maps, we get $G(p_1, \tilde{x}_1) = G(p_2, \tilde{x}_2)$.

Conversely, suppose $G(p_1, \tilde{x}_1) = G(p_2, \tilde{x}_2)$. By the lifting criterion, we can lift $p_1 : \tilde{X}_1 \longrightarrow X$ to a map $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \tilde{p}_1 = p_1$. In the other direction we obtain $\tilde{p}_2$ with $p_1 \tilde{p}_2 = p_2$. The composition $\tilde{p}_1 \tilde{p}_2$ is then a lift of p_2 which agrees with the Identity lift at the basepoint, hence it must be the identity. similarly for $\tilde{p}_2 \tilde{p}_1$. ☐

Finally, to show that there is a covering space corresponding to each subgroup $G \subset \pi_1(X, x_0)$, we give a construction. The first step is to construct a simply-connected covering space, corresponding to the trivial subgroup. Note that for such a covering to exist, X must have the property of being semi-locally simply connected, i.e. each point x must have a neighbourhood U such that the inclusion $\iota_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial. In fact this property is equivalent to the requirement that $\pi_1(X, x)$ be *discrete* as a topological group. We prove the existence of a simply-connected covering space when X is path-connected, locally path-connected, and semi-locally simply connected.

Existence of simply-connected covering. Let X be as above, with basepoint x_0. Define

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

and let $\tilde{x}_0$ be the trivial path at x_0. Define also the map $p : \tilde{X} \longrightarrow X$ by $p([\gamma]) = \gamma(1)$. p is surjective, since X is path-connected.

We need to define a topology on $\tilde{X}$, show that p is a covering map, and that it is simply-connected.

Topology: Since X is locally path-connected and semilocally simply-connected, it follows that the collection $\mathcal{U}$ of path-connected open sets $U \subset X$ with $\pi_1(U) \longrightarrow \pi_1(X)$ trivial forms a basis for the topology of X. We now lift this collection to a basis for a topology on $\tilde{X}$: Given $U \in \mathcal{U}$ and $[\gamma] \in p^{-1}(U)$, define

$$U_{[\gamma]} = \{[\gamma\eta] \mid \eta \text{ is a path in } U \text{ starting at } \gamma(1)\}$$

Note that $p : U_{[\gamma]} \longrightarrow U$ is surjective since U path-connected and injective since $\pi_1(U) \longrightarrow \pi_1(X)$ trivial. Using the fact that $[\gamma'] \in U_{[\gamma]} \Rightarrow U_{[\gamma]} = U_{[\gamma']}$, we obtain that the sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$. With respect to this topology, $p : U_{[\gamma]} \longrightarrow U$ gives a homeomorphism, since it gives a bijection between subsets $V_{[\gamma']} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U ($p(V_{[\gamma']}) = V$ and also $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V).

Hence $p : \tilde{X} \longrightarrow X$ is continuous, and it is a covering map, since for fixed $U \in \mathcal{U}$, the sets $\{U_{[\gamma]}\}$ partition $p^{-1}(U)$.

To see that $\tilde{X}$ is simply-connected: Note that for any point $[\gamma] \in \tilde{X}$, we can shrink the path to give a homotopy $t \mapsto [\gamma_t]$ to the constant path $[x_0]$ (this shows $\tilde{X}$ is path-connected). If $[\gamma] \in \pi_1(X, x_0)$ is in the image of p_*, it means that the lift $[\gamma_t]$ is a loop, meaning that $[\gamma_1] = [x_0]$. But $\gamma_1 = \gamma$, this means that $[\gamma] = [x_0]$, hence the image of p_* is trivial. By injectivity of p_*, we get that $\tilde{X}$ is simply-connected. $\square$

Under assumptions on X (connected, local simple-connected, and semi-locally simply connected, in order to define the topology of $\tilde{X}$) we constructed a universal covering $(\tilde{X}, p)$, by setting

$$\tilde{X} = \{[\gamma] \ : \ \gamma \text{ is a path in } X \text{ starting at } x_0\}.$$

We also saw that this space has trivial fundamental group, as follows: Any path γ in X may be lifted to $\tilde{X}$ by defining $\tilde{\gamma}(t)$ to be the path γ up to time t (and constant afterwards). If $[\gamma]$ is in the image of p_*, this means that there is a loop in this class, say γ, which lifts to a loop $\tilde{\gamma}$ in $\tilde{X}$. But this means that γ up to time 1 is equal in $\tilde{X}$ (i.e. homotopic to) to γ up to time 0, i.e. $[\gamma] = 0$ in $\pi_1(X)$. Since p_* is injective, it must be that $\pi_1(\tilde{X}) = 0$.

Having the universal cover, we can produce all other coverings via quotients of it, as follows:

surjectivity of functor. Suppose now that (X, x_0) has a (path-connected) universal covering space $(\tilde{X}, \tilde{x}_0)$, and suppose a subgroup $H \subset \pi_1(X, x_0)$ is specified. Then we define an equivalence relation on $\tilde{X}$ as follows: given points $[\gamma], [\gamma'] \in \tilde{X}$, we define $[\gamma] \sim [\gamma']$ iff $\gamma(1) = \gamma'(1)$ and $[\gamma'\gamma^{-1}] \in H$. Because H is a subgroup, this is an equivalence relation. Now set $X_H = \tilde{X}/\sim$. Note that this equivalence relation holds for nearby paths in the sense $[\gamma] \sim [\gamma']$ iff $[\gamma\eta] \sim [\gamma'\eta]$. Therefore, if any two points in $U_{[\gamma]}, U_{[\gamma']}$ are equivalent, then so is every other point in the neighbourhood. This shows that the projection $p : X_H \longrightarrow X$ via $[\gamma] \mapsto \gamma(1)$ is a covering map.

As a basepoint in X_H, pick $[x_0]$, the constant path at x_0. Then the image of p_* is$H$, since the lift of the loop γ is a path beginning at $[x_0]$ and ending at $[\gamma]$, and this is a loop exactly when $[\gamma] \sim [x_0]$, i.e. $[\gamma] \in H$. $\square$

Example 1.37 (Diagram: page 58)**.** *Consider the wedge $S^1 \vee S^1$. Recall that $\pi_1(S^1 \vee S^1) = F_2 =< a, b >$. View it as a graph with one vertex and two edges, labeled by a, b with their appropriate orientations. We can then take any other graph $\tilde{X}$, labeled in this way, and such that each vertex is locally isomorphic to the given vertex, and define a covering map to $S^1 \vee S^1$. The resulting graph $\tilde{X}$ is itself a wedge of k circles, with fundamental group F_k. Hence we obtain a map $F_k \longrightarrow F_2$ which is an injection. Examples (1), (2)*

In fact, every 4-valent graph can be labeled in the way required above: if the graph is finite, take an Eulerian circuit and label the edges $a, b, a, b \ldots$. Then the a edges are a collection of disjoint circles: orient them and do the same for the b edges.

An infinite 4-valent graph may be constructed which is a simply-connected covering space for $S^1 \vee S^1$: it is a fractal 4-branched tree (drawing).

Not only can we have a free group on any number of generators as a subgroup of F_2, but also we can have infinitely many generators (drawing of (10), (11))

Note that changing the basepoint vertex of a covering simply conjugates $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ inside $\pi_1(X, x_0)$. (draw (3), (4)). Isomorphism of coverings (without fixing basepoints) is just a graph isomorphism preserving labeling and orientation.

Note also that characteristic subgroups may be isomorphic *without being* conjugate*. (draw (5),(6)), these are homeomorphic graphs, but not isomorphic as covering spaces.*

Example 1.38. *If X is a path-connected space with fundamental group $\pi_1(X, x_0)$, then by attaching 2-cells e^2_α via maps $\varphi_\alpha : S^1 \longrightarrow X$, then the resulting space Y will have fundamental group which is a quotient of $\pi_1(X, x_0)$ by the normal subgroup N generated by loops of the form $\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}$, for any γ_α chosen to join x_0 to $\varphi_\alpha(1)$. This is seen by Van Kampen's theorem applied to a thickened version Z of Y where the paths γ_α are thickened to intervals attached to the discs e_α.*

We can use this construction to obtain any group as a fundamental group. Choose a presentation

$$G = \langle g_\alpha \mid r_\beta \rangle.$$

This is possible since any group is a quotient of a free group. Then we construct X_G from $\vee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β by loops specified by the words r_β. (for example, to obtain $\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle$, attach a single 2-cell to S^1 via the map $z \mapsto z^n$. For $n = 2$ we obtain $\mathbb{R}P^2$.

The Cayley complex is one way of describing the universal cover of X_G. It is a cell complex $\tilde{X}_G$ constructed as follows: The vertices are the elements of G itself. Then at each vertex $g \in G$, attach an edge joining g to gg_α for each generator g_α. The resulting graph is the Cayley graph of G with respect to the generators g_α. Then, each relation r_β determines a loop starting at any $g \in G$, and we attach a 2-cell to all these loops. There is an obvious map to X_G given by quotienting by the action of G on the left, which sends all points to the basepoint, each edge $g \longrightarrow gg_\alpha$ to the edge S^1_α, and each 2-cell associated to r_β to that attached in the construction of X_G.

For example, consider $G = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1\rangle$. then the Cayley graph has vertices $\{\ldots, bab, ba, b, e, a, ab, aba, \ldots\}$, and two generators so there will be four edges coming in/out of each vertex g: two outward edges corresponding to right multiplication by a, b to ga, gb, and two inward coming from ga^{-1}, gb^{-1}. We therefore obtain an infinite sequence of tangent circles. We produce the Cayley complex by attaching a 2-cell corresponding to a^2 to the loop produced at each vertex g by following the loop $g \longrightarrow ga \longrightarrow ga^2$. This attaches two 2-cells to each circle, yielding a sequence of tangent 2-spheres, clearly a simply-connected space. The action of G corresponds to an action by even translations (ab) and the antipodal maps, giving the quotient space $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

1.7 Group actions and Deck transformations

In many cases we obtain covering spaces $\tilde{X} \longrightarrow X$ from group actions; if a group A acts on $\tilde{X}$, the quotient map $\tilde{X} \longrightarrow \tilde{X}/A$ may, under some assumptions on A and its action, be a covering.

For example, we can define the n-fold covering $S^1 \longrightarrow S^1$ as simply the quotient of S^1 by the action of $\mathbb{Z}_n$ via $x \mapsto x\sigma^n$ for $\sigma = e^{2\pi i/n}$, or even $\mathbb{R} \longrightarrow S^1$ via the quotient by the $\mathbb{Z}$ action $x \mapsto x + n$.

In general, if $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a universal cover, then we can obtain X as a quotient of $\tilde{X}$ by the action of the fundamental group $\pi_1(X, x_0)$ as follows:

Given an element $[\gamma] \in \pi_1(X, x_0)$, γ lifts to a path terminating in $\tilde{x}_0'$ over x_0. Now the covering p has a unique lift to $\tilde{X}$, sending $\tilde{x}_0$ to the alternative basepoint $\tilde{x}_0'$. This lift is a homeomorphism $\tilde{X} \longrightarrow \tilde{X}$, and this defines an action of $\pi_1(X, x_0)$ on $\tilde{X}$. We'll be careful in a moment to show the quotient is X.

In general, not all covering maps p will be the quotient by the action of a group: this will only be the case for *normal* covering maps, i.e. those for which $p_*(\pi_1(\tilde{X}))$ is a normal subgroup N; Then $\pi_1(X, x_0)/N$ is a group, and this will act in the same way as above, with quotient X.

Example 1.39 (Coverings of surfaces)**.** *There are many interesting coverings of surfaces, which can be constructed by acting by symmetry groups:*

An example of a covering of a compact surface: take a genus $mn+1$ surface, draw it as a surface with m genus n legs and a hole in the center. There is an obvious $\mathbb{Z}_n$ symmetry by rotating by $2\pi/m$. The quotient map is then a m-fold covering map to a surface of genus $n+1$.

Consider a genus g surface in $\mathbb{R}^3$ with the holes along an axis, and consider the rotation about this axis by π, giving a $\mathbb{Z}_2$ action with $2(g+1)$ fixed points. Remove the fixed points. The punctured surface then is a 2-sheeted cover of S^2 punctured in $2(g+1)$ points. This is the topological description of an equation $y^2 = f(z)$ with f of degree $2g+1$ (this way, $y^2 = f$ has exactly two solutions except at the $2g+1$ zeros of f and the point at infinity where $f = \infty$. The particular case where f has degree 3 defines a genus 1 surface, which is called an elliptic curve once a complex structure is chosen on it.

Example 1.40. *The antipodal map on S^n is an action of $\mathbb{Z}_2$ with no fixed points; the quotient map is a covering of $\mathbb{R}P^n$. This will imply that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$. In the case $n = 3$, this 2:1 cover is also known as the sequence of groups*

$$0 \longrightarrow \mathbb{Z}_2 = \{\pm 1\} \longrightarrow SU(2) \xrightarrow{\pi} SO(3) \longrightarrow 0$$

Note that $SO(3)$ has several famous finite subgroups: the cyclic groups A_n, the dihedral groups D_n, and the symmetry groups of the tetrahedron, octahedron, and dodecahedron, E_6, E_7, E_8. In this way we can construct other covering spaces, e.g. $S^3 \longrightarrow S^3/\pi^{-1}(E_8)$, the Poincaré dodecahedral space, a homology 3-sphere.

To formalize the observations above, we wish to answer the following questions: Given a connected covering space (without basepoint), what is its group of automorphisms (deck transformations), and when does this group define the covering as a quotient? And, more generally, when is a group action defining a covering map?

Definition 10. A covering map $p : \tilde{X} \longrightarrow X$ is called normal when, for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}'$ of x, there is an automorphism of p taking $\tilde{x}$ to $\tilde{x}'$.

Theorem 1.41. *If $p : \tilde{X} \longrightarrow X$ is a path-connected covering (of X path-connected and locally path-connected), with characteristic subgroup H, then the group of automorphisms of p is $A = N(H)/H$, and the quotient $\tilde{X}/A$ is the covering with characteristic subgroup $N(H)$. Therefore, a covering is normal precisely when H is normal, and in this case the automorphism group is $A = \pi_1(X)/H$ and $\tilde{X}/A = X$.*

Proof. Changing the basepoint from $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ corresponds to conjugating H by $[\gamma] \in \pi_1(X, x_0)$ which lifts to a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Therefore, $[\gamma] \in N(H)$ iff $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$, which is the case (by the lifting of maps) iff there is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$. Therefore $\tilde{X}$ is normal iff $N(H) = \pi_1(X, x_0)$, i.e. H is already normal in $\pi_1(X, x_0)$.

In general there is a group homomorphism $\varphi : N(H) \longrightarrow A$, sending $[\gamma]$ to the deck transformation mapping $\tilde{x}_0 \mapsto \tilde{x}_1$ as above. It is surjective by the argument above, and its kernel is precisely the classes $[\gamma]$ lifting to loops, i.e. the elements of H itself. ☐

Theorem 1.42. *Suppose G acts on Y in a properly discontinuous way, i.e. each $y \in Y$ has a neighbourhood U such that gU are disjoint for all $g \in G$. Then the quotient of Y by G is a normal covering map, and if Y is path-connected then G is the automorphism group of the cover.*

Proof. First we remark that deck transformations of a covering space obviously have the properly discontinuous property.

To prove the result, take any open set U as in the definition of proper discontinuity. Then the quotient map identifies the disjoint homeomorphic neighbourhoods $\{g(U) \ : \ g \in G\}$ with $p(U) \subset Y/G$. By the definition of the quotient topology, this gives a homeomorphism on each component, and hence we have a covering.

Certainly G is a subgroup of the deck transformations, and the covering space is normal since $g_2 g_1^{-1}$ takes $g_1(U)$ to $g_2(U)$, and if Y is path-connected then G equals the deck transformations, since if a deck transformation f sends y to $f(y)$, we may simply lift the covering to the alternative point $f(y)$ (the lifting criterion is satisfied since the cover is normal) and this deck transformation must coincide with f by uniqueness. $\square$

Remark 1. *Suppose $p : \tilde{X} \longrightarrow X$ is a finite covering. Fixing $x_0 \in X$, we have two natural permutation actions on the finite set $p^{-1}(x_0)$: one is by $\pi_1(X, x_0)$, via lifting of loops, i.e. given $[\gamma] \in \pi_1(X, x_0)$, the permutation $\sigma([\gamma])$ acts on $\tilde{x}_0$ by $\sigma(\tilde{x}_0) = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of γ starting at $\tilde{x}_0$. The study of this permutation action is an alternative approach to classifying covering spaces, and this is described in Hatcher. It is useful to understand both approaches.*

The second action is by the group of deck transformations $A = N(H)/H$ (for the characteristic subgroup H). These actions commute. Interestingly, when $\tilde{X}$ is the universal cover, A is $\pi_1(X, x_0)$ as well, and so we have the same group acting in two ways– these actions need not coincide.

2 Homology

We now turn to Homology, a functor which associates to a topological space X a sequence of abelian groups $H_k(X)$. We will investigate several important related ideas:

- Homology, relative homology, axioms for homology, Mayer-Vietoris
- Cohomology, coefficients, Poincaré Duality
- Relation to de Rham cohomology (de Rham theorem)
- Applications

The basic idea of homology is quite simple, but it is a bit difficult to come up with a proper definition. In the definition of the homotopy group, we considered loops in X, considering loops which could be "filled in" by a disc to be trivial.

In homology, we wish to generalize this, considering loops to be trivial if they can be "filled in" by any surface; this then generalizes to arbitrary dimension as follows (let X be a manifold for this informal discussion).

A k-dimensional chain is defined to be a k-dimensional submanifold $S \subset X$ with boundary, equipped with a chosen orientation σ on S. A chain is called a cycle when its boundary is empty. Then the k^{th} homology group is defined as the free abelian group generated by the k-cycles (where we identify (S, σ) with $-(S, -\sigma)$), modulo those k-cycles which are boundaries of $k+1$-chains. Whenever we take the boundary of an oriented manifold, we choose the boundary orientation given by the outward pointing normal vector.

Example 2.1. *Consider an oriented loop separating a genus 2 surface into two genus 1 punctured surfaces. This loop is nontrivial in the fundamental group, but is trivial in homology, i.e. it is* homologous *to zero.*

Example 2.2. *Consider two parallel oriented loops L_1, L_2 on T^2. Then we see that $L_1 - L_2 = 0$, i.e. L_1 is homologous to L_2.*

Example 2.3. *This definition of homology is not well-behaved: if we pick any embedded submanifold S in a manifold and slightly deform it to S' which still intersects S, then there may be no submanifold with $S \cup S'$ as its boundary. We want such deformations to be homologous, so we slightly relax our requirements: we allow the k-chains to be smooth maps $\iota : S \longrightarrow M$ which needn't be embeddings.*

This definition is still problematic: it's not clear what to do about non-smooth topological spaces, and also the definition seems to require knowledge of all possible manifolds mapping into M. We solve both problems by cutting S into triangles (i.e. simplices) and focusing only on maps of simplices into M.

2.1 Simplicial homology

Definition 11. An n-simplex $[v_0, \cdots, v_n]$ is the convex hull of $n+1$ ordered points (called *vertices*)in $\mathbb{R}^m$ for which $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

The standard n-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \ \forall i\},$$

and there is a canonical map $\Delta^n \longrightarrow [v_0, \cdots, v_n]$ via

$$(t_0, \ldots, t_n) \mapsto \sum_i t_i v_i,$$

called *barycentric coordinates* on $[v_0, \cdots, v_n]$. A *face* of $[v_0, \cdots, v_n]$ is defined as the simplex obtained by deleting one of the v_i, we denote it $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. The union of all faces is the *boundary* of the simplex, and its complement is called the *interior*, or the *open simplex*.

Definition 12. A Δ-complex decomposition of a topological space X is a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$ (n depending on α) such that σ_α is injective on the open simplex Δ^n_o, every point is in the image of exactly one $\sigma_\alpha|_{\Delta^n_o}$, and each restriction of σ_α to a face of $\Delta^{n(\alpha)}$ coincides with one of the maps σ_β, under the canonical identification of Δ^{n-1} with the face (which preserves ordering). We also require the topology to be compatible: $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in the simplex for each α.

It is easy to see that such a structure on X actually expresses it as a cell complex.

Example 2.4. *Give the standard decomposition of 2-dimensional compact manifolds.*

We may now define the simplicial homology of a Δ-complex X. We basically want to mod out cycles by boundaries, except now the chains will be made of linear combinations of the n-simplices which make up X.

Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices $e^n_\alpha = \sigma_\alpha(\Delta^n_o)$ of X. Elements $\sum_\alpha n_\alpha \sigma_\alpha \in \Delta_n(X)$ are called n-chains (finite sums).

Each n-simplex has a natural orientation based on its ordered vertices, and its boundary obtains a natural orientation from the outward-pointing normal vector field. Algebraically, this induced orientation is captured by the following formula (which captures the interior product by the outward normal vector to the i^{th} face):

$$\boxed{\partial[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n].}$$

This allows us to define the boundary homomorphism:

Definition 13. The boundary homomorphism $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ is determined by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}.$$

This definition of boundary is clearly a triangulated version of the usual boundary of manifolds, and satisfies $\partial \circ \partial = \emptyset$, i.e.

Lemma 2.5. *The composition $\partial_{n-1} \circ \partial_n = 0$.*

Proof.

$$\partial\partial[v_0 \cdots v_n] = \sum_{j<i} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n] + \sum_{j>i} (-1)^{i+j-1} [v_0, \cdots, \hat{v}_i, \cdots \hat{v}_j, \cdots, v_n]$$

the two displayed terms cancel. □

Now we have produced an algebraic object: a chain complex (just as we saw in the case of the de Rham complex). Let C_n be the abelian group $\Delta_n(X)$; then we get the simplicial chain complex:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

and the homology is defined as the simplicial homology

$$H^\Delta_n(X) := \frac{Z_n = \ker \partial_n}{B_n = \mathrm{im}\ \partial_{n+1}}$$

Example 2.6. *The circle is a Δ-complex with one vertex and one 1-simplex. so $\Delta_0(S^1) = \Delta_1(S^1) = \mathbb{Z}$ and $\partial_1 = 0$ since $\partial e = v - v$. hence $H^\Delta_0(S^1) = \mathbb{Z} = H^\Delta_1(S^1)$ and $H^\Delta_k(S^1) = 0$ otherwise.*

Example 2.7. *For T^2 and Klein bottle: $\Delta_0 = \mathbb{Z}$, $\Delta_1 = \langle a, b, c\rangle$ and $\Delta_2 = \langle P, Q\rangle$. For $\mathbb{R}P^2$, same except $\Delta_0 = \mathbb{Z}^2$.*

2.2 Singular homology

Simplicial homology, while easy to calculate (at least by computer!), is not entirely satisfactory, mostly because it is so rigid - it is not clear, for example, that the groups do not depend on the triangulation. We therefore relax the definition and describe singular homology.

Definition 14. A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \longrightarrow X$. The free abelian group on the set of n-simplices is called $C_n(X)$, the group of n-chains.

There is a linear boundary homomorphism $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ given by

$$\partial_n \sigma = \sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]},$$

where $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ is canonically identified with Δ^{n-1}. The homology of the chain complex $(C_\bullet(X), \partial)$ is called the *singular homology* of X:

$$H_n(X) := \frac{\ker \partial : C_n(X) \longrightarrow C_{n-1}(X)}{\mathrm{im}\partial : C_{n+1}(X) \longrightarrow C_n(X)}.$$

We would like to justify the statement that the homology is a functor. In fact we would like to show that our assigning, to every space X, the complex of singular chains

$$X \mapsto (C_\bullet(X), \partial)$$

is actually a functor from topological spaces to the category of chain complexes of abelian groups, where the latter category has morphisms given by chain homomorphisms, just as in the case for the de Rham complex $(\Omega^\bullet(M), d)$. By actually taking homology, we then obtain a functor to abelian groups. We would actually like to show even more: that the functor $X \mapsto (C_\bullet(X), \partial)$ can be made into a 2-functor, sending homotopies of continuous maps to chain homotopies: this will allow us to show that $H_\bullet(X)$ is a homotopy invariant.

Given a singular n-simplex $\sigma : \Delta^n \longrightarrow X$ and a map $f : X \longrightarrow Y$, the composition $f \circ \sigma$ defines a simplex for the space Y. In this way we define

$$f_\sharp : C_n(X) \longrightarrow C_n(Y),$$

and we may verify that $f_\sharp \partial = \partial f_\sharp$, implying that $f_\sharp$ is a morphism of chain complexes, defining a functor since $(f \circ g)_\sharp = f_\sharp \circ g_\sharp$. As a consequence, this induces a homomorphism

$$f_* : H_n(X) \longrightarrow H_n(Y).$$

Now we see how $f_\sharp$ behaves for homotopic maps:

Theorem 2.8. *The chain maps $f_\sharp, g_\sharp$ induced by homotopic maps $f, g : X \longrightarrow Y$ are chain homotopic, i.e. there exists $P : C_n(X) \longrightarrow C_{n+1}(Y)$ such that*

$$g_\sharp - f_\sharp = P\partial + \partial P.$$

Hencce, $f_* = g_*$, i.e. the induced maps on homology are equal for homotopic maps.

Proof. The proof is completely analogous to the same result for the de Rham complex. Given a homotopy $F : X \times I \longrightarrow Y$ from f to g, define the *Prism operators* $P : C_n(X) \longrightarrow C_{n+1}(Y)$ as follows: for any n-simplex $\sigma : [v_0, \cdots, v_n] \longrightarrow X$, form the prism $[v_0, \cdots, v_n] \times I$, name the vertices $v_i = (v_i, 0)$ and $w_i = (v_i, 1)$, and decompose this prism in terms of $n+1$-simplices as follows:

$$[v_0, \cdots, v_n] \times I = \bigcup_{i=0}^{n} [v_0, \cdots, v_i, w_i, \cdots w_n].$$

Then we define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{Id})|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]} \quad \in C_{n+1}(Y)$$

Now we show that $\partial P = g_\sharp - f_\sharp - P\partial$, which expresses the fact that the boundary of the prism (left hand) consists of the top $\Delta^n \times 1$, bottom $\Delta^n \times 0$, and sides $\partial\Delta^n \times I$ of the prism.

$$\begin{aligned}\partial P(\sigma) = &\sum_{j\leq i} (-1)^i(-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0\cdots,\hat{v}_j,\cdots v_i,w_i,\cdots,w_n]} \\ &+ \sum_{j\geq i} (-1)^i(-1)^{j+1} F \circ (\sigma \times \mathrm{Id})|_{[v_0\cdots,v_i,w_i,\cdots \hat{w}_j,\cdots,w_n]}\end{aligned}$$

The terms with $i = j$ in the two lines cancel except for $i = j = 0$ and $i = j = n$, giving $g_\sharp(\sigma) - f_\sharp(\sigma)$. The terms with $i \neq j$ are $-P\partial(\sigma)$ by expressing it as a sum

$$\begin{aligned}P\partial(\sigma) = &\sum_{i<j} (-1)^i(-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0\cdots,v_i,w_i,\cdots \hat{w}_j,\cdots,w_n]} \\ &+ \sum_{i>j} (-1)^{i-1}(-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}\end{aligned}$$

$\square$

Corollary 2.9. *$C_\bullet$ is a 2-functor and $H_\bullet$ is homotopy invariant.*

2.3 H_0 and H_1

Proposition 2.10. *If X has path components X_α, then $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$.*

Proof. A singular simplex always has path-connected image. Hence $C_n(X)$ is the direct sum of $C_n(X_\alpha)$. The boundary maps preserve this decomposition. So $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$. (Since chains are finite sums, we use the direct sum). ☐

Proposition 2.11. *If X is path-connected (and nonempty) then $H_0(X) \cong \mathbb{Z}$.*

Proof. Define $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$ via $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. This is surjective if X nonempty. We must show that $\ker\epsilon = \mathrm{im}\partial_1$.

For any singular 1-simplex $\sigma : \Delta^1 \longrightarrow X$, we have $\epsilon(\partial\sigma) = \epsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$. Hence $\mathrm{im}\partial_1 \subset \ker\epsilon$.

For the reverse inclusion: if $\sum_i n_i = 0$, we wish to show tha $\sum_i n_i\sigma_i$ is a boundary of a singular 1-simplex. Choose a path $\tau_i : I \longrightarrow X$ from a basepoint x_0 to $\sigma_i(v_0)$ and let σ_0 be the 0-simplex with image x_0. Then $\partial\tau_i = \sigma_i - \sigma_0$, viewing τ_i as a singular 1-simplex. Then $\partial\sum_i n_i\tau_i = \sum_i n_i\sigma_i$. ☐

Later, we will axiomatize homology as a functor from spaces to abelian groups: there are many different such functors, corresponding to different *homology theories*. To understand any homology theory it is fundamental to compute its value on the one-point space.

Proposition 2.12. *If $X = \{*\}$ then $H_n(X) = 0$ for $n > 0$ (and $H_0(X) = \mathbb{Z}$ by the above result).*

Proof. When the target is a single point, there can be only one singular n-simplex for each n, namely, the map sending Δ^n to the point $*$. Hence the chain groups are all $\mathbb{Z}$, generated by σ_n. The boundary map is $\partial\sigma_n = \sum_{i=0}^n(-1)^i\sigma_{n-1}$, which vanishes for n odd and is equal to σ_{n-1} for $n \neq 0$ and even. Hence the singular chain complex is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

which has homology

$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad 0$$

☐

Note that the map $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$ defined above may be viewed as an extension of the singular chain complex (with $C_{-1}(X) = \mathbb{Z}$). The homology groups of this augmented chain complex are called the *reduced homology* of X, and denoted $\tilde{H}_n(X)$. Clearly $H_n(X) \cong \tilde{H}_n(X) \oplus \mathbb{Z}$ and $\tilde{H}_n(X) \cong H_n(X)$ for all $n > 0$.

Theorem 2.13 (Hurewicz isomorphism)**.** *The natural map $h : \pi_1(X, x_0) \longrightarrow H_1(X)$, given by regarding loops as singular 1-cycles, is a homomorphism. If X is path-connected, h induces an isomorphism $\pi_1(X)/[\pi_1(X), \pi_1(X)] \longrightarrow H_1(X)$, i.e. $H_1(X)$ is the abelianization of the fundamental group.*

In higher dimension, the Hurewicz theorem states that if the path-connected space X is $n-1$ connected for $n \geq 2$ (i.e. $\pi_k(X) = 0\ \forall k < n$), then $\pi_n(X)$ is isomorphic to $H_n(X)$.

Proof. First we describe some properties of the homology relation on paths $f \sim g \Leftrightarrow \exists\tau\ :\ f - g = \partial\tau$, as opposed to the homotopy of paths relation $f \simeq g$.

- if f is a constant path, then $f \sim 0$ since $H_1(*) = 0$.
- $f \simeq g \Rightarrow f \sim g$ since we can write the homotopy $I \times I \longrightarrow X$ as a singular 2-chain (with two singular 2-simplices – cut the square by the diagonal) with boundary $f - g + x_0 - x_1$, and since the constant paths x_0, x_1 are boundaries, so is $f - g$.
- $f \cdot g \sim f + g$, since we can define a singular 2-chain with boundary $f + g - f \cdot g$ by letting $\sigma : [v_0, v_1, v_2] \longrightarrow X$ be the composition of orthogonal projection onto $[v_0, v_2]$ followed by $f \cdot g : [v_0, v_2] \longrightarrow X$.

- $f^{-1} \sim -f$, since $f + f^{-1} \sim f \cdot f^{-1} \sim 0$.

Applying these properties to loops, we obtain that h is a homomorphism. Clearly $[\pi_1, \pi_1] \subset \ker h$, since H_1 is abelian. Hence h induces a homomorphism $\pi_1^{ab}(X) \longrightarrow H_1$.

A map in the opposite direction is given as follows: if f is a loop representative for a class in H_1, choose any path γ from x_0 to $f(0)$. Then $\psi : [f] \mapsto [\gamma f \gamma^{-1}]$ is well-defined when taking values in π_1^{ab}.

Furthermore, it vanishes on boundaries: check on a singular 2-simplex, and view the 2-simplex as a homotopy. It remains to show that $\psi \circ h = h \circ \psi = 1$. $\square$

2.4 Relative homology and the excision property

It is natural to expect that the homology of a space X is related to the homology of one of its subspaces $A \subset X$; relative homology is a systematic way of analyzing this idea. Under some conditions on the pair (X, A), we will also investigate the relationship to the homology of X/A. This will also lead us to the Excision property and the Mayer-Vietoris sequence.

Definition 15. Let X be a space and $A \subset X$ a subspace. The *relative chains* $C_n(X, A)$ are chains in X modulo chains in A, i.e.

$$C_n(X, A) := \frac{C_n(X)}{C_n(A)}.$$

Since the boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it descends to a boundary map, also called $\partial : C_n(X, A) \longrightarrow C_{n-1}(X, A)$. We therefore get a chain complex

$$\cdots \longrightarrow C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \longrightarrow \cdots$$

whose cohomology gives the relative homology groups $H_n(X, A)$. Intuitively, relative homology is the homology of X modulo A.

It is clear that our previous functoriality results on $H_n(X)$ (sometimes called the *absolute* homology of X) carry over to the relative homology. For example:

Proposition 2.14. *if two maps of pairs $f, g : (X, A) \longrightarrow (Y, B)$ are homotopic through maps of pairs $(X, A) \longrightarrow (Y, B)$, then $f_* = g_*$ on relative cohomology.*

The first result about relative homology groups is an algebraic fact which follows directly from their definition. Since $C_n(X, A)$ is by definition the quotient of $C_n(X)$ by $C_n(A)$, let $i : C_n(A) \longrightarrow C_n(X)$ be the inclusion and j be the quotient map, so that we have the exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

We have this exact sequence for each n, and it also commutes with the boundary operator. Hence we get an exact sequence of *chain complexes*:

$$0 \longrightarrow (C_\bullet(A), \partial) \xrightarrow{i} (C_\bullet(X), \partial) \xrightarrow{j} (C_\bullet(X, A), \partial) \longrightarrow 0$$

Just as we saw for the de Rham complex, a short exact sequence of chain complexes gives a long exact sequence of homology groups. Since we are dealing with chain complexes, not cochain complexes, the connecting homomorphism δ coming from the boundary map ∂ is of degree -1. In this case, we obtain

Proposition 2.15 (Exactness). *Given $A \subset X$, we have the following exact sequence:*

$$\begin{tikzcd} H_\bullet(A) \arrow[rr, "i_*"] & & H_\bullet(X) \arrow[dl, "j_*"] \\ & H_\bullet(X, A) \arrow[ul, "\delta(-1)"] & \end{tikzcd}$$

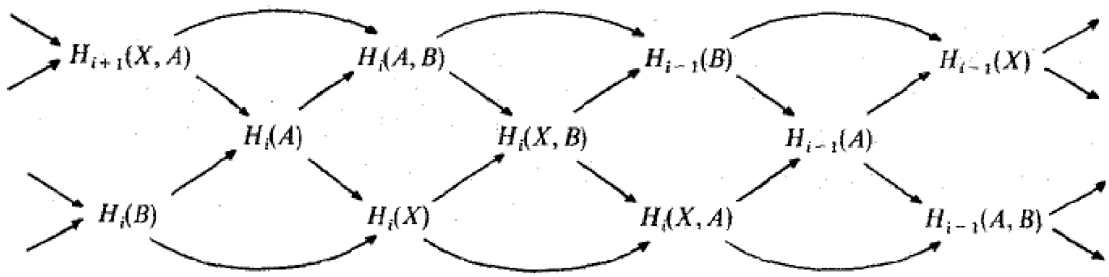

Figure 1: Braid diagram for triple

In fact, the boundary map δ has an obvious description in this application to relative homology: if $\alpha \in C_n(X, A)$ is a relative cycle, then $\delta[\alpha]$ is the $n-1$-homology class given by $[\partial\alpha] \in H_{n-1}(A)$.

Example 2.16. *Let $x_0 \in X$ and consider the pair (X, x_0). Then the long exact sequence in relative homology implies $H_n(X, x_0) \cong H_n(X)$ for all $n > 0$, while for $n = 0$ we have*

$$0 \longrightarrow H_0(x_0) \longrightarrow H_0(X) \longrightarrow H_0(X, x_0) \longrightarrow 0 \ ,$$

showing that $H_0(X, x_0) \cong \tilde{H}_0(X)$ and hence $H_n(X, x_0) \cong \tilde{H}_n(X)$ for all n.

Formal consequences of subspace inclusion for relative homology can be more complicated: for instance, suppose we have a triple (X, A, B) where $B \subset A \subset X$. Then we have short exact sequences

$$0 \longrightarrow C_n(A, B) \longrightarrow C_n(X, B) \longrightarrow C_n(X, A) \longrightarrow 0 \ ,$$

inducing the long exact sequence in homology:

$$\begin{tikzcd}
H_\bullet(A, B) \arrow[rr] & & H_\bullet(X, B) \arrow[dl] \\
& H_\bullet(X, A) \arrow[ul, "\delta(-1)"] &
\end{tikzcd}$$

In fact, this long exact sequence couples with the long exact sequences for each pair to form a braid diagram– see Fig. 2.4

The main result on relative homology is the *excision property*, which states that the homology of X relative to $A \subset X$ remains the same after deleting a subset Z whose closure sits in the interior of A. The property is so fundamental that it has been promoted to an axiom defining a homology theory, as we shall see.

Theorem 2.17 (Excision)**.** *Let $Z \subset A \subset X$, with $\overline{Z} \subset A^{int}$. Then the inclusion $(X\backslash Z, A\backslash Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_n(X\backslash Z, A\backslash Z) \longrightarrow H_n(X, A) \quad \forall n.$$

An equivalent formulation is that if $A, B \subset X$ have interiors which cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \longrightarrow H_n(X, A)$ for all n (simply set $B = X\backslash Z$ or $Z = X\backslash B$).

Proof of Excision. Consider X as a union of A and B with interiors covering X. Then we have natural inclusion maps

$$
\begin{tikzcd}
 & C_\bullet(X) & \\
 & C_\bullet(A)+C_\bullet(B) \arrow[u, "\iota"] & \\
C_\bullet(A) \arrow[ur] & & C_\bullet(B) \arrow[ul] \\
 & C_\bullet(A\cap B) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

If the map ι were an isomorphism, then we would have $C_\bullet(X)/C_\bullet(A) = C_\bullet(B)/C_\bullet(A\cap B)$, giving the result. But the problem is that ι is not an isomorphism; there are "bad" simplices which can have nonempty intersection with $A - A\cap B$ and $B - A\cap B$. We would like to show that we can subdivide the bad simplices into smaller good ones via a chain map $\rho : C_\bullet(X) \longrightarrow C_\bullet(A) + C_\bullet(B)$, in such a way that it doesn't change the homology. In fact, we show that $C_\bullet(A) + C_\bullet(B)$ is a *deformation retract* of $C_\bullet(X)$, in the sense that $\rho\circ\iota = \mathrm{Id}$ and $\mathrm{Id} - \iota\circ\rho = \partial D + D\partial$ for some chain homotopy D. In fact we will choose D to preserve the subcomplexes $C_\bullet(A)$ and $C_\bullet(B)$, implying that we obtain a chain homotopy equivalence

$$C_\bullet(X)/C_\bullet(A) \longrightarrow C_\bullet(B)/C_\bullet(A\cap B),$$

yielding the proof of the theorem.

The map ρ will essentially be an iteration of the barycentric subdivision map S, which we now define (we will be a little sloppy to speed things up - see Hatcher for a full treatment).

Definition 16 (Subdivision operator)**.** If $w_0,\dots w_n$ are points in a vector space and b is any other point, then b can be added to a simplex, forming a cone: $b\cdot[w_0,\cdots,w_n] = [b,w_0,\cdots,w_n]$. Note that $\partial b = \mathrm{Id} - b\partial$, i.e. the boundary of a cone consists of the base together with the cone on the boundary. Given any simplex λ, let b_λ be the barycenter. Then we define inductively the *barycentric subdivision* $S\lambda = b_\lambda \cdot S(\partial\lambda)$, with the initial step $S[\emptyset] = [\emptyset]$ on the empty simplex. Note that the diameter of each simplex in the barycentric subdivision of $[v_0,\cdots,v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0,\cdots,v_n]$, so that they approach zero size as $n\to\infty$.

Now, given a singular n-simplex $\sigma : \Delta^n \longrightarrow X$, define $S\sigma = \sigma|_{S\Delta^n}$, in the sense that it is a signed sum of restrictions of σ to the n-simplices of the barycentric subdivision of Δ^n. $S : C_n(X) \longrightarrow C_n(X)$ is a chain map, since

$$
\begin{aligned}
\partial S\lambda &= \partial(b_\lambda(S\partial\lambda))\\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \text{ since } \partial b_\lambda + b_\lambda\partial = 1\\
&= S\partial\lambda - b_\lambda(S\partial\partial\lambda) \text{ by induction}\\
&= S\partial\lambda.
\end{aligned}
$$

This subdivision operator is chain homotopic to the identity, via the map $T : C_n(X) \longrightarrow C_{n+1}(X)$ given as follows: Subdivide $\Delta^n\times I$ into simplices inductively by joining all simplices in $\Delta^n\times\{0\}\cup\partial\Delta^n\times I$ to the barycenter of $\Delta^n\times\{1\}$. Projecting $\Delta^n\times I\longrightarrow\Delta^n$, we may compose with any singular simplex $\sigma : \Delta^n\longrightarrow X$ to obtain a sum of $n+1$-simplices. Formalizing this, we have $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ and $T[\emptyset] = 0$. We may then check the formula $\partial T + T\partial = \mathrm{Id} - S$:

$$
\begin{aligned}
\partial T\lambda &= \partial(b_\lambda(\lambda - T\partial\lambda))\\
&= \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda)) \text{ using } \partial B_\lambda = \mathrm{Id} - b_\lambda\partial\\
&= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda) \text{ by induction}\\
&= \lambda - T\partial\lambda - S\lambda \text{ since } S\lambda = b_\lambda(S\partial\lambda)
\end{aligned}
$$

Note that T also preserves $C_\bullet(A), C_\bullet(B)$.

For each singular n-simplex $\sigma : \Delta^n \longrightarrow X$, there exists a minimal $m(\sigma)$ such that $S^{m(\sigma)}(\sigma)$ lies in $C_n(A) + C_n(B)$. Note that the chain homotopy between S^m and Id is given by

$$D_m = T(1 + S + S^2 + \cdots + S^{m-1}).$$

We then define $D : C_n(X) \longrightarrow C_{n+1}(X)$ via $D\sigma = D_{m(\sigma)}\sigma$, and then we compute

$$\partial D\sigma + D\partial\sigma = \sigma - [S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)],$$

and finally we define $\rho(\sigma)$ to be the bracketed term. Claim: ρ maps $C_\bullet(X)$ to $C_\bullet(A)+C_\bullet(B)$. The first term $S^{m(\sigma)}$ clearly does, and since $m(\partial\sigma) \leq m(\sigma)$, it follows that $(D_{m(\sigma)} - D)(\partial\sigma)$ consists of terms $TS^i(\partial\sigma)$ for $i \geq m(\partial\sigma)$, which all lie in $C_\bullet(A) + C_\bullet(B)$.

Finally, we have constructed ρ, D such that $\rho\iota = \text{Id}$ (since m is zero) and $\partial D + D\partial = \text{Id} - \iota\rho$, with D preserving the subcomplex $C_\bullet(A) + C_\bullet(B)$. As explained earlier, this proves the result.

□

Let $A \subset X$ be a nonempty closed subspace which is a deformation retract of some neighbourhood in X. We call such a pair (X, A) a *good pair* (CW pairs are automatically good pairs, see the Appendix in Hatcher).

Corollary 2.18. *If (X, A) is a good pair, then the quotient map $q : (X, A) \longrightarrow (X/A, A/A)$ induces isomorphisms*

$$q_* : H_n(X, A) \longrightarrow H_n(X/A, A/A) \cong \tilde{H}_n(X/A) \qquad \forall n.$$

Proof. Let V be a neighbourhood of A in X which deformation retracts onto A and let $\iota : A \hookrightarrow V$ be the inclusion. Then we have the diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\iota_*"] \arrow[d, "q_*"'] & H_n(X,V) \arrow[d, "q'_*"] \\ H_n(X/A, A/A) \arrow[r, "\iota'_*"'] & H_n(X/A, V/A) \end{tikzcd}$$

The map ι_* is an isomorphism, as follows: $H_n(V, A)$ are zero for all n, since the deformation retraction gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$ and $H_n(A, A) = 0$. Then using the long exact sequence for the triple (X, V, A) we see that ι_* is an iso.

ι'_* is also an iso, since the deformation retraction induces a deformation retraction of V/A onto A/A, so by the same argument we get ι'_* is an iso.

The groups on the right can be obtained by excision:

$$\begin{tikzcd} H_n(X,V) \arrow[d, "q'_*"'] & H_n(X\backslash A, V\backslash A) \arrow[l, "j_*"] \arrow[d, "q''_*"] \\ H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "j'_*"'] \end{tikzcd}$$

The maps j_*, j'_* are iso by the excision property, and q''_* is an iso, since q restricted to the complement of A is a homeo. This implies q'_* is an iso, and hence q_* is an iso, as required. □

Corollary 2.19. *If (X, A) is a good pair, then the exact sequence for relative homology may be written as*

$$\begin{tikzcd} \tilde{H}_\bullet(A) \arrow[rr] & & \tilde{H}_\bullet(X) \arrow[dl] \\ & \tilde{H}_\bullet(X/A) \arrow[ul, "\delta(-1)"] & \end{tikzcd}$$

The above long exact sequence may be applied to the pair $(D^n, \partial D^n)$, where D^n is the closed unit n-ball; Note that $D^n \simeq *$ and hence $H_k(D^n) = 0\ \forall k$. Also note that $D^n/\partial D^n \simeq S^n$. Hence we have isomorphisms $\tilde{H}_i(S^n) \cong H_{i-1}(S^{n-1})$, implying that $\tilde{H}_k(S^n)$ vanishes for $k \neq n$ and is isomorphic to $\mathbb{Z}$ for $k = n$.

Corollary 2.20. *$H_k(S^n) \cong \mathbb{Z}$ for $k = 0, n$ and $H_i(S^n) = 0$ otherwise.*

We also get Brouwer's theorem from this:

Corollary 2.21. *∂D^n is not a retract of D^n, and hence every map $f : D^n \longrightarrow D^n$ has a fixed point.*

Proof. Let r be such a retraction, so that $ri = \mathrm{Id}$ for the inclusion $i : \partial D^n \hookrightarrow D^n$. Then the composition

$$H_{n-1}(\partial D^n) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(\partial D^n)$$

is the identity map on $H_{n-1}(\partial D^n) \cong \mathbb{Z}$. Of course this is absurd since $H_{n-1}(D^n) = 0$. □

Another easy consequence is the computation of $H_\bullet(X \wedge Y)$: if the wedge sum is formed at points $x \in X$ and $y \in Y$ such that $(x, X), (y, Y)$ are good pairs, then the inclusions $i : X \longrightarrow X \wedge Y$ and $j : Y \longrightarrow X \wedge Y$ induce isomorphisms

$$\tilde{H}_k(X) \oplus \tilde{H}_k(Y) \longrightarrow \tilde{H}_k(X \wedge Y).$$

This follows from the fact that $(X \sqcup Y, \{x, y\})$ is a good pair and $H(X \sqcup Y/\{x, y\}) \cong \tilde{H}(X \sqcup Y, \{x, y\}) = \tilde{H}(X) \oplus \tilde{H}(Y)$.

Yet another result which we may now prove easily: Brouwer's invariance of dimension.

Corollary 2.22. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic and nonempty, then $n = m$.*

This result is easily obtained with the definition of *local homology groups*

Definition 17. Let $x \in X$. Then the local homology groups of X at x are $H_n(X, X \backslash \{x\})$.

For any open neighbourhood U of x, excision gives isomorphisms

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\}),$$

hence the local homology groups only depend locally on x. For instance, a homeomorphism $f : X \longrightarrow Y$ must induce an isomorphism from the local homology of x to that of $f(x)$.

For topological n-manifolds, $H_k(X, X - \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \tilde{H}_{k-1}(\mathbb{R}^n - \{0\}) \cong \tilde{H}_{k-1}(S^{n-1})$ and hence it vanishes unless $k = n$, in which case it is isomorphic to $\mathbb{Z}$. Note that we obtain a fiber bundle over X with fiber above x given by $H_n(X, X - \{x\})$ and isomorphic to $\mathbb{Z}$. Is this a trivial fiber bundle?

2.5 Excision implies Simplicial = Singular homology

Recall that simplicial homology was defined in terms of a Δ-complex decomposition of X, via a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$. We then define the chains to be the free abelian group on the n-simplices, i.e. $\Delta_n(X)$. We would like to show that if a Δ-complex structure is chosen, then its simplicial homology coincides with the singular homology of the space X. It will be useful to do this by induction on the k-skeleton X^k consisting of all simplices of dimension k or less, and so we would like to use a relative version of simplicial homology:

Define relative simplicial homology for any sub-Δ-complex $A \subset X$ as usual, using relative chains

$$\Delta_n(X, A) = \frac{\Delta_n(X)}{\Delta_n(A)},$$

and denote it by $H_n^\Delta(X, A)$.

Theorem 2.23. *Any n-simplex in a Δ-complex decomposition of X may be viewed as a singular n-simplex, hence we have a chain map*

$$\Delta_n(X, A) \longrightarrow C_n(X, A).$$

The induced homomorphism $H_n^\Delta(X, A) \longrightarrow H_n(X, A)$ is an isomorphism. Taking $A = \emptyset$, we obtain the equivalence of absolute singular and simplicial homology.

Lemma 2.24. *The identity map $i_n : \Delta^n \longrightarrow \Delta^n$ is a cycle generating $H_n(\Delta^n, \partial\Delta^n)$.*

Proof. Certainly i_n defines a cycle, and it clearly generates for $n = 0$. We do an induction by relating i_n to i_{n-1} by killing $\Lambda \subset \Delta^n$, the union of all but one $n-1$-dimensional face of Δ^n and considering the triple $(\Delta^n, \partial\Delta^n, \Lambda)$. Since $H_i(\Delta^n, \Lambda) = 0$ by deformation retraction, we get isomorphism

$$H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda).$$

But $(\partial\Delta^n, \Lambda)$ and $(\Delta^{n-1}, \partial\Delta^{n-1})$ are good pairs and hence the relative homologies equal the reduced homology of the quotients, which are *homeomorphic*. Hence we have

$$H_{n-1}(\partial\Delta^n, \Lambda) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

Under the first iso, i_n is sent to ∂i_n which in the relative complex is $\pm i_{n-1}$, so we see that i_n generates iff i_{n-1} generates. ☐

proof of theorem. First suppose that X is finite dimensional, and $A = \emptyset$. Then the map of simplicial to singular gives a morphism of relative homology long exact sequences:

$$\begin{tikzcd}
H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^k) \arrow[r] \arrow[d] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_{n-1}^\Delta(X^{k-1}) \arrow[d] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

We will show that most of the vertical maps are isos and then deduce the center map is an iso.

First the maps on relative homology: the group $\Delta_n(X^k, X^{k-1})$ is free abelian on the k-simplices, and hence it vanishes for $n \neq k$. Therefore the only nonvanishing homology group is $H_k^\Delta(X^k, X^{k-1})$, which is free abelian on the k-simplices. To compute the singular group $H_n(X^k, X^{k-1})$, consider all the simplices together as a map

$$\Phi : \sqcup_\alpha(\Delta_\alpha^k, \partial\Delta_\alpha^k) \longrightarrow (X^k, X^{k-1})$$

and note that it gives a homeomorphism of quotient spaces. Hence we have

$$\begin{tikzcd} H_\bullet(\Delta^k_\alpha, \partial\Delta^k_\alpha) \arrow[r] \arrow[d, equal] & H_\bullet(X^k, X^{k-1}) \arrow[d] \\ \tilde{H}_\bullet(\Delta^k_\alpha/\partial\Delta^k_\alpha) \arrow[r, equal] & \tilde{H}_\bullet(X^k/X^{k-1}) \end{tikzcd}$$

which shows that the top is an iso. Using the previous lemma which tells us that the generators of $H_\bullet(\Delta^k, \partial\Delta^k)$ are the same as the simplicial generators, we get that the maps

$$H^\Delta_k(X^k, X^{k-1}) \longrightarrow H_k(X^k, X^{k-1})$$

are isomorphisms. The second and fifth vertical maps are isomorphisms by induction, and then by the Five-Lemma, we get the central map is an iso.

What about if X is not finite-dimensional? Use the fact that a compact set in X may only meet finitely many open simplices (i.e. simplices with proper faces deleted) of X (otherwise we would have an infinite sequence (x_i) such that $U_i = X - \cup_{j\neq i}\{x_j\}$ give an open cover of the compact set with no finite subcover.

To prove $H^\Delta_n(X) \longrightarrow H_n(X)$ is surjective, let $[z] \in H_n(X)$ for z a singular n-cycle. It meets only finitely many simplices hence it must be in X^k for some k. But we showed that $H^\Delta_n(X^k) \longrightarrow H_n(X^k)$ is an isomorphism, so this shows that z must be homologous in X^k to a simplicial cycle. For injectivity: if z is a boundary of some chain, this chain must have compact image and lie in some X^k, so that $[z]$ is in the kernel $H^\Delta_n(X^k) \longrightarrow H_n(X)$. But this is an injection, so that z is a simplicial boundary in X^k (and hence in X).

All that remains is the case where $A \neq \emptyset$, which follows by applying the Five-Lemma to both long exact sequences of relative homology, for each of the simplicial and singular homology theories. $\square$

Lemma 2.25 (Five-Lemma)**.** *If $\alpha, \beta, \delta, \epsilon$ are isos in the diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

and the rows are exact sequences, then γ is an iso.

Proof. γ surjective: take $c' \in C'$. Then $k'(c') = \delta(d) = \delta k(c) = k'\gamma(c)$ for some c. Therefore $k'(c' - \gamma(c)) = 0$, which implies $c' - \gamma(c) = j'(b') = j'(\beta(b)) = \gamma j(b)$ for some b, showing that $c' = \gamma(c + j(b))$.

γ injective: $\gamma(c) = 0$ implies $c = j(b)$ for some b with $\beta(b) = i'(a') = i'\alpha(a) = \beta i(a)$ for some a, so that $b = i(a)$, showing that $c = 0$. $\square$

The previous theorem allows us to conclude that for X a Δ-complex with finitely many n-simplices, $H_n(X)$ is finitely generated, and hence it is given by the direct sum of $\mathbb{Z}^{b_n}$ and some finite cyclic groups. b_n is called the n^{th} Betti number, and the finite part of the homology is called the *torsion*.

2.6 Axioms for homology

Eilenberg, Steenrod, and Milnor obtained a system of axioms which characterize homology theories without bothering with simplices and singular chains. Be warned: not all "homology theories" satisfy these axioms precisely: Čech homology fails exactness and Bordism and K-theory fail dimension (without dimension, the homology theory is called *extraordinary*).

If we restrict our attention to Cell complexes (i.e. CW complexes), then singular homology is the *unique* functor up to isomorphism which satisfies these axioms (we won't have time to prove this).

Definition 18. A *homology theory* is a functor H from topological pairs (X, A) to graded abelian groups $H_\bullet(X, A)$ together with a natural transformation $\partial_* : H_p(X, A) \longrightarrow H_{p-1}(A)$ called the *connecting homomorphism*[9] (note that $H_p(A) := H_p(A, \emptyset)$) such that

i) (Homotopy) $f \simeq g \Rightarrow H(f) = H(g)$

ii) (Exactness) For $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, the following is an exact sequence of groups:

$$\begin{tikzcd} H_\bullet(A) \arrow[rr, "i_*=H(i)"] & & H_\bullet(X) \arrow[dl, "j_*=H(j)"] \\ & H_\bullet(X,A) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

iii) (Excision) Given $Z \subset A \subset X$ with $\overline{Z} \subset A^{int}$, the inclusion $k : (X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism
$$H(k) : H_\bullet(X - Z, A - Z) \xrightarrow{\cong} H_\bullet(X, A).$$

iv) (Dimension) For the one-point space $*$, $H_i(*) = 0$ for all $i \neq 0$.

v) (Additivity) H preserves coproducts, i.e. takes arbitrary disjoint unions to direct sums[10].

Finally, the *coefficient group* of the theory is defined to be $G = H_0(*)$.

Note: There are natural shift functors S, s acting on topological pairs and graded abelian groups, respectively, given by $S : (X, A) \mapsto (A, \emptyset)$ and $(s(G_\bullet))_n = G_{n+1}$. The claim that ∂_* is natural is properly phrased as
$$\partial_* : H \Rightarrow s^{-1} \circ H \circ S.$$

Note: If the coefficient group G is not $\mathbb{Z}$, then the theorem mentioned above for CW complexes says that the homology functor must be isomorphic to $H_\bullet(X, A; G)$, singular homology with coefficients in G, meaning that chains consist of linear combinations of simplices with coefficients in G instead of $\mathbb{Z}$.

There is a sense in which homology with coefficients in $\mathbb{Z}$ is more fundamental than homology with coefficients in some other abelian group G. The result which explains this assertion is called the "universal coefficient theorem for homology". Let's describe this briefly, because it is the first example we encounter of a *derived functor*.

The chains $C_n(X, A; G)$ with coefficients in G is naturally isomorphic to the tensor product $C_n(X, A) \otimes_{\mathbb{Z}} G$, and the boundary map is nothing but
$$\partial \otimes \mathrm{Id} : C_n(X, A) \otimes G \longrightarrow C_{n-1}(X, A) \otimes G.$$

So, instead of computing the homology of the chain complex C_n, we are computing the homology of $C_n \otimes G$.

[9] This natural transformation ∂_* is the only remnant of chains, boundary operators, etc. All that is gone, but we retain the categorical notion defined by ∂_*.

[10] recall that coproduct of X_i is the universal object with maps from X_i, whereas the product is the universal object with projections to X_i

Theorem 2.26. *If C is a chain complex of abelian groups, then there are natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0$$

and these sequences split but not naturally.

Here, $\mathrm{Tor}(A, G)$ is an abelian group (always torsion, it turns out) which depends on the abelian groups A, G, and is known as the first derived functor of the functor $A \mapsto A \otimes G$. In particular, the following rules will help us compute the Tor group: $\mathrm{Tor}(A, G) = 0$ if A is free. $\mathrm{Tor}(A_1 \oplus A_2, G) \cong \mathrm{Tor}(A_1, G) \oplus \mathrm{Tor}(A_2, G)$, and most importantly $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \cong \ker(G \xrightarrow{n\cdot} G)$. Clearly under many circumstances $\mathrm{Tor}(H_{n-1}(C), G)$ will vanish and in this case $H_n(C;G) = H_n(C) \otimes G$. For example, although $H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, multiplication by 2 has trivial kernel on $\mathbb{Z}/3\mathbb{Z}$, hence $H_n(\mathbb{R}P^2, \mathbb{Z}/3\mathbb{Z}) = H_n(\mathbb{R}P^2) \otimes \mathbb{Z}/3\mathbb{Z}$. On the other hand, with $\mathbb{Z}/2\mathbb{Z}$ coefficients, $\mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence $H_2(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

If we triangulate $\mathbb{R}P^2$ with two 2-simplices, we can check that the sum of the two 2-simplices can't have zero boundary with $\mathbb{Z}$ coefficients. Certainly it is zero with $\mathbb{Z}/2\mathbb{Z}$ coefficients. We can interpret this to mean that when coefficients $\mathbb{Z}/2\mathbb{Z}$ are chosen, orientation ceases to be meaningful and a compact manifold then has a cycle in top dimension, even though it may have no oriented cycle in top dimension.

2.7 Mayer-Vietoris sequence

The Mayer-Vietoris sequence is often more convenient to use than the relative homology exact sequence and excision. As in the case for de Rham cohomology, it is particularly useful for deducing a property of a union of sets, given the property holds for each component and each intersection.

Theorem 2.27. *Let X be covered by the interiors of subsets $A, B \subset X$. Then we have a canonial long exact sequence of homology groups*

$$\begin{tikzcd}
H_\bullet(A \cap B) \arrow[rr, "\Phi"] & & H_\bullet(A) \oplus H_\bullet(B) \arrow[dl, "\Psi"] \\
& H_\bullet(X) \arrow[ul, "\partial_{-1}"] &
\end{tikzcd}$$

Proof. The usual inclusions induce the following short exact sequence of chain complexes

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A) + C_n(B) \subset C_n(X) \longrightarrow 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$. Why is it exact? $\ker \varphi = 0$ since any chain in $A \cap B$ which is zero as a chain in A or B must be zero. Then $\psi\varphi = 0$, proving that $\mathrm{im}\varphi \subset \ker \psi$. Also, $\ker \psi \subset \mathrm{im}\varphi$, since if $(x, y) \in C_n(A) \oplus C_n(B)$ satisfies $x + y = 0$, then $x = -y$ must be a chain in A and in B, i.e. $x \in C_n(A \cap B)$ and $(x, y) = (x, -x)$ is in $\mathrm{im}\varphi$. Exactness at the final step is by definition of $C_n(A) + C_n(B)$.

The long exact sequence in homology which obtains from this short exact sequence of chain complexes almost gives the result, except it involves the homology groups of the chain complex $C_n(A) + C_n(B)$. We showed in the proof of excision that the inclusion $\iota : C_\bullet(A) + C_\bullet(B) \longrightarrow C_\bullet(X)$ is a deformation retract i.e. we found a subdivision operator ρ such that $\rho \circ \iota = \mathrm{Id}$ and $\mathrm{Id} - \iota \circ \rho = \partial D + D\partial$ for a chain homotopy D. So ι is an isomorphism on homology, and we obtain the result. □

The connecting homomorphism $H_n(X) \longrightarrow H_{n-1}(A \cap B)$ can be described as follows: take a cycle $z \in Z_n(X)$, produce the homologous subdivided cycle $\rho(z) = x + y$ for x, y chains in A, B – these need not be cycles but $\partial x = -\partial y$. $\partial[z]$ is defined to be the class $[\partial x]$.

Often we would like to use Mayer-Vietoris when the interiors of A and B don't cover, but A and B are deformation retracts of neighbourhoods U, V with $U \cap V$ deformation retracting onto $A \cap B$. Then the Five-Lemma implies that the maps $C_n(A) + C_n(B) \longrightarrow C_n(U) + C_n(V)$ are isomorphisms on homology and therefore so are the maps $C_n(A) + C_n(B) \longrightarrow C_n(X)$, giving the Mayer-Vietoris sequence.

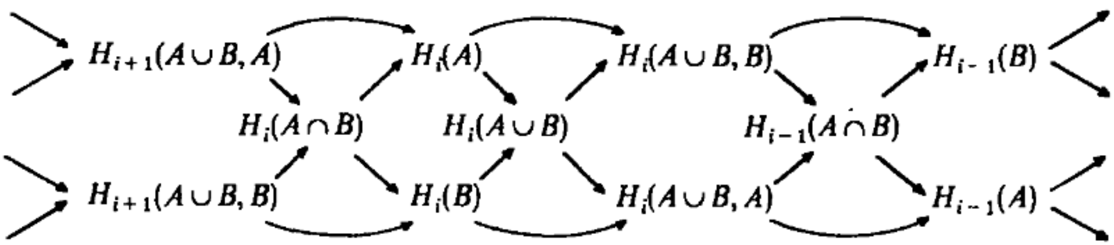

Figure 2: Braid diagram for $A \cap B$ (Bredon)

Example 2.28. *write $S^n = A \cup B$ with A, B the northern and southern closed hemispheres, so that $A \cap B = S^{n-1}$. Then $H_k(A) \oplus H_k(B)$ vanish for $k \neq 0$, and we obtain isos $H_n(S^n) = H_{n-1}(S^{n-1})$.*

Example 2.29. *Write the Klein bottle as the union of two Möbius bands A, B glued by a homeomorphism of their boundary circles. A, B, and $A \cap B$ are homotopy equivalent to circles, and so we obtain by Mayer-Vietoris*

$$0 \longrightarrow H_2(K) \longrightarrow H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \longrightarrow H_1(K) \longrightarrow 0$$

(The sequence ends in zero since the next map $H_0(A \cap B) \longrightarrow H_0(A) \oplus H_0(B)$ is injective.) The map $\Phi : \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is $1 \mapsto (2, -2)$ since the boundary circle wraps twice around the core circle. Φ is injective, so $H_2(K) = 0$ (c.f. orientable surface!) Then we obtain $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ since we can choose $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}(1, 0) + \mathbb{Z}(1, -1)$.

Example 2.30. *Compute homology for $\mathbb{R}P^2$.*

The Mayer-Vietoris sequence can also be deduced from the axioms for homology (the way we did it above used a short exact sequence of chain complexes). Let X be covered by the interiors of A, B. Then by the exactness axiom applied to $(A, A \cap B)$ and $(B, A \cap B)$, we obtain two long exact sequences. Applying the excision axiom to the inclusion $(A, A \cap B) \hookrightarrow (A \cup B, B)$ and similarly for $(B, A \cap B) \hookrightarrow (A \cup B, A)$, we can modify the relative homology groups in the previous sequences to involve $A \cup B$. Then observe that these two sequences combine to form the braid diagram of 4 commuting exact sequences in Figure 2.

By a diagram chase, we then obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_i(A \cap B) \xrightarrow{\Phi = i_*^A \oplus -i_*^B} H_i(A) \oplus H_i(B) \xrightarrow{\Psi = j_*^A + j_*^B} H_i(A \cup B) \xrightarrow{\partial} H_{i-1}(A \cap B) \longrightarrow \cdots$$

Define ∂ by either composition in the braid (they coincide). Check that it's a complex at $H_i(A \cup B)$ and that it is exact. Similar arguments prove exactness at each step.

2.8 Degree

While we will only use the degree of a map $f : S^n \longrightarrow S^n$, the degree of a continuous map of orientable, compact n-manifolds $f : M \longrightarrow N$ is an integer defined as follows: one can show that $H_n(M) = \mathbb{Z}$ for any compact orientable n-manifold and that this is generated by the "fundamental class" which we denote $[M]$. This class would be represented by, for example, the sum of simplices in an oriented triangulation of M. See Chapter 3 of Hatcher for the details.

Definition 19. Let M, N be compact, oriented n-manifolds and $f : M \longrightarrow N$ a continuous map. Then the map $f_* : H_n(M) \longrightarrow H_n(N)$ sends $[M]$ to $d[N]$, for some integer $d = \deg(f)$, which we call the *degree* of f.

Degree is easiest for maps $f : S^n \longrightarrow S^n$, where we showed $H_n(S^n) = \mathbb{Z}$, so that $f_*(\alpha) = d\alpha$, and we then put $\deg(f) = d$. As listed in Hatcher, here are some properties of $\deg f$ for spheres:

- $\deg \mathrm{Id} = 1$
- if f is not surjective, then $\deg f = 0$, since f can be written as a composition $S^n \longrightarrow S^n - \{x_0\} \longrightarrow S^n$ for some point x_0, and $H_n(S^n - \{x_0\}) = 0$.
- If $f \simeq g$, then $\deg f = \deg g$, since $f_* = g_*$. The converse statement follows from $\pi_n(S^n) = \mathbb{Z}$.
- $\deg fg = \deg f \deg g$, since $(fg)_* = f_* g_*$, and hence $\deg f = \pm 1$ if it is a homotopy equivalence.
- a reflection of S^n has $\deg = -1$. A simple way of seeing this is to write S^n as the union of two n-simplices Δ_1, Δ_2 so that $[S^n] = \Delta_1 - \Delta_2$ and the reflection then exchanges Δ_i, acting by -1.
- The antipodal map on S^n, denoted by $-\mathrm{Id}$, has degree $(-1)^{n+1}$, since it is the reflection in all $n+1$ coordinate axes.
- If f has no fixed points, then the line segment from $f(x)$ to $-x$ avoids the origin, so that if we define $g_t(x) = (1-t)f(x) - tx$, then $g_t(x)/|g_t(x)|$ is a homotopy of maps from f to the antipodal map. Hence $\deg(f) = n + 1$.

The degree was historically used to study zeros of vector fields, since for example a sphere around an isolated zero is mapped via the vector field to another sphere of the same dimension (after normalizing the vector field). Hence the degree may be used to assign an integer to any vector field. A related result is the theorem which says you can't comb the hair on a ball flat.

Theorem 2.31. *A nonvanishing continuous vector field may only exist on S^n if n is odd.*

Proof. View the vector field as a map from S^n to itself. If the vector field is nonvanishing, we may normalize it to unit length. Call the resulting map $x \mapsto v(x)$. Then $f_t(x) = \cos(t)x + \sin(t)v(x)$ for $t \in [0, \pi]$ defines a homotopy from Id to the antipodal map $-\mathrm{Id}$. Hence by homotopy invariance of degree, $(-1)^{n+1} = 1$, as required.

To see that odd spheres do have nonvanishing vector fields, view $S^{2n-1} \subset \mathbb{C}^n$, and if ∂_r is the unit radial vector field, then $i\partial_r$ is a vector field of unit length everywhere tangent to S^{2n-1}. $\square$

Recall that when we studied differentiable maps, we defined the degree of a map $f : M^n \longrightarrow N^n$ of n-manifolds where M is compact and N connected; it was defined as $I_2(f, p)$ for a point $p \in N$. Note that this is simply the cardinality mod 2 of the inverse image $f^{-1}(p)$, for p a regular value of f. A similar formula may be used to compute the integer degree of a map (See Bredon for a detailed, but elementary, proof)

Let $f : S^n \longrightarrow S^n$ be a smooth map and $p \in S^n$ a regular value, so that $f^{-1}(p) = \{q_1, \ldots, q_k\}$. Then for each q_i, the derivative gives a map

$$D_{q_i} f : T_{q_i} S^n \longrightarrow T_p S^n,$$

with determinant

$$\det(D_{q_i} f) : \wedge^n T_{q_i} S^n \longrightarrow \wedge^n T_p S^n.$$

Since S^n is orientable, we can choose an identification $\wedge^n T S^n = \mathbb{R}$, and the sign of $\det(D_{q_i} f)$ is independent of this identification.

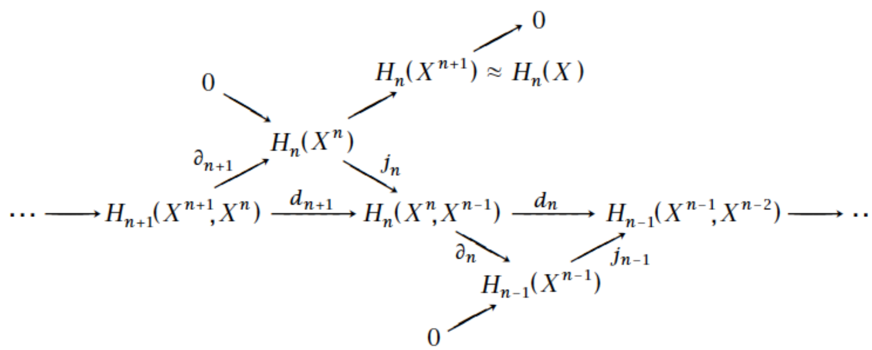

Figure 3: Diagram defining cellular homology (Hatcher)

Theorem 2.32. *With the above hypotheses,*

$$\deg f = \sum_{i=1}^{k} \operatorname{sgn} \det(D_{q_i} f)$$

2.9 Cellular homology

Cellular homology is tailor made for computing homology of cell complexes, based on simple counting of cells and computing degrees of attaching maps. Recall that a cell complex is defined by starting with a discrete set X^0 and inductively attaching n-cells $\{e^n_\alpha\}$ to the n-skeleton X^{n-1}. The weak topology says $A \subset X$ is open if it is open in each of the X^n.

The nice thing about cell complexes is that the boundary map is nicely compatible with the relative homology sequences of the inclusions $X^n \subset X^{n+1}$, and that these are all good pairs.

The relative homology sequence for $X^{n-1} \subset X^n$ is simplified by the fact that $H_k(X^n, X^{n+1})$ vanishes for $k \neq n$, and for $k = n$, X^n/X^{n-1} is a wedge of n-spheres indexed by the n-cells. Since the pair is good, we see

$$H_n(X^n, X^{n-1}) = \text{free abelian group on } n\text{-cells}$$

Then by the long exact sequence in relative homology for this pair (n fixed!), namely

$$H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1})$$

we see that $H_k(X^{n-1})$ is isomorphic to $H_k(X^n)$ for all $k > n$. Hence we can let n drop down to zero, and we obtain $H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) = 0$. Hence

$$H_k(X^n) = 0 \ \ \forall k > n.$$

Finally we observe using the same sequence but letting n increase, that if $n > k$ then

$$H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \ \ \forall n > k.$$

In particular, if X is finite dimensional then we see $H_k(X^n)$ computes $H_k(X)$ for any $n > k$. See Hatcher for a proof of this fact for X infinite dimensional.

Now we combine the long exact sequences for $(X^{n-1}, X^{n-2}), (X^n, X^{n-1})$, and (X^{n+1}, X^n) to form the diagram in Figure 3, where d_i are defined by the composition of the boundary and inclusion maps. clearly $d^2 = 0$. This chain complex, i.e.

$$C^{CW}_n(X) := H_n(X^n, X^{n-1}),$$

fashioned from the relative homologies (which are free abelian groups, recall) of the successive skeleta, is the *cellular chain complex* and its homology is $H^{CW}_\bullet(X)$, the cellular homology.

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\approx"'] \arrow[d, "\Phi_{\alpha *}"] & \widetilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\approx"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\approx"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

Proof.

Figure 4: Diagram computing differential d_n in terms of degree(Hatcher)

Theorem 2.33. $H_n^{CW}(X) \cong H_n(X)$.

Proof. From the diagram, we see that $H_n(X) = \frac{H_n(X^n)}{\mathrm{im}\partial_{n+1}} = \frac{\mathrm{im} j_n = \ker d_n}{\mathrm{im} j \partial_{n+1} = \mathrm{im} d_{n+1}} = H_n^{CW}(X)$ □

We can immediately conclude, for example, that if we have no k-cells, then $H_k(X) = 0$. Or, similarly, if no two cells are adjacent in dimension, then $H_\bullet(X)$ is free on the cells.

Example 2.34. *Recall that $\mathbb{C}P^n$ is a cell complex*

$$\mathbb{C}P^n = e^0 \sqcup e^2 \sqcup \cdots \sqcup e^{2n},$$

so that $H_\bullet(\mathbb{C}P^n) = \mathbb{Z}\ 0\ \mathbb{Z}\ 0\ \mathbb{Z}\ \cdots \mathbb{Z}$.

For more sophisticated calculations, we need an explicit description of the differential d_n in the cell complex. Essentially it just measures how many times the attaching map wraps around its target cycle.

Proposition 2.35 (Cellular differential)**.** *Let e^n_α and e^{n-1}_β be cells in adjacent dimension, and let ϕ_α be the attaching map $S^{n-1}_\alpha \longrightarrow X^{n-1}$ for e^n_α. Also we have the canonical collapsing $\pi : X^{n-1} \longrightarrow X^{n-1}/(X^{n-1} - e^{n-1}_\beta) \cong S^{n-1}_\beta$. Let $d_{\alpha\beta}$ be the degree of the composition*

$$\Delta_{\alpha\beta} : S^{n-1}_\alpha \xrightarrow{\phi_\alpha} X^{n-1} \xrightarrow{\pi} S^{n-1}_\beta.$$

Then

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta.$$

In Figure 4, we see the lower left triangle defines d_n. To determine $d_n(e^n_\alpha)$, take $[e^n_\alpha] \in H_n(D^n_\alpha, \partial D^n_\alpha)$ on the top left, which is sent to the basis element corresponding to e^n_α by Φ_α (the characteristic inclusion map, with associated attaching map φ_α), and we use excision/good pairs to identify its image in $H_{n-1}(X^{n-1}, X^{n-2})$ with the image by the quotient projection q to $\tilde{H}_{n-1}(X^{n-1}/X^{n-2})$. Then the further quotient map $q_\beta : X^{n-1}/X^{n-2} \longrightarrow S^{n-1}_\beta$ collapses the complement of e^{n-1}_β to a point, so it picks out the coefficient we need, which then by the commutativity of the diagram is the degree of $\Delta_{\alpha\beta}$, as required. □

Example 2.36 (orientable genus g 2-manifold)**.** *If M_g is a compact orientable surface of genus g, with usual CW complex with 1 0-cell, 2g 1-cells and 1 2-cell whose attaching map sends the boundary circle to the concatenated path $[a_1, b_1] \cdots [a_g, b_g]$, we have the chain complex*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where $d_2(e^2) = 0$, since for example the coefficient for a_1 would be $+1 - 1 = 0$ sincea_1 appears twice with opposite signs in the boundary, hence we would be measuring the degree of a map which goes once around the circle and then once in the opposite direction around the same circle - such a map is homotopic to the constant map, and has degree 0. Hence $d_2 = 0$. The differential d_1 is also zero. Hence the chain complex is exactly the same as the homology itself: $H_\bullet(M_g) = [\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}]$.

Example 2.37 (nonorientable surface, genus g). *Similarly, for a nonorientable surface N_g, we may choose a cell structure with one 0-cell, g 1-cells and one 2-cells attached via $a_1^2 \cdots a_g^2$ Hence $d_2 : \mathbb{Z} \longrightarrow \mathbb{Z}^g$ is given by $1 \mapsto (2, \cdots, 2)$. Hence d_2 is injective and $H_2(N_g) = 0$. Choosing $(1, \cdots, 1)$ as a basis element, we see immediately that $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Example 2.38 (Real projective space). *Recall that the cell complex structure on $\mathbb{R}P^n$ may be viewed as attaching $\mathbb{R}^n \cong D^n$ to the $\mathbb{R}P^{n-1}$ at infinity via the attaching map $S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$ given by the canonical projection (view $S^{n-1} \subset \mathbb{R}^n$, whereas $\mathbb{R}P^{n-1}$ is the lines through 0 in $\mathbb{R}^n$).*

The chain complex is $\mathbb{Z}$ in each degree from 0 to n. The differential is given by computing the degree of the map $S^{n-1} \longrightarrow \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}$. This map may be factored via $S^{n-1} \longrightarrow S_+^{n-1} \wedge S_-^{n-1} \xrightarrow{\nu} S^{n-1}$, where $S_\pm^{n-1} = S^{n-1}/D_\pm$, with $D_\pm$ the closed north/south hemisphere and ν given by the identity map on one factor and the antipodal map on the other (which is which depends on the choice of identification of $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}$ with S^{n-1}. Hence $\nu_* : (1, 1) \mapsto 1 + (-1)^n$, and we have $d_k = 1 + (-1)^k$, alternating between $0, 2$. It follows that

$$H_k(\mathbb{R}P^n) = [\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \cdots, \mathbb{Z}_2, 0] \text{ for } n \text{ even}$$
$$H_k(\mathbb{R}P^n) = [\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \cdots, \mathbb{Z}_2, \mathbb{Z}] \text{ for } n \text{ odd}$$

Note that with $\mathbb{Z}_2$ coefficients we have $H_k(\mathbb{R}P^n) = \mathbb{Z}_2$ for all $n, 0 \le k \le n$.

One can show that for a compact, connected n-manifold, it is orientable if and only if its n-th homology is $\mathbb{Z}$; otherwise it vanishes. So we see from the previous computation that $\mathbb{R}P^{2n-1}$ is orientable; this is easy to see as follows: If A is the antipodal map $x \mapsto -x$ on $\mathbb{R}^{2n}$, and if $v \in \Omega^{2n}(\mathbb{R}^{2n})$ is the standard Euclidean volume form, then $A^* v = v$, whereas $A_* X = X$ for $X = \sum_i x^i \frac{\partial}{\partial x_i}$, so that $A^*(i_X v) = i_X v$, showing that $i_X v$ defines a volume form on S^{2n-1} invariant under the antipodal map – hence it descends to a volume form on $\mathbb{R}P^{2n-1}$.

Understanding the homology of projective spaces can help us understand the behaviour of maps on spheres with respect to the antipodal map; for example, an even map f, i.e. satisfying $f \circ A = f$, must have degree zero on an even sphere, since we have $\deg f \cdot (-1)^{n+1} = \deg f$. It needn't have degree zero on an odd sphere, but it must have *even* degree, since f may be expressed as a composition

$$S^n \xrightarrow{\pi} \mathbb{R}P^n \xrightarrow{f'} S^n,$$

and so $\deg f = \deg \pi \cdot \deg f' = 2 \deg f'$ must be even.

We can also understand odd maps: Suppose that $f \circ A = A \circ f$. This means that f induces a map $f' : \mathbb{R}P^n \longrightarrow \mathbb{R}P^n$. We will show that odd maps must have odd degree, a fact which implies the Borsuk-Ulam theorem (we proved this in dimension 2, using the fact that any odd map $S^1 \longrightarrow S^1$ must be homotopically nontrivial, which in this case would follow from having odd degree)

Theorem 2.39. *An odd map $f : S^n \longrightarrow S^n$ must have odd degree.*

Proof. The proof will be to show that f_* is an isomorphism on $H_n(S^n; \mathbb{Z}_2)$. The proof will exploit the fact that while the homology of S^n is empty between $0, n$, the $\mathbb{Z}_2$ homology of $\mathbb{R}P^n$ is $\mathbb{Z}_2$ in every degree k, and we will show f induces an isomorphism on $H_k(\mathbb{R}P^n)$ for all k.

To understand the relationship between the homology of $\mathbb{R}P^n$ and its double cover, we use the transfer sequence relating them (there are transfer sequences for covering spaces in general):

$$0 \longrightarrow C_n(X; \mathbb{Z}_2) \xrightarrow{\tau} C_n(\tilde{X}; \mathbb{Z}_2) \xrightarrow{p_*} C_n(X, \mathbb{Z}_2) \longrightarrow 0$$

Here, τ sends each simplex to the sum of its two lifts to $\tilde{X}$, and p_* is the map induced by the covering map. This is a short exact sequence of chain groups, and induces a "transfer" long exact sequence (coefficients in

$\mathbb{Z}_2$!)

$$0 \longrightarrow H_n(\mathbb{R}P^n) \xrightarrow{\tau_*} H_n(S^n) \xrightarrow{p_*=0} H_n(\mathbb{R}P^n) \xrightarrow{\cong} H_{n-1}(\mathbb{R}P^{n-1}) \longrightarrow 0 \longrightarrow \cdots$$

$$H_1(\mathbb{R}P^n) \xrightarrow{\cong} H_0(\mathbb{R}P^n) \xrightarrow{0} H_0(S^n) \xrightarrow{p_*} H_0(\mathbb{R}P^n) \longrightarrow 0$$

Now we use the fact that f induces a map $\bar{f} : \mathbb{R}P^n \longrightarrow \mathbb{R}P^n$, and $(\bar{f}, f, \bar{f})$ define a chain map of the transfer short exact sequence to itself, and we observe by induction from dimension zero that they induce isomorphisms on the long exact sequences in homology. In particular we obtain that the last map, $f_* : H_n(S^n, \mathbb{Z}_2) \longrightarrow H_n(S^n, \mathbb{Z}_2)$, is an isomorphism. □

As a final comment, we can easily show that the Euler characteristic of a finite cell complex, usually defined as an alternating sum $\chi(X) = \sum_n (-1)^n c_n$ where c_n is the number of n-cells, can be defined purely homologically, and is hence independent of the CW decomposition:

Theorem 2.40.

$$\chi(X) = \sum_n (-1)^n \text{rank } H_n(X),$$

where rank is the number of $\mathbb{Z}$ summands.

Proof. The CW homology gives us short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, where $C_n = H_n(X^n, X^{n-1})$, etc. For such sequences, the alternating sum of ranks is always zero. Summing over n, we obtain the result. □

3 Cohomology

Cohomology $H^\bullet$ is in many ways *dual* to homology, but not (always) literally so. Just as homology, it can be characterized by a set of axioms, but the functor $H^\bullet$ is *contravariant* instead of covariant: it reverses arrows. One can think of cohomology as a theory of functions on chains; as is normally the case, functions may be pulled back in a contravariant way. Functions on spaces have many additional algebraic properties, and this is also true of cohomology: unlike homology, the cohomology forms a graded commutative *ring*, under the *cup product*. Also it acts on the homology via the *cap product*, and this action is usually used to describe Poincaré duality.

Singular cohomology is defined as the homology of the singular cochain complex, which itself is produced by applying the dualization functor $\mathrm{Hom}(-,\mathbb{Z})$ to the singular chain complex:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & C_\bullet(X) \arrow[r] \arrow[d, "\mathrm{Hom}(-{,}\mathbb{Z})"] & H_\bullet(X) \\ & C^\bullet(X) \arrow[r] & H^\bullet(X) \end{tikzcd}$$

This will define cohomology *with integral coefficients*, often written $H^\bullet(X;\mathbb{Z})$; for another abelian group G, we define $H^\bullet(X;G)$ via the dualization functor $\mathrm{Hom}(-,G)$ applied to $C_\bullet(X)$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & C_\bullet(X) \arrow[r] \arrow[d, "\mathrm{Hom}(-{,}G)"] & H_\bullet(X) \\ & C^\bullet(X;G) \arrow[r] & H^\bullet(X;G) \end{tikzcd}$$

will apply a functor, not to homology, but rather start with the singular chain complex $(C_n,\partial) = (C_n(X),\partial)$ and form the cochain complex $(C^n,\delta) = (\mathrm{Hom}(C_n,\mathbb{Z}),\delta=\partial^*)$:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

$$\cdots \longleftarrow C^{n+1} \xleftarrow[\delta]{} C^n \xleftarrow[\delta]{} C^{n-1} \longleftarrow \cdots$$

The coboundary map δ has the following nice description: for $\varphi \in C^n(X;G)$, the coboundary $\delta\varphi$ is the composition

$$C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G,$$

so that for a singular $n+1$-simplex $\sigma : \Delta^{n+1} \longrightarrow X$,

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{n+1}]})$$

One is tempted to think that the cohomology groups are also simply the duals of the homology groups - this is wrong for a simple reason: consider the simple chain complex

$$0 \longleftarrow \mathbb{Z}[0] \xleftarrow{m} \mathbb{Z}[1] \longleftarrow 0,$$

where the brackets denote the grading. Then $H^0 = \mathbb{Z}_m$ and $H^1 = 0$. Dualizing, we obtain

$$0 \longrightarrow \mathbb{Z}[0] \xrightarrow{m} \mathbb{Z}[1] \longrightarrow 0,$$

so that now $H_0 = 0, H_1 = \mathbb{Z}_m$. Clearly the torsion group has moved up one degree when passing to cohomology.

Just as when applying the functor $-\otimes_{\mathbb{Z}} G$, the functor $\mathrm{Hom}(-,G)$ is not well-behaved on exact sequences, and it is necessary to use its "derived functor" $\mathrm{Ext}(-,G)$ to figure out the cohomology.

Theorem 3.1 (universal coefficients for cohomology)**.** *The cohomology groups $H^n(X;G)$ are determined by the split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \xrightarrow{h} \mathrm{Hom}(H_n(X), G) \longrightarrow 0,$$

where h is given by evaluation on a cycle.

Just as for the derived functor $Tor(-, G)$, we can give simple rules which enable us to calculate Ext in all common circumstances. First $\mathrm{Ext}(A \oplus B, G) \cong \mathrm{Ext}(A, G) \oplus \mathrm{Ext}(B, G)$. Also $\mathrm{Ext}(H, G) = 0$ if H is free, and finally $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$. This means, for example, that $\mathrm{Ext}(H, \mathbb{Z})$ is isomorphic to the torsion subgroup of H, when H is finitely generated. As a a result, we have the following formalization of our earlier observation about torsion moving up one degree:

Proposition 3.2. *Assuming $H_n(X)$ and $H_{n-1}(X)$ are finitely generated, then $H^n(X) \cong (H_n(X)/T_n) \oplus T_{n-1}$, for T_i the torsion in H_i.*

Proposition 3.3 (Field coefficients)**.** *When we take coefficients in a field F, for example $\mathbb{Z}_2$, $\mathbb{Q}$, or $\mathbb{R}, \mathbb{C}$, we get a lot of simplification: the chain groups $\mathrm{Hom}(C_n(X), F)$ are F-modules (vector spaces!) which can be written as $\mathrm{Hom}_F(C_n(X;F), F)$, and the derived functors of Hom_F are zero, so all the Ext groups are zero. Hence we get that* **with field coefficients,** *$H^n(X, F)$* **is precisely dual to** *$H_n(X, F)$.*

The de Rham theorem states that de Rham cohomology $H^k_{dR}(M)$ coincides with the singular cohomology with coefficients in $\mathbb{R}$. Then, the duality relation with homology may be interpreted as the integration pairing: any k-form may be integrated on an oriented k-submanifold, and if this k-form is closed*, then the resulting number will be independent of the homology class of the k-submanifold.*

Example 3.4. *Compute the singular cohomology of $\mathbb{R}P^n$. Note that since $H_k(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 \leq k \leq n$, and since $\mathbb{Z}_2$ is a field, we have $H^k(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2$ as well.*

3.1 Cup product

To define the cup product, we take advantage of the ring structure of $\mathbb{Z}$; we could also use another ring such as $\mathbb{Z}_n$ or $\mathbb{Q}, \mathbb{R}$. Given chains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, define the cup product $\varphi \cup \psi$ so its value on $\sigma : \Delta^{k+l} \longrightarrow X$ is

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0 \cdots v_k]})\psi(\sigma|_{[v_k \cdots v_{k+l}]}).$$

While this product is defined on cochains, it has the following compatibility with δ:

$$\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup \delta\psi,$$

i.e. δ is a graded derivation of the cup product on cochains. Hence the product descends to cohomology, rendering $H^\bullet(M;\mathbb{Z})$ into a graded ring.

The cup product corresponds to the product on de Rham cohomology induced by wedge product of differential forms. This correspondence is part of de Rham's theorem, stating that the de Rham cohomology is isomorphic to the singular cohomology, but with coefficients in $\mathbb{R}$. Also, for spaces satisfying Poincaré duality, the cup product induces a product on homology called the *intersection product*, which correctly computes transverse geometric intersections, weighted by orientation (for example, if we take coefficients in $\mathbb{Z}_2$, the intersection product of submanifolds of complementary dimension recovers our definition last semester of the intersection number mod 2).

Sometimes this ring structure on $H^\bullet(M, \mathbb{Z})$ can be used to distinguish spaces which may have the same cohomology groups. For example, $\mathbb{C}P^2$ and $S^2 \wedge S^4$ have the same cohomology groups $[\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}]$ but in $\mathbb{C}P^2$ the degree 2 generator squares to a degree 4 generator, whereas in $S^2 \wedge S^4$ it squares to zero.

Example 3.5. *For an oriented genus g surface, we have*

$$H^\bullet(\Sigma_g, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}],$$

with the first cohomology generated by $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ which are a dual basis to the usual choice of basis for the first homology. Check that $\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0$ and $\alpha_i \cup \beta_j = \delta_{ij} v$, where v is the generator of $H^2(\Sigma_g, \mathbb{Z})$. Notice that this defines a nondegenerate skew-symmetric inner product on $H^1(\Sigma_g, \mathbb{Z})$, making it a symplectic vector space.

Example 3.6. *For $\mathbb{R}P^2$, if we compute the cup product on $H^\bullet(\mathbb{R}P^2, \mathbb{Z}_2)$, we get that the generator in degree 1 squares to the generator in degree 2. Hence we have*

$$H^\bullet(\mathbb{R}P^2; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^3),$$

where we view the polynomial ring as a graded ring with the element x having degree 1.

In fact, this is a general occurrence; one can show

$$H^\bullet(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1}),$$

and for the infinite projective space we have $H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[x]$.

A similar calculation gives the cohomology rings for complex projective space:

$$H^\bullet(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}),$$

but in this case, x has degree 2, and is dual to a $\mathbb{C}P^1 \subset \mathbb{C}P^n$. Similarly, one can show $H^\bullet(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}[x]$.

3.2 Contravariance of cohomology

The contravariance of cohomology is often useful from the point of view of moduli theory; we can give a brief explanation here. Any real projective space $\mathbb{R}P^n$ has a natural line bundle on it, where the line above a point is the line represented by the point itself. Call this line bundle U.

Given any map $\phi : X \longrightarrow \mathbb{R}P^n$, we may pull back the bundle U, and get a line bundle $\phi^* U \longrightarrow X$ over X. In fact, allowing arbitrarily high n, any real line bundle may be obtained this way, and homotopic maps $\phi \simeq \phi'$ yield isomorphic line bundles and vice versa.

The cohomology of $\mathbb{R}P^\infty$ is generated as a ring by the element x in the previous example. Therefore, given any line bundle L over X, represent it by $\phi_L : X \longrightarrow \mathbb{R}P^\infty$, and then we may pull back x, yielding $\phi_L^* x \in H^2(X; \mathbb{Z}_2)$. This is a cohomology class on X which depends on the isomorphism class of L; it is called the *first Stiefel-Whitney characteristic class* of L, and it actually classifies real line bundles up to isomorphism.

Similarly, complex line bundles may be described as maps $\phi : X \longrightarrow \mathbb{C}P^\infty$, and therefore they have a characteristic class $\phi^* x \in H^2(X, \mathbb{Z})$, which is called the *first Chern class* of the complex line bundle; this cohomology class completely classifies complex line bundles.

The theory of characteristic classes provides a means of characterizing some of the topology of vector bundles in terms of cohomology of the base manifold.

3.3 Cap product

The cap product is an operation of $H^\bullet(X, \mathbb{Z})$ on $H_\bullet(X, \mathbb{Z})$, making the latter a right module over the cohomology ring. It is defined, for $\sigma : \Delta^k \longrightarrow X$ and $\varphi \in C^l(X; \mathbb{Z})$ via

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0, \cdots, v_l]}) \sigma|_{[v_l, \cdots, v_k]}.$$

The functoriality of this operation for maps $f : X \longrightarrow Y$ may be written as

$$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^* \varphi).$$

Using the cap product, we can state Poincaré duality.

Theorem 3.7 (Poincaré duality)**.** *If M is a compact orientable n-manifold, then the map $H^k(M; \mathbb{Z}) \longrightarrow H_{n-k}(M; \mathbb{Z})$ given by capping against the fundamental class is an isomorphism for all k.*

Often one uses this theorem for $\mathbb{R}$ coefficients, in which case $H_{n-k} = H^{n-k}$. In this case, if we are working on a smooth manifold, de Rham's theorem gives $H^{n-k} = H^{n-k}_{dR}$, and then Poincaré plus de Rham allows us to write the cap map as

$$H^k_{dR}(M) \longrightarrow H^{n-k}_{dR}(M).$$

This map, in the presence of a Riemannian metric, is given by the Hodge star operator on differential forms $\star : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$.

These are only some of the many topics we leave for future courses.