**Exercise 1.** Let $\Gamma$ be a discrete group (a group with a countable number of elements, each one of which is an open set). Show (easy) that $\Gamma$ is a zero-dimensional Lie group. Suppose that $\Gamma$ acts smoothly on a manifold $\tilde{M}$, meaning that the action map

$$\begin{aligned} \theta : & \Gamma \times \tilde{M} \longrightarrow \tilde{M} \\ & (h, x) \mapsto h \cdot x \end{aligned}$$

is $C^\infty$. Suppose also that the action is *free* , i.e. that the only group element with a fixed point is the identity. Finally suppose the action is *properly discontinuous*, meaning that the following conditions hold:

i) Each $x \in \tilde{M}$ has a nbhd $U$ s.t. $\{h \in \Gamma \ : \ (h \cdot U) \cap U \neq \emptyset\}$ is finite.

ii) If $x, y \in \tilde{M}$ are not in the same orbit, then there are nbhds $U, V$ containing $x, y$ respectively, such that $U \cap (\Gamma \cdot V) = \emptyset$.

**Then show the following:**

a) Show that the quotient topological space $M = \tilde{M}/\Gamma$ is Hausdorff and has a countable basis of open sets.

b) Show that $M$ naturally inherits the structure of a smooth manifold.

c) Let $\tilde{M} = S^n$ and Let $\Gamma = \mathbb{Z}/2\mathbb{Z}$ act via $h \cdot x = -x$, where $h$ is the generator of $\Gamma$. Show that the hypotheses above are satisfied, and identify the resulting quotient manifold.

d) Let $\tilde{M} = \mathbb{C}^n - \{0\}$ and let the generator of $\Gamma = \mathbb{Z}$ act via $x \mapsto 2x$, for $x \in \tilde{M}$. Verify the hypotheses above and show the quotient manifold is diffeomorphic to $S^{2n-1} \times S^1$.

d) Show that any discrete subgroup of a Lie group $G$ acts freely and properly discontinuously on $G$ by left multiplication.

e) Show that the unipotent $3 \times 3$ matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \ : \ x, y, z \in \mathbb{R} \right\}$$

form a Lie group $G$, and that the unipotent matrices with integer entries forms a discrete subgroup $\Gamma$. Show that $G/\Gamma$ is a compact smooth 3-dimensional manifold, where $\Gamma$ acts by left multiplication.

**Exercise 2.** Show that the orthogonal, unitary and symplectic groups $O(n, \mathbb{R}), U(n)$ and $Sp(2n, \mathbb{R})$ are smooth manifolds and compute their dimension. Is $U(n)$ diffeomorphic to $SU(n) \times S^1$? Are they isomorphic as groups?

**Exercise 3.** Determine the natural smooth manifold structure in each case, and establish which of the following are diffeomorphic:

- The intersection of the sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ in $\mathbb{C}^3$ with the complex cone $z_1^2 + z_2^2 + z_3^2 = 0$.
- The unit tangent vectors to the 2-sphere in $\mathbb{R}^3$
- The Lie group $SO(3, \mathbb{R})$ of $3 \times 3$ real special orthogonal matrices, i.e. $SO(3, \mathbb{R}) = \{T \in SL(3, \mathbb{R}) \ : \ TT^\top = 1\}$
- The solid unit 3-ball with antipodal boundary points identified
- $\mathbb{R}P^3$

**Exercise 4.** Prove that $S^2, S^1 \times S^1, S^3, S^1 \times S^2$, and $\mathbb{R}P^3$ are each null-cobordant, i.e. cobordant with the empty set. Prove that if $M_i$ are compact smooth manifolds, then $M_1 \sqcup M_2$ is cobordant with the connect sum $M_1 \sharp M_2$.

**Exercise 5.** Prove that a smooth manifold with boundary is the disjoint union of its interior and its boundary. I.e. show that $M = \text{Int}(M) \sqcup \partial M$.