**Exercise 1.** Let $M, N$ be compact manifolds with boundary, and let $\psi : \partial M \longrightarrow \partial N$ be a diffeomorphism. Explain how to define a smooth structure on $M \sqcup_\psi N = M \sqcup N / \sim$, where $x \sim y$ iff $\psi(x) = y$ or $x = y$. Is the manifold resulting from your procedure uniquely specified (up to diffeomorphism) by the data $(M, N, \psi)$ provided above? Give a simple example where the diffeomorphism class of $M \sqcup_\psi N$ depends on $\psi$.

**Exercise 2.** Let $f : M \longrightarrow M$ be a smooth map and suppose $p$ is a fixed point under $f$, i.e. $f(p) = p$. The point $p$ is called a *Lefschetz fixed point* when the derivative map $Df(p) : T_pM \longrightarrow T_pM$ does not have $+1$ as an eigenvalue.

Show that if $M$ is compact and all fixed points for $f$ are Lefschetz, then there are only finitely many fixed points for $f$.

**Exercise 3.** Prove that there are no smooth functions on a compact manifold $M$ without critical points.

**Exercise 4.** A *Morse function* on a manifold $M$ is a real-valued function all of whose critical points are nondegenerate, in the sense that the Hessian matrix at every critical point $p$ is nondegenerate (this is independent of which chart is used to compute the Hessian).

*Note: Morse functions are important because, while they are not regular everywhere, they do have a local classification near each critical point - look up the "Morse Lemma" if you are interested, it says that near each critical point there is a coordinate chart for which $f = \sum_i \pm x_i^2$.*

Show that if $U \subset \mathbb{R}^n$ is an open set and $f : U \longrightarrow \mathbb{R}$ is a smooth function, then for almost all $n$-tuples $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the modified function

$$f_a = f + \sum_{i=1}^{n} a_i x_i$$

is a Morse function. (Note: "almost all" means that the set of $a$ for which $f_a$ fails to be Morse is of measure zero in $\mathbb{R}^n$.)

**Exercise 5** (Stability of Morse functions)**.** Let $f_0$ be a Morse function on a compact manifold $M$, and suppose that $f : (-1,1) \times M \longrightarrow \mathbb{R}$ is a smooth function with $f|_{\{0\}\times M} = f_0$. Show that for $\epsilon$ sufficiently small, $f|_{\{\epsilon\}\times M}$ is also Morse.

*(Intuitively, we think of $f$ as giving a smooth family of maps $M \longrightarrow \mathbb{R}$, parametrized by $t \in (-1, 1)$.*

**Exercise 6.** Consider the function $\mathbb{C}^2 \longrightarrow \mathbb{C}$ given by $f(z_1, z_2) = z_1^p + z_2^q$, for integers $p, q$ which are relatively prime and $\geq 2$. Describe the regular and critical points and values of this map.

Show that the intersection $K = f^{-1}(0) \cap S^3$ is transverse, where we view $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$, and identify the manifold $K$. By considering the 2-tori $\{(z_1, z_2) : |z_1| = c_1 \text{ and } |z_2| = c_2\}$ for constants $c_1, c_2$, describe the way in which $K$ is embedded in $S^3$, perhaps including a diagram.