Convince yourself that you understand the classification of 1-dimensional manifolds: namely, if the manifold M is connected and 1-dimensional, then it is diffeomorphic either to $[0,1]$, $(0,1]$, $(0,1)$, or S^1.

Exercise 1. Let X_1, X_2 be sets of n_1, n_2 points respectively. What conditions on n_1, n_2 are there which guarantee that there exists a (compact!) cobordism between X_1, X_2? How many labeled cobordisms are there between X_1, X_2 up to equivalence? Assume each connected component of the cobordism has non-empty boundary.

A labeled cobordism between X and $\emptyset$ is a manifold Y (with boundary), together with a labeling diffeomorphism $\ell_Y : \partial Y \longrightarrow X$. Pairs $(Y, \ell_Y), (Y', \ell_{Y'})$ of labeled cobordisms from X to $\emptyset$ are equivalent when there is a diffeomorphism $\psi : Y \longrightarrow Y'$ such that $\ell_Y = \ell_{Y'} \circ \partial\psi$. You are asked to count equivalence classes of cobordisms labeled by the set $X_1 \sqcup X_2$, excluding circles.

Exercise 2. A smooth map $f : X \longrightarrow Y$ is said to be transverse to a regular submanifold $Z \subset Y$ when f is transverse to the inclusion map $\iota : Z \subset Y$. Assuming X is compact and Z closed, show that the transversality of f to Z is stable under perturbations of f.

Exercise 3. Let $f : M \longrightarrow N$ be a smooth map of manifolds with the same dimension, and suppose M is compact. Let $\iota : [0,1] \longrightarrow N$ be an embedding such that both ι and $\partial\iota$ are transverse to f. Show first that $f^{-1}(\iota(0))$ and $f^{-1}(\iota(1))$ are finite sets and second that $\sharp(f^{-1}(\iota(0))) \equiv \sharp(f^{-1}(\iota(1))) \pmod 2$.

For any $m \equiv n \pmod 2$, give an example of a map $f : S^1 \longrightarrow S^1$ such that $f^{-1}(1)$ has n elements and $f^{-1}(-1)$ has m elements, and give an example of an embedding ι as above with $\iota(0) = 1$ and $\iota(1) = -1$.

Exercise 4. Show that having an isolated zero of a smooth function $f : T^2 \longrightarrow \mathbb{R}$ is not stable under perturbations of f. Show also that if a function $g : T^2 \longrightarrow \mathbb{R}^2$ has only regular zeros, then there must be a finite even number of them. Give an example of such a function with 4 zeros.

Exercise 5. Show that a compact n-manifold M cannot be embedded in $\mathbb{R}^n$. Can it be immersed? Can it be submerged?

Exercise 6. Show using the Brouwer fixed point theorem that any $n \times n$ matrix A with non-negative real entries has a non-negative eigenvalue.

Exercise 7. Show that the fixed point in the Brouwer fixed point theorem need not be an interior point.

Exercise 8. Let X be a manifold with boundary and $x \in \partial X$ be a boundary point. Show there exists a smooth non-negative function on some neighbourhood U of x with 0 as a regular value and $f^{-1}(0) = \partial U$. Then show that there exists a smooth non-negative function F on all of X with 0 a regular value and such that $F^{-1}(0) = \partial X$. Hint: in going from local to global, a partition of unity is often useful.