Exercise 1. Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

where (r, ϕ, θ) are standard Cartesian coordinates on $\mathbb{R}^3$.

- Show that φ is a diffeomorphism onto its image when restricted to $U = \{(r, \phi, \theta) \ : \ 0 < r, 0 < \phi < \pi, 0 < \theta < 2\pi\}$.
- Compute $\varphi^* dx, \varphi^* dy, \varphi^* dz$ where (x, y, z) are Cartesian coordinates for $\mathbb{R}^3$.
- Compute $\varphi^*(dx \wedge dy \wedge dz)$.
- For any vector field X, define ι_X to be the unique degree -1 (i.e. it reduces degree by 1) derivation (i.e. $\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \iota_X(\beta)$) of the algebra of differential forms such that $i_X(f) = 0$ and $i_X df = X(f)$ for $f \in \Omega^0(M)$. Compute the integral

  $$\int_{S^2_r} \iota_X(dx \wedge dy \wedge dz),$$

  for the vector field $X = \varphi_* \frac{\partial}{\partial r}$, where S^2_r is the sphere of radius r.

Exercise 2. Let z be the standard complex coordinate on $\mathbb{C}$, i.e. $z = x + iy$, and form the complex differential form $\frac{dz}{z}$. Where is this well-defined? Decompose it into real and imaginary parts. Are these closed forms? Compute the integrals

$$\int_{S^1} \iota^* \mu,$$

for $\iota : S^1 \longrightarrow \mathbb{C}$ the standard inclusion and μ the real or imaginary part of $\frac{dz}{z}$. What consequence does Stokes' theorem have in this context, for each part of $\frac{dz}{z}$?

Exercise 3. Let $f_0 : M \longrightarrow N$, $f_1 : M \longrightarrow N$ be smoothly homotopic maps, i.e. there exists a smooth map

$$h : M \times \mathbb{R} \longrightarrow N$$

such that $h(x, i) = f_i(x)$ for $i = 0, 1$. Then show that $(f_1^* - f_0^*)\alpha$ is exact when α is closed.

Exercise 4. Let $\{U_i\}$, $i = 1, \ldots, N$ be a finite cover of a compact, oriented n-manifold M, and let $\alpha \in \Omega^n(M)$. Express $\int_M \alpha$ in terms of the integrals

$$\int_{U_{i_1} \cap \cdots \cap U_{i_k}} \alpha$$

for k ranging from 1 to N.

Exercise 5. Compute all these de Rham cohomology groups for all degrees.

- What is the de Rham cohomology of $\mathbb{R}^3 - \{p_1 \cup \cdots \cup p_k\}$ where p_i are a collection of k distinct points?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup \cdots \cup l_m\}$ where l_i are a collection of m non-intersecting lines?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{p_1 \cup \cdots \cup p_k \cup l_1 \cup \cdots \cup l_m\}$, assuming no p_i lies on a l_j, $\{p_i\}$ are distinct, and $\{l_j\}$ are non-intersecting?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup l_2\}$, assuming that l_1 intersects l_2 in exactly one point?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup \cdots \cup l_m\}$, assuming that all the lines intersect the origin but are distinct?

- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup l_2 \cup l_3\}$, assuming the lines intersect in exactly three distinct points?
- What is the de Rham cohomology of $\mathbb{R}^n - \{X_i\}$, where X_i is a i-dimensional linear subspace?

Note: The preceding question is similar to John Nash's blackboard question in the movie "A Beautiful Mind".

Exercise 6. Let $N = T^*M$ be the total space of the cotangent bundle of a smooth manifold M, and let $\pi : N \longrightarrow M$ be the usual bundle projection. We now describe a natural 1-form $\theta \in \Omega^1(N)$. At each point $p = (x, \xi) \in N$ (here $x \in M$ is a point and $\xi \in T_x^*M$ is a covector at x), the 1-form takes the following value on a vector $V \in T_pN$:

$$\theta(V) = \xi(\pi_* V).$$

i) Choosing coordinates $(x^1, ..., x^n)$ for an open set U containing x and using coordinates $(x^1, \ldots, x^n, \xi_1, \ldots \xi_n)$ to represent the points $(x \in U, \xi = \sum \xi_i dx^i) \in T^*U$, write the coordinate expression of θ and verify that it is smooth.

ii) Compute $\omega = d\theta \in \Omega^2(N)$. View the result as a smooth family of skew-symmetric 2-forms on N. Compute the rank of this 2-form at the point $p = (x, \xi)$ (i.e. if we write $\omega = \sum \omega_{ij} dx^i \wedge dx^j$, the rank is the rank of the matrix ω_{ij}).

iii) Let $\mu \in \Omega^1(M)$ be a 1-form on M, and view it as a smooth section $\mu : M \longrightarrow T^*M$ of the cotangent bundle. Therefore it defines a smooth map $\mu : M \longrightarrow N$. Compute the pullbacks $\mu^*(\theta) \in \Omega^1(M)$ and $\mu^*(\omega) \in \Omega^2(M)$ as a function of μ.

iv) Just as a natural 1-form θ was defined on T^*M, define a natural k-form $\theta \in \Omega^k(N_k)$ on the total space of the bundle $N_k = \wedge^k T^*M$. If $\mu \in \Omega^k(M)$, view it as a smooth map $\mu : M \longrightarrow N_k$ and compute $\mu^*(d\theta)$. Does all this work for $k = 0$?