You may only quote theorems we have covered in the class so far.

Question 1. Let $\mu \in \Omega^1(M,\mathbb{R})$ be a 1-form on a smooth manifold M such that $d\mu = 0$ and $\mu(p) \neq 0$ for some $p \in M$. Show there exists a neighbourhood U of p and coordinates $(x_1,\dots,x_n)$ in this neighbourhood such that $\mu = dx_1$ in U.

Question 2. Consider the differential 2-form on $\mathbb{R}^3$ given by

$$\rho = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy,$$

and let $\iota : S^2 \hookrightarrow \mathbb{R}^3$ be the usual inclusion of the unit sphere. Show that $\iota^*\rho$ determines an orientation on S^2, and compute $\int_{S^2} \iota^*\rho$ with respect to this orientation.

Question 3. Computations of de Rham cohomology: justify all computations.

i) Determine the de Rham cohomology groups of $D^2\backslash\{(0,0)\}$, where D^2 is the open unit disc in $\mathbb{R}^2$. Give differential forms representing a basis for each cohomology group. Which of these forms can be chosen to have compact support[1]?

ii) Using Mayer-Vietoris, determine the de Rham cohomology groups of $D^2\backslash\{p_1,\dots,p_k\}$, for $\{p_i\}$ distinct points in D^2.

iii) Determine the de Rham cohomology groups of $\mathbb{R}^3\backslash Z$, where Z is the union of three distinct rays emanating from the origin.

iv) Using Mayer-Vietoris, determine the de Rham cohomology groups of Σ_g, the compact orientable surface of genus g.

Question 4. Computations of fundamental groups: give justification for all computations.

i) S^{n-1} is naturally included in S^n via the inclusion of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$; by composition this defines a natural inclusion $S^1 \subset S^3$. Compute $\pi_1(S^3\backslash S^1)$. Hint: write $S^3 = \mathbb{R}^3 \sqcup \{\infty\}$.

ii) By composing the above inclusions we also have $S^m \subset S^n$ for $m < n$. Compute $\pi_1(S^n\backslash S^m)$.

iii) The above inclusions induce inclusions $\mathbb{R}P^m \subset \mathbb{R}P^n$ for $m < n$; compute $\pi_1(\mathbb{R}P^n\backslash \mathbb{R}P^m)$.

Question 5. Give an example of a space with fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_4$ (hint: build a cell complex)

Question 6. Give an example of a covering which is not normal (abnormal?)

Bonus question: Compute $\pi_1((S^2 \times S^2)\backslash(S^1 \times S^1))$, where $S^1 \times S^1$ includes via the product of the standard inclusion (as in Q. 4) with itself.

[1] ρ has compact support when the closure of $\{x : \rho(x) \neq 0\}$ is compact.