2.5 Excision implies Simplicial = Singular homology

Recall that simplicial homology was defined in terms of a Δ-complex decomposition of X, via a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$. We then define the chains to be the free abelian group on the n-simplices, i.e. $\Delta_n(X)$. We would like to show that if a Δ-complex structure is chosen, then its simplicial homology coincides with the singular homology of the space X. It will be useful to do this by induction on the k-skeleton X^k consisting of all simplices of dimension k or less, and so we would like to use a relative version of simplicial homology:

Define relative simplicial homology for any sub-Δ-complex $A \subset X$ as usual, using relative chains

$$\Delta_n(X, A) = \frac{\Delta_n(X)}{\Delta_n(A)},$$

and denote it by $H_n^\Delta(X, A)$.

Theorem 2.23. *Any n-simplex in a Δ-complex decomposition of X may be viewed as a singular n-simplex, hence we have a chain map*

$$\Delta_n(X, A) \longrightarrow C_n(X, A).$$

The induced homomorphism $H_n^\Delta(X, A) \longrightarrow H_n(X, A)$ is an isomorphism. Taking $A = \emptyset$, we obtain the equivalence of absolute singular and simplicial homology.

Lemma 2.24. *The identity map $i_n : \Delta^n \longrightarrow \Delta^n$ is a cycle generating $H_n(\Delta^n, \partial\Delta^n)$.*

Proof. Certainly i_n defines a cycle, and it clearly generates for $n = 0$. We do an induction by relating i_n to i_{n-1} by killing $\Lambda \subset \Delta^n$, the union of all but one $n-1$-dimensional face of Δ^n and considering the triple $(\Delta^n, \partial\Delta^n, \Lambda)$. Since $H_i(\Delta^n, \Lambda) = 0$ by deformation retraction, we get isomorphism

$$H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda).$$

But $(\partial\Delta^n, \Lambda)$ and $(\Delta^{n-1}, \partial\Delta^{n-1})$ are good pairs and hence the relative homologies equal the reduced homology of the quotients, which are *homeomorphic*. Hence we have

$$H_{n-1}(\partial\Delta^n, \Lambda) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

Under the first iso, i_n is sent to ∂i_n which in the relative complex is $\pm i_{n-1}$, so we see that i_n generates iff i_{n-1} generates. $\square$

proof of theorem. First suppose that X is finite dimensional, and $A = \emptyset$. Then the map of simplicial to singular gives a morphism of relative homology long exact sequences:

$$\begin{tikzcd}
H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^k) \arrow[r] \arrow[d] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_{n-1}^\Delta(X^{k-1}) \arrow[d] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

We will show that most of the vertical maps are isos and then deduce the center map is an iso.

First the maps on relative homology: the group $\Delta_n(X^k, X^{k-1})$ is free abelian on the k-simplices, and hence it vanishes for $n \neq k$. Therefore the only nonvanishing homology group is $H_k^\Delta(X^k, X^{k-1})$, which is free abelian on the k-simplices. To compute the singular group $H_n(X^k, X^{k-1})$, consider all the simplices together as a map

$$\Phi : \sqcup_\alpha(\Delta_\alpha^k, \partial\Delta_\alpha^k) \longrightarrow (X^k, X^{k-1})$$

and note that it gives a homeomorphism of quotient spaces. Hence we have

$$\begin{tikzcd} H_\bullet(\Delta^k_\alpha, \partial\Delta^k_\alpha) \arrow[r] \arrow[d, equal] & H_\bullet(X^k, X^{k-1}) \arrow[d] \\ \tilde{H}_\bullet(\Delta^k_\alpha/\partial\Delta^k_\alpha) \arrow[r, equal] & \tilde{H}_\bullet(X^k/X^{k-1}) \end{tikzcd}$$

which shows that the top is an iso. Using the previous lemma which tells us that the generators of $H_\bullet(\Delta^k, \partial\Delta^k)$ are the same as the simplicial generators, we get that the maps

$$H^\Delta_k(X^k, X^{k-1}) \longrightarrow H_k(X^k, X^{k-1})$$

are isomorphisms. The second and fifth vertical maps are isomorphisms by induction, and then by the Five-Lemma, we get the central map is an iso.

What about if X is not finite-dimensional? Use the fact that a compact set in X may only meet finitely many open simplices (i.e. simplices with proper faces deleted) of X (otherwise we would have an infinite sequence (x_i) such that $U_i = X - \cup_{j\neq i}\{x_j\}$ give an open cover of the compact set with no finite subcover.

To prove $H^\Delta_n(X) \longrightarrow H_n(X)$ is surjective, let $[z] \in H_n(X)$ for z a singular n-cycle. It meets only finitely many simplices hence it must be in X^k for some k. But we showed that $H^\Delta_n(X^k) \longrightarrow H_n(X^k)$ is an isomorphism, so this shows that z must be homologous in X^k to a simplicial cycle. For injectivity: if z is a boundary of some chain, this chain must have compact image and lie in some X^k, so that $[z]$ is in the kernel $H^\Delta_n(X^k) \longrightarrow H_n(X)$. But this is an injection, so that z is a simplicial boundary in X^k (and hence in X).

All that remains is the case where $A \neq \emptyset$, which follows by applying the Five-Lemma to both long exact sequences of relative homology, for each of the simplicial and singular homology theories. $\square$

Lemma 2.25 (Five-Lemma)**.** *If $\alpha, \beta, \delta, \epsilon$ are isos in the diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

and the rows are exact sequences, then γ is an iso.

Proof. γ surjective: take $c' \in C'$. Then $k'(c') = \delta(d) = \delta k(c) = k'\gamma(c)$ for some c. Therefore $k'(c' - \gamma(c)) = 0$, which implies $c' - \gamma(c) = j'(b') = j'(\beta(b)) = \gamma j(b)$ for some b, showing that $c' = \gamma(c + j(b))$.

γ injective: $\gamma(c) = 0$ implies $c = j(b)$ for some b with $\beta(b) = i'(a') = i'\alpha(a) = \beta i(a)$ for some a, so that $b = i(a)$, showing that $c = 0$. $\square$

The previous theorem allows us to conclude that for X a Δ-complex with finitely many n-simplices, $H_n(X)$ is finitely generated, and hence it is given by the direct sum of $\mathbb{Z}^{b_n}$ and some finite cyclic groups. b_n is called the n^{th} Betti number, and the finite part of the homology is called the *torsion*.

2.6 Axioms for homology

Eilenberg, Steenrod, and Milnor obtained a system of axioms which characterize homology theories without bothering with simplices and singular chains. Be warned: not all "homology theories" satisfy these axioms precisely: Čech homology fails exactness and Bordism and K-theory fail dimension (without dimension, the homology theory is called *extraordinary*).

If we restrict our attention to Cell complexes (i.e. CW complexes), then singular homology is the *unique* functor up to isomorphism which satisfies these axioms (we won't have time to prove this).

Definition 18. A *homology theory* is a functor H from topological pairs (X, A) to graded abelian groups $H_\bullet(X, A)$ together with a natural transformation $\partial_* : H_p(X, A) \longrightarrow H_{p-1}(A)$ called the *connecting homomorphism*[9] (note that $H_p(A) := H_p(A, \emptyset)$) such that

i) (Homotopy) $f \simeq g \Rightarrow H(f) = H(g)$

ii) (Exactness) For $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, the following is an exact sequence of groups:

$$\begin{tikzcd} H_\bullet(A) \arrow[rr, "i_*=H(i)"] & & H_\bullet(X) \arrow[dl, "j_*=H(j)"] \\ & H_\bullet(X,A) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

iii) (Excision) Given $Z \subset A \subset X$ with $\overline{Z} \subset A^{int}$, the inclusion $k : (X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism
$$H(k) : H_\bullet(X - Z, A - Z) \xrightarrow{\cong} H_\bullet(X, A).$$

iv) (Dimension) For the one-point space $*$, $H_i(*) = 0$ for all $i \neq 0$.

v) (Additivity) H preserves coproducts, i.e. takes arbitrary disjoint unions to direct sums[10].

Finally, the *coefficient group* of the theory is defined to be $G = H_0(*)$.

Note: There are natural shift functors S, s acting on topological pairs and graded abelian groups, respectively, given by $S : (X, A) \mapsto (A, \emptyset)$ and $(s(G_\bullet))_n = G_{n+1}$. The claim that ∂_* is natural is properly phrased as
$$\partial_* : H \Rightarrow s^{-1} \circ H \circ S.$$

Note: If the coefficient group G is not $\mathbb{Z}$, then the theorem mentioned above for CW complexes says that the homology functor must be isomorphic to $H_\bullet(X, A; G)$, singular homology with coefficients in G, meaning that chains consist of linear combinations of simplices with coefficients in G instead of $\mathbb{Z}$.

There is a sense in which homology with coefficients in $\mathbb{Z}$ is more fundamental than homology with coefficients in some other abelian group G. The result which explains this assertion is called the "universal coefficient theorem for homology". Let's describe this briefly, because it is the first example we encounter of a *derived functor*.

The chains $C_n(X, A; G)$ with coefficients in G is naturally isomorphic to the tensor product $C_n(X, A) \otimes_{\mathbb{Z}} G$, and the boundary map is nothing but
$$\partial \otimes \mathrm{Id} : C_n(X, A) \otimes G \longrightarrow C_{n-1}(X, A) \otimes G.$$

So, instead of computing the homology of the chain complex C_n, we are computing the homology of $C_n \otimes G$.

[9]This natural transformation ∂_* is the only remnant of chains, boundary operators, etc. All that is gone, but we retain the categorical notion defined by ∂_*.

[10]recall that coproduct of X_i is the universal object with maps from X_i, whereas the product is the universal object with projections to X_i

Theorem 2.26. *If C is a chain complex of abelian groups, then there are natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0$$

and these sequences split but not naturally.

Here, $\mathrm{Tor}(A,G)$ is an abelian group (always torsion, it turns out) which depends on the abelian groups A, G, and is known as the first derived functor of the functor $A \mapsto A \otimes G$. In particular, the following rules will help us compute the Tor group: $\mathrm{Tor}(A,G) = 0$ if A is free. $\mathrm{Tor}(A_1 \oplus A_2, G) \cong \mathrm{Tor}(A_1, G) \oplus \mathrm{Tor}(A_2, G)$, and most importantly $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \cong \ker(G \xrightarrow{n\cdot} G)$. Clearly under many circumstances $\mathrm{Tor}(H_{n-1}(C), G)$ will vanish and in this case $H_n(C;G) = H_n(C) \otimes G$. For example, although $H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, multiplication by 2 has trivial kernel on $\mathbb{Z}/3\mathbb{Z}$, hence $H_n(\mathbb{R}P^2, \mathbb{Z}/3\mathbb{Z}) = H_n(\mathbb{R}P^2) \otimes \mathbb{Z}/3\mathbb{Z}$. On the other hand, with $\mathbb{Z}/2\mathbb{Z}$ coefficients, $\mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence $H_2(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

If we triangulate $\mathbb{R}P^2$ with two 2-simplices, we can check that the sum of the two 2-simplices can't have zero boundary with $\mathbb{Z}$ coefficients. Certainly it is zero with $\mathbb{Z}/2\mathbb{Z}$ coefficients. We can interpret this to mean that when coefficients $\mathbb{Z}/2\mathbb{Z}$ are chosen, orientation ceases to be meaningful and a compact manifold then has a cycle in top dimension, even though it may have no oriented cycle in top dimension.

2.7 Mayer-Vietoris sequence

The Mayer-Vietoris sequence is often more convenient to use than the relative homology exact sequence and excision. As in the case for de Rham cohomology, it is particularly useful for deducing a property of a union of sets, given the property holds for each component and each intersection.

Theorem 2.27. *Let X be covered by the interiors of subsets $A, B \subset X$. Then we have a canonial long exact sequence of homology groups*

$$\begin{tikzcd} H_\bullet(A \cap B) \arrow[rr, "\Phi"] & & H_\bullet(A) \oplus H_\bullet(B) \arrow[dl, "\Psi"] \\ & H_\bullet(X) \arrow[ul, "\partial_{-1}"] & \end{tikzcd}$$

Proof. The usual inclusions induce the following short exact sequence of chain complexes

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A) + C_n(B) \subset C_n(X) \longrightarrow 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x,y) = x + y$. Why is it exact? $\ker \varphi = 0$ since any chain in $A \cap B$ which is zero as a chain in A or B must be zero. Then $\psi\varphi = 0$, proving that $\mathrm{im}\varphi \subset \ker \psi$. Also, $\ker \psi \subset \mathrm{im}\varphi$, since if $(x,y) \in C_n(A) \oplus C_n(B)$ satisfies $x + y = 0$, then $x = -y$ must be a chain in A and in B, i.e. $x \in C_n(A \cap B)$ and $(x,y) = (x,-x)$ is in $\mathrm{im}\varphi$. Exactness at the final step is by definition of $C_n(A) + C_n(B)$.

The long exact sequence in homology which obtains from this short exact sequence of chain complexes almost gives the result, except it involves the homology groups of the chain complex $C_n(A) + C_n(B)$. We showed in the proof of excision that the inclusion $\iota : C_\bullet(A) + C_\bullet(B) \longrightarrow C_\bullet(X)$ is a deformation retract i.e. we found a subdivision operator ρ such that $\rho \circ \iota = \mathrm{Id}$ and $\mathrm{Id} - \iota \circ \rho = \partial D + D\partial$ for a chain homotopy D. So ι is an isomorphism on homology, and we obtain the result. ☐

The connecting homomorphism $H_n(X) \longrightarrow H_{n-1}(A \cap B)$ can be described as follows: take a cycle $z \in Z_n(X)$, produce the homologous subdivided cycle $\rho(z) = x + y$ for x, y chains in A, B – these need not be cycles but $\partial x = -\partial y$. $\partial[z]$ is defined to be the class $[\partial x]$.

Often we would like to use Mayer-Vietoris when the interiors of A and B don't cover, but A and B are deformation retracts of neighbourhoods U, V with $U \cap V$ deformation retracting onto $A \cap B$. Then the Five-Lemma implies that the maps $C_n(A) + C_n(B) \longrightarrow C_n(U) + C_n(V)$ are isomorphisms on homology and therefore so are the maps $C_n(A) + C_n(B) \longrightarrow C_n(X)$, giving the Mayer-Vietoris sequence.

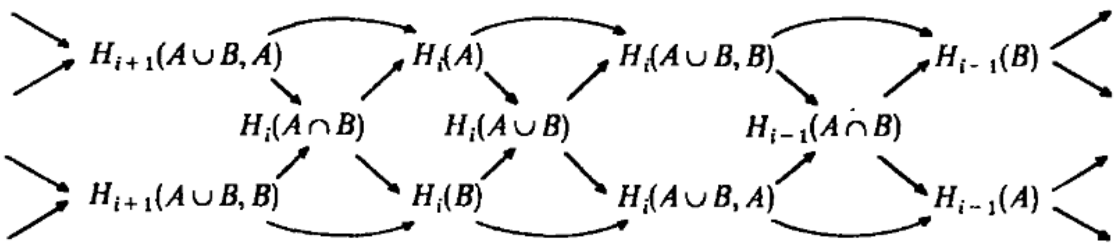

Figure 2: Braid diagram for $A\cap B$ (Bredon)

Example 2.28. *write $S^n = A\cup B$ with A, B the northern and southern closed hemispheres, so that $A\cap B = S^{n-1}$. Then $H_k(A)\oplus H_k(B)$ vanish for $k\neq 0$, and we obtain isos $H_n(S^n) = H_{n-1}(S^{n-1})$.*

Example 2.29. *Write the Klein bottle as the union of two Möbius bands A, B glued by a homeomorphism of their boundary circles. A, B, and $A\cap B$ are homotopy equivalent to circles, and so we obtain by Mayer-Vietoris*

$$0 \longrightarrow H_2(K) \longrightarrow H_1(A\cap B) \xrightarrow{\Phi} H_1(A)\oplus H_1(B) \longrightarrow H_1(K) \longrightarrow 0$$

(The sequence ends in zero since the next map $H_0(A\cap B) \longrightarrow H_0(A)\oplus H_0(B)$ is injective.) The map $\Phi : \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$ is $1\mapsto (2,-2)$ since the boundary circle wraps twice around the core circle. Φ is injective, so $H_2(K) = 0$ (c.f. orientable surface!) Then we obtain $H_1(K)\cong \mathbb{Z}\oplus\mathbb{Z}_2$ since we can choose $\mathbb{Z}\oplus\mathbb{Z} = \mathbb{Z}(1,0)+\mathbb{Z}(1,-1)$.

Example 2.30. *Compute homology for $\mathbb{R}P^2$.*

The Mayer-Vietoris sequence can also be deduced from the axioms for homology (the way we did it above used a short exact sequence of chain complexes). Let X be covered by the interiors of A, B. Then by the exactness axiom applied to $(A, A\cap B)$ and $(B, A\cap B)$, we obtain two long exact sequences. Applying the excision axiom to the inclusion $(A, A\cap B)\hookrightarrow (A\cup B, B)$ and similarly for $(B, A\cap B)\hookrightarrow (A\cup B, A)$, we can modify the relative homology groups in the previous sequences to involve $A\cup B$. Then observe that these two sequences combine to form the braid diagram of 4 commuting exact sequences in Figure 2.

By a diagram chase, we then obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_i(A\cap B) \xrightarrow{\Phi = i_*^A \oplus -i_*^B} H_i(A)\oplus H_i(B) \xrightarrow{\Psi = j_*^A + j_*^B} H_i(A\cup B) \xrightarrow{\partial} H_{i-1}(A\cap B) \longrightarrow \cdots$$

Define ∂ by either composition in the braid (they coincide). Check that it's a complex at $H_i(A\cup B)$ and that it is exact. Similar arguments prove exactness at each step.

2.8 Degree

While we will only use the degree of a map $f : S^n \longrightarrow S^n$, the degree of a continuous map of orientable, compact n-manifolds $f : M \longrightarrow N$ is an integer defined as follows: one can show that $H_n(M) = \mathbb{Z}$ for any compact orientable n-manifold and that this is generated by the "fundamental class" which we denote $[M]$. This class would be represented by, for example, the sum of simplices in an oriented triangulation of M. See Chapter 3 of Hatcher for the details.

Definition 19. Let M, N be compact, oriented n-manifolds and $f : M \longrightarrow N$ a continuous map. Then the map $f_* : H_n(M) \longrightarrow H_n(N)$ sends $[M]$ to $d[N]$, for some integer $d = \deg(f)$, which we call the *degree* of f.

Degree is easiest for maps $f : S^n \longrightarrow S^n$, where we showed $H_n(S^n) = \mathbb{Z}$, so that $f_*(\alpha) = d\alpha$, and we then put $\deg(f) = d$. As listed in Hatcher, here are some properties of $\deg f$ for spheres:

- $\deg \mathrm{Id} = 1$
- if f is not surjective, then $\deg f = 0$, since f can be written as a composition $S^n \longrightarrow S^n - \{x_0\} \longrightarrow S^n$ for some point x_0, and $H_n(S^n - \{x_0\}) = 0$.
- If $f \simeq g$, then $\deg f = \deg g$, since $f_* = g_*$. The converse statement follows from $\pi_n(S^n) = \mathbb{Z}$.
- $\deg fg = \deg f \deg g$, since $(fg)_* = f_* g_*$, and hence $\deg f = \pm 1$ if it is a homotopy equivalence.
- a reflection of S^n has $\deg = -1$. A simple way of seeing this is to write S^n as the union of two n-simplices Δ_1, Δ_2 so that $[S^n] = \Delta_1 - \Delta_2$ and the reflection then exchanges Δ_i, acting by -1.
- The antipodal map on S^n, denoted by $-\mathrm{Id}$, has degree $(-1)^{n+1}$, since it is the reflection in all $n+1$ coordinate axes.
- If f has no fixed points, then the line segment from $f(x)$ to $-x$ avoids the origin, so that if we define $g_t(x) = (1-t)f(x) - tx$, then $g_t(x)/|g_t(x)|$ is a homotopy of maps from f to the antipodal map. Hence $\deg(f) = n + 1$.

The degree was historically used to study zeros of vector fields, since for example a sphere around an isolated zero is mapped via the vector field to another sphere of the same dimension (after normalizing the vector field). Hence the degree may be used to assign an integer to any vector field. A related result is the theorem which says you can't comb the hair on a ball flat.

Theorem 2.31. *A nonvanishing continuous vector field may only exist on S^n if n is odd.*

Proof. View the vector field as a map from S^n to itself. If the vector field is nonvanishing, we may normalize it to unit length. Call the resulting map $x \mapsto v(x)$. Then $f_t(x) = \cos(t)x + \sin(t)v(x)$ for $t \in [0, \pi]$ defines a homotopy from Id to the antipodal map $-\mathrm{Id}$. Hence by homotopy invariance of degree, $(-1)^{n+1} = 1$, as required.

To see that odd spheres do have nonvanishing vector fields, view $S^{2n-1} \subset \mathbb{C}^n$, and if ∂_r is the unit radial vector field, then $i\partial_r$ is a vector field of unit length everywhere tangent to S^{2n-1}. □

Recall that when we studied differentiable maps, we defined the degree of a map $f : M^n \longrightarrow N^n$ of n-manifolds where M is compact and N connected; it was defined as $I_2(f, p)$ for a point $p \in N$. Note that this is simply the cardinality mod 2 of the inverse image $f^{-1}(p)$, for p a regular value of f. A similar formula may be used to compute the integer degree of a map (See Bredon for a detailed, but elementary, proof)

Let $f : S^n \longrightarrow S^n$ be a smooth map and $p \in S^n$ a regular value, so that $f^{-1}(p) = \{q_1, \dots, q_k\}$. Then for each q_i, the derivative gives a map

$$D_{q_i} f : T_{q_i} S^n \longrightarrow T_p S^n,$$

with determinant

$$\det(D_{q_i} f) : \wedge^n T_{q_i} S^n \longrightarrow \wedge^n T_p S^n.$$

Since S^n is orientable, we can choose an identification $\wedge^n T S^n = \mathbb{R}$, and the sign of $\det(D_{q_i} f)$ is independent of this identification.

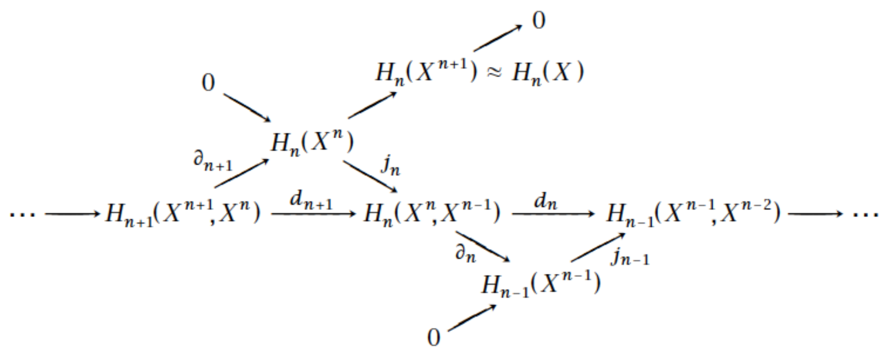

Figure 3: Diagram defining cellular homology (Hatcher)

Theorem 2.32. *With the above hypotheses,*

$$\deg f = \sum_{i=1}^{k} \operatorname{sgn} \det(D_{q_i} f)$$

2.9 Cellular homology

Cellular homology is tailor made for computing homology of cell complexes, based on simple counting of cells and computing degrees of attaching maps. Recall that a cell complex is defined by starting with a discrete set X^0 and inductively attaching n-cells $\{e^n_\alpha\}$ to the n-skeleton X^{n-1}. The weak topology says $A \subset X$ is open if it is open in each of the X^n.

The nice thing about cell complexes is that the boundary map is nicely compatible with the relative homology sequences of the inclusions $X^n \subset X^{n+1}$, and that these are all good pairs.

The relative homology sequence for $X^{n-1} \subset X^n$ is simplified by the fact that $H_k(X^n, X^{n+1})$ vanishes for $k \neq n$, and for $k = n$, X^n/X^{n-1} is a wedge of n-spheres indexed by the n-cells. Since the pair is good, we see

$$H_n(X^n, X^{n-1}) = \text{free abelian group on } n\text{-cells}$$

Then by the long exact sequence in relative homology for this pair (n fixed!), namely

$$H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1})$$

we see that $H_k(X^{n-1})$ is isomorphic to $H_k(X^n)$ for all $k > n$. Hence we can let n drop down to zero, and we obtain $H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) = 0$. Hence

$$H_k(X^n) = 0 \ \ \forall k > n.$$

Finally we observe using the same sequence but letting n increase, that if $n > k$ then

$$H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \ \ \forall n > k.$$

In particular, if X is finite dimensional then we see $H_k(X^n)$ computes $H_k(X)$ for any $n > k$. See Hatcher for a proof of this fact for X infinite dimensional.

Now we combine the long exact sequences for $(X^{n-1}, X^{n-2}), (X^n, X^{n-1})$, and (X^{n+1}, X^n) to form the diagram in Figure 3, where d_i are defined by the composition of the boundary and inclusion maps. clearly $d^2 = 0$. This chain complex, i.e.

$$C^{CW}_n(X) := H_n(X^n, X^{n-1}),$$

fashioned from the relative homologies (which are free abelian groups, recall) of the successive skeleta, is the *cellular chain complex* and its homology is $H^{CW}_\bullet(X)$, the cellular homology.

Proof.

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\approx"'] \arrow[d, "\Phi_{\alpha *}"] & \widetilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\approx"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\approx"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

Figure 4: Diagram computing differential d_n in terms of degree(Hatcher)

Theorem 2.33. $H_n^{CW}(X) \cong H_n(X)$.

Proof. From the diagram, we see that $H_n(X) = \frac{H_n(X^n)}{\mathrm{im}\partial_{n+1}} = \frac{\mathrm{im} j_n = \ker d_n}{\mathrm{im} j \partial_{n+1} = \mathrm{im} d_{n+1}} = H_n^{CW}(X)$ $\square$

We can immediately conclude, for example, that if we have no k-cells, then $H_k(X) = 0$. Or, similarly, if no two cells are adjacent in dimension, then $H_\bullet(X)$ is free on the cells.

Example 2.34. *Recall that $\mathbb{C}P^n$ is a cell complex*

$$\mathbb{C}P^n = e^0 \sqcup e^2 \sqcup \cdots \sqcup e^{2n},$$

so that $H_\bullet(\mathbb{C}P^n) = \mathbb{Z}\ 0\ \mathbb{Z}\ 0\ \mathbb{Z}\ \cdots \mathbb{Z}$.

For more sophisticated calculations, we need an explicit description of the differential d_n in the cell complex. Essentially it just measures how many times the attaching map wraps around its target cycle.

Proposition 2.35 (Cellular differential)**.** *Let e^n_α and e^{n-1}_β be cells in adjacent dimension, and let ϕ_α be the attaching map $S^{n-1}_\alpha \longrightarrow X^{n-1}$ for e^n_α. Also we have the canonical collapsing $\pi : X^{n-1} \longrightarrow X^{n-1}/(X^{n-1} - e^{n-1}_\beta) \cong S^{n-1}_\beta$. Let $d_{\alpha\beta}$ be the degree of the composition*

$$\Delta_{\alpha\beta} : S^{n-1}_\alpha \xrightarrow{\phi_\alpha} X^{n-1} \xrightarrow{\pi} S^{n-1}_\beta.$$

Then

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta.$$

In Figure 4, we see the lower left triangle defines d_n. To determine $d_n(e^n_\alpha)$, take $[e^n_\alpha] \in H_n(D^n_\alpha, \partial D^n_\alpha)$ on the top left, which is sent to the basis element corresponding to e^n_α by Φ_α (the characteristic inclusion map, with associated attaching map φ_α), and we use excision/good pairs to identify its image in $H_{n-1}(X^{n-1}, X^{n-2})$ with the image by the quotient projection q to $\tilde{H}_{n-1}(X^{n-1}/X^{n-2})$. Then the further quotient map $q_\beta : X^{n-1}/X^{n-2} \longrightarrow S^{n-1}_\beta$ collapses the complement of e^{n-1}_β to a point, so it picks out the coefficient we need, which then by the commutativity of the diagram is the degree of $\Delta_{\alpha\beta}$, as required. $\square$

Example 2.36 (orientable genus g 2-manifold)**.** *If M_g is a compact orientable surface of genus g, with usual CW complex with 1 0-cell, $2g$ 1-cells and 1 2-cell whose attaching map sends the boundary circle to the concatenated path $[a_1, b_1] \cdots [a_g, b_g]$, we have the chain complex*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where $d_2(e^2) = 0$, since for example the coefficient for a_1 would be $+1 - 1 = 0$ sincea_1 appears twice with opposite signs in the boundary, hence we would be measuring the degree of a map which goes once around the circle and then once in the opposite direction around the same circle - such a map is homotopic to the constant map, and has degree 0. Hence $d_2 = 0$. The differential d_1 is also zero. Hence the chain complex is exactly the same as the homology itself: $H_\bullet(M_g) = [\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}]$.

Example 2.37 (nonorientable surface, genus g). *Similarly, for a nonorientable surface N_g, we may choose a cell structure with one 0-cell, g 1-cells and one 2-cells attached via $a_1^2 \cdots a_g^2$ Hence $d_2 : \mathbb{Z} \longrightarrow \mathbb{Z}^g$ is given by $1 \mapsto (2, \cdots, 2)$. Hence d_2 is injective and $H_2(N_g) = 0$. Choosing $(1, \cdots, 1)$ as a basis element, we see immediately that $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Example 2.38 (Real projective space). *Recall that the cell complex structure on $\mathbb{R}P^n$ may be viewed as attaching $\mathbb{R}^n \cong D^n$ to the $\mathbb{R}P^{n-1}$ at infinity via the attaching map $S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$ given by the canonical projection (view $S^{n-1} \subset \mathbb{R}^n$, whereas $\mathbb{R}P^{n-1}$ is the lines through 0 in $\mathbb{R}^n$).*

The chain complex is $\mathbb{Z}$ in each degree from 0 to n. The differential is given by computing the degree of the map $S^{n-1} \longrightarrow \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}$. This map may be factored via $S^{n-1} \longrightarrow S_+^{n-1} \wedge S_-^{n-1} \xrightarrow{\nu} S^{n-1}$, where $S_\pm^{n-1} = S^{n-1}/D_\pm$, with $D_\pm$ the closed north/south hemisphere and ν given by the identity map on one factor and the antipodal map on the other (which is which depends on the choice of identification of $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}$ with S^{n-1}. Hence $\nu_* : (1,1) \mapsto 1 + (-1)^n$, and we have $d_k = 1 + (-1)^k$, alternating between $0, 2$. It follows that

$$H_k(\mathbb{R}P^n) = [\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \cdots, \mathbb{Z}_2, 0] \text{ for } n \text{ even}$$
$$H_k(\mathbb{R}P^n) = [\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \cdots, \mathbb{Z}_2, \mathbb{Z}] \text{ for } n \text{ odd}$$

Note that with $\mathbb{Z}_2$ coefficients we have $H_k(\mathbb{R}P^n) = \mathbb{Z}_2$ for all $n, 0 \le k \le n$.

As a final comment, we can easily show that the Euler characteristic of a finite cell complex, usually defined as an alternating sum $\chi(X) = \sum_n (-1)^n c_n$ where c_n is the number of n-cells, can be defined purely homologically, and is hence independent of the CW decomposition:

Theorem 2.39.
$$\chi(X) = \sum_n (-1)^n \text{rank } H_n(X),$$
where rank is the number of $\mathbb{Z}$ summands.

Proof. The CW homology gives us short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, where $C_n = H_n(X^n, X^{n-1})$, etc. For such sequences, the alternating sum of ranks is always zero. Summing over n, we obtain the result. $\square$