These properties descend to the fundamental groupoid, as well as to the fundamental group, implying that for any continuous map of pointed spaces $f : (X, x_0) \longrightarrow (Y, y_0)$, we obtain a homomorphism of groups $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$, given simply by composition $[\gamma] \mapsto [f \circ \gamma]$. This last fact is usually proven directly, since it is so simple.

1.3 $\pi_1(S^1) = \mathbb{Z}$

In this section we will compute the fundamental group of S^1. The method we use will help us develop the theory of covering spaces. We essentially follow Hatcher, Chapter 1.

Theorem 1.12. *The map $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$ given by $n \mapsto [\omega_n]$, for $\omega_n(s) = e^{2\pi i n s}$, is an isomorphism.*

Proof. Consider the map $p : \mathbb{R} \longrightarrow S^1$ defined by $p(s) = e^{2\pi i s}$. It can be viewed as a projection of a single helix down to a circle. The loop ω_n may be factored as a linear path $\tilde{\omega}_n(s) = ns$ in $\mathbb{R}$, composed with p:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{\omega}_n"] \arrow[r, "\omega_n"'] & S^1 \end{tikzcd}$$

We say that $\tilde{\omega}_n$ is a "lift" of ω_n to the "covering space" $\mathbb{R}$. Note that $\Phi(n)$ could be defined as $[p \circ \tilde{f}]$ for any path $\tilde{f}$ in $\mathbb{R}$ joining 0 to n. This is because $\tilde{f} \simeq \tilde{\omega}_n$ via the homotopy $(1-t)\tilde{f} + t\tilde{\omega}_n$.

To check that Φ is a homomorphism, note that $\Phi(m+n)$ is represented by the loop $p \circ (\tilde{\omega}_m \cdot (\tau_m \circ \tilde{\omega}_n))$, where $\tau_m : \mathbb{R} \longrightarrow \mathbb{R}$ is the translation $\tau_m(x) = x + m$. But since[5] $p \circ \tau_m = p$, we see that the loop is equal to the concatenation $\omega_m \cdot \omega_n$. Thus $\Phi(m+n) = \Phi(m)\Phi(n)$.

To prove that Φ is surjective, we do it by taking any loop $f : I \longrightarrow S^1$ and lifting it to $\tilde{f}$ starting at 0, which then must go to an integer n. Then $\Phi(n) = [f]$ as required. For this to work, we need to prove:

a) For each path $f : I \longrightarrow S^1$ with $f(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \longrightarrow \mathbb{R}$ with $f(0) = \tilde{x}_0$.

To prove that Φ is injective, suppose that $\Phi(m) = \Phi(n)$. This means that there is a homotopy $f_t : \omega_m = f_0 \Rightarrow \omega_n = f_1$. Let us lift this homotopy to a homotopy $\tilde{f}_t$ of paths starting at 0. By uniqueness it must be that $\tilde{f}_0 = \tilde{\omega}_0$ and similarly $\tilde{f}_1 = \tilde{\omega}_1$. Since $\tilde{f}_t$ is a homotopy of paths, its endpoint is the same for all t, hence $m = n$. For this to work, we need to be able to lift the homotopy via the statement:

b) For each homotopy $f_t : I \longrightarrow S^1$ of paths starting at $x_0 \in S^1$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy $\tilde{f}_t : I \longrightarrow \mathbb{R}$ of paths starting at $\tilde{x}_0$.

Both statements $a), b)$ are lifting results and can be absorbed in the statement of the following lemma. □

Lemma 1.13 (Lifting lemma)**.** *Given a map $F : Y \times I \longrightarrow S^1$ and a "initial lift" $\tilde{F}_0 : Y \times \{0\} \longrightarrow \mathbb{R}$ lifting $F|_{Y \times \{0\}}$, there is a unique "complete lift" $\tilde{F} : Y \times I \longrightarrow \mathbb{R}$ lifting F and agreeing with $\tilde{F}_0$.*

Proof. The main ingredient of the proof is to use the fact that $p : \mathbb{R} \longrightarrow S^1$ is a covering space, meaning that there is an open cover $\{U_\alpha\}$ of S^1 such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each mapped homeomorphically onto U_α by p. For example, we could take the usual cover U_0, U_1 by two open arcs.

To construct the lift $\tilde{F}$, we first lift the homotopy for small neighbourhoods $N \subset Y$, producing $\tilde{F} : N \times I \longrightarrow \mathbb{R}$. We then observe that these lifts on neighbourhoods glue together to give a complete lift.

Fix $y_0 \in Y$. By compactness of $y_0 \times I$, there is a neighbourhood N of y_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of the interval such that $F(N \times [t_i, t_{i+1}])$ is contained in some U_α for each i (call this open

[5] We see here that τ_m is a "deck transformation", an automorphism of the covering space fixing the base.

set U_i). The lift on $N \times [0, t_0]$ is given as $\tilde{F}|_{N\times\{0\}}$. Assume inductively that $\tilde{F}$ has been constructed on $N \times [0, t_i]$. For the next segment, $F(N \times [t_i, t_{i+1}]) \subset U_i$ and $\tilde{F}(y_0, t_i)$ lies inside $\tilde{U}_i$. Replacing N by a smaller neighbourhood of y_0, we may assume that $\tilde{F}(N \times \{t_i\}) \subset \tilde{U}_i$. Now we simply define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be $p|_{\tilde{U}_i}^{-1} \circ F$. In this way we get a lift $\tilde{F} : N \times I \longrightarrow \mathbb{R}$ for some neighbourhood N of y_0.

The fact that these local lifts glue to a global lift stems from the uniqueness of the lift at each point y_0 (hence two local lifts for neighbourhoods N, N' must agree on their intersection. Furthermore, the uniqueness of the complete lift is also implied by the uniqueness of the lift at each point y_0, which we now show.

Let Y be a point. Suppose $\tilde{F}, \tilde{F}'$ are two lifts of $F : I \longrightarrow S^1$ with $\tilde{F}(0) = \tilde{F}'(0)$. Choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ compatible with $\{U_i\}$ as before. Assume that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, $\tilde{F}([t_i, t_{i+1}])$ is also, and must lie in a single one of the lifts $\tilde{U}_i$ of U_i, in fact the same one which $\tilde{F}'([t_i, t_{i+1}])$ is in, since these share the same value at t_i. Since p is an isomorphism on this open set, we obtain $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$, completing the proof. □

Corollary 1.14. *Any nonconstant complex polynomial $f(z)$ must have a zero.*

Proof. If f has no zeros, then f must take $\mathbb{C}\backslash\{0\}$ into $\mathbb{C}\backslash\{0\}$, both homotopic to S^1. For sufficiently small R, the loop $\gamma_R(t) = f(Re^{2\pi it})$ is homotopic to a constant loop ω_0. Letting R grow sufficiently large, $f(z)$ behaves as z^n for n the degree of f, and so $\gamma_R(t)$ is homotopic to ω_n. By the theorem, $n = 0$, a contradiction. □

Using the same arguments you can show that f must have $n = \deg f$ zeros, counted with multiplicity.

Corollary 1.15 (Brouwer fixed point theorem)**.** *Every continuous map $h : D^2 \longrightarrow D^2$ has a fixed point.*

Proof. If h has no fixed point, then we obtain a map $r : D^2 \longrightarrow S^1$ by intersecting the ray from $h(x)$ to x with the boundary circle. This is a retraction onto the circle. But a retract $r : X \longrightarrow A$ to a subspace $A \overset{i}{\hookrightarrow} X$ satisfiees $r \circ i = \mathrm{Id}$, implying $r_* \circ i_* = \mathrm{Id}$, implying that i_* must be an injection. Contradiction. □

Corollary 1.16 (Borsuk-Ulam)**.** *Every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$ takes the same value on at least one pair of antipodal points.*

Proof. If not, then $\tilde{g}(x) = f(x) - f(-x)$ is an odd function $S^2 \longrightarrow \mathbb{R}^2$ with no zeros, so that $g(x) = \tilde{g}(x)/|\tilde{g}(x)|$ is well defined and still odd. Composing with the equatorial path $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$, we obtain an odd function $h : S^1 \longrightarrow S^1$. We prove that h is nontrivial in $\pi_1(S^1)$: lift h to $\tilde{h} : S^1 \longrightarrow \mathbb{R}$; since $h(s + 1/2) = -h(s)$ for $s \in [0, 1/2]$, it follows that $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$ for some odd integer q (q must be constant since it depends continuously on s but is an integer). In particular $\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q$. In other words, h is homotopic to an odd multiple of the generator of $\pi_1(S^1)$ and hence must be nontrivial. On the other hand, since η is nullhomotopic in S^2, $h = g \circ \eta$ must also be nullhomotopic, a contradiction. □

Borsuk-Ulam can be used to prove the famous "Ham Sandwich theorem", stating that bread, ham, and cheese, can always be cut with one slice in such a way so that all three quantities are halved. This is proved by starting with the bread: for each direction $v \in S^2$, let $P(v)$ be the plane normal to v which cuts the bread in half (the middle such plane if there is an interval of these). Then define a map $S^2 \longrightarrow \mathbb{R}^2$ via $f(v) = (c(v), h(v))$, where $c(v)$ is the volume of cheese on the side of $P(v)$ in the direction of v, and similarly for the ham $h(v)$. Borsuk-Ulam then implies that there is a plane which ensures a well-balanced meal.

Before we discuss the computation of $\pi_1(X)$ for other, more complicated examples, let's try to understand the *fundamental groupoid* of S^1.

As we saw before, any paths $\gamma, \gamma' \in \mathcal{P}(\mathbb{R})$ joining $p, q \in \mathbb{R}$ must be homotopic, i.e. there is a single homotopy class of paths joining points in $\mathbb{R}$, and so the fundamental groupoid of $\mathbb{R}$ is simply $\mathbb{R} \times \mathbb{R}$, with groupoid law $(x, y) \circ (y, z) = (x, z)$.

Now let $a, b \in S^1$ and let γ be a path from a to b. Choose $\tilde{a} \in p^{-1}(a)$, so that γ may be lifted to $\tilde{\gamma}$, starting at $\tilde{a}$ and ending at $\tilde{b} := \tilde{\gamma}(1)$. Of course $\tilde{\gamma}$ is homotopic to a unique linear path, and similarly for γ; and two such linear paths $p \circ \tilde{\gamma}, p \circ \tilde{\gamma}'$ coincide iff $\gamma' = \gamma + n,\ \ n \in \mathbb{Z}$. As a result, we see that $\Pi_1(S^1) = \mathbb{R} \times \mathbb{R}/\sim$, where $(x, y) \sim (x + n, y + n),\ \ n \in \mathbb{Z}$. Therefore we obtain that $\Pi_1(S^1)$ has a cylinder as its space of arrows, which then maps to S^1 via the source and target maps (s, t). Note also that for $p \in S^1$, $s^{-1}(p)$ is homeomorphic to $\mathbb{R}$, and t maps this to S^1 as a covering map, precisely the same one as $p : \mathbb{R} \longrightarrow S^1$ from earlier.

1.4 Further computations of π_1

The main technique for computing $\pi_1(X)$ is the Van Kampen theorem, which is an analog of the Mayer-Vietoris sequence which we learned about for de Rham cohomology. Before we get to it, we will cover some more elementary facts about computing π_1.

Proposition 1.17. *Let X, Y be path-connected. Then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Proof. Recall that a map $f : Z \longrightarrow X \times Y$ is continuous iff the projections $g : Z \longrightarrow X$, $h : Z \longrightarrow Y$ are separately continuous. Therefore if f is a loop based at (x_0, y_0), it is nothing more than a pair of loops in X and Y based at x_0 and y_0. Similarly homotopies of loops are nothing but pairs of homotopies of pairs of loops, and so $[f] \mapsto ([g], [h])$ defines the obvious isomorphism. $\square$

A natural example to consider, given that $\pi_1(S^1) \cong \mathbb{Z}$, is the torus $T = S^1 \times S^1$. Then $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

Proposition 1.18. *$\pi_1(S^n) = \{0\}$ for $n > 2$.*

Proof. Any continuous map of smooth manifolds is homotopic to a smooth map: given $f : S^1 \longrightarrow S^n$, we may find a smooth approximation $\tilde{f} : S^1 \longrightarrow \mathbb{R}^{n+1}$ which lies in a small tubular neighbourhood U of S^n. Then form $H(p, t) = r((1 - t)f(p) + t\tilde{f}(p))$, for $r : U \longrightarrow S^n$ the retraction.

By Sard's theorem, $\tilde{f}$ is not surjective for $n \geq 2$, failing to take $q \in S^n$ as a value. $S^n \backslash \{q\}$ is contractible, hence $\tilde{f}$ is homotopic to the trivial path. $\square$

Corollary 1.19. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*