1.5 The Van Kampen theorem

There are many versions of the Van Kampen theorem; all of them help us to do the following: determine the fundamental group of a space X which has been expressed as a union $\bigcup_\alpha U_\alpha$ of open sets, given the fundamental groups of each U_α and $U_\alpha \cap U_\beta$, as well as the induced maps on fundamental groups given by the inclusion (or *fibered coproduct*) diagram

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{i_{\alpha\beta}} & U_\alpha \\ {\scriptstyle i_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle i_\alpha} \\ U_\beta & \xrightarrow[i_\beta]{} & U_\alpha \cup U_\beta \end{array} \tag{1}$$

Before we begin to state the theorem, we briefly review the idea of the free product of groups. Given groups G_1, G_2, we may form the *free product* $G_1 * G_2$, defined as follows: $G_1 * G_2$ is the group of equivalence classes of finite words made from letters chosen from $G_1 \sqcup G_2$, where the equivalence relation is finitely generated by $a * b \sim ab$ for a, b both in G_1 or G_2, and the identity elements $e_i \in G_i$ are equivalent to the empty word. The group operation is juxtaposition. For example, $\mathbb{Z} * \mathbb{Z}$ is the free group on two generators:

$$\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle = \{a^{i_1} b^{j_1} a^{i_2} b^{j_2} \cdots a^{i_k} b^{j_k} \ : \ i_p, j_p \in \mathbb{Z}, \ \ k \geq 0\}$$

Note that from a categorical point of view[6], $G_1 * G_2$ is the *coproduct* or *sum* of G_1 and G_2 in the following sense: not only does it fit into the following diagram of groups:

$$\begin{array}{ccc} & & G_2 \\ & & \downarrow{\scriptstyle \iota_2} \\ G_1 & \xrightarrow[\iota_1]{} & G_1 * G_2 \end{array}$$

but $(\iota_1, \iota_2, G_1 * G_2)$ is the "most general" such object, i.e. any other triple (j_1, j_2, G) replacing it in the diagram must factor through it, via a unique map $G_1 * G_2 \longrightarrow G$.

The simplest version of Van Kampen is for a union $X = U_1 \cup U_2$ of two path-connected open sets such that $U_1 \cap U_2$ is path-connected and simply connected. Note that the injections ι_1, ι_2 give us induced homomorphisms $\pi_1(U_i) \longrightarrow \pi_1(X)$. By the coproduct property, this map must factor through a group homomorphism

$$\Phi : \pi_1(U_1) * \pi_1(U_2) \longrightarrow \pi_1(X).$$

Theorem 1.20 (Van Kampen, version 1)**.** *If $X = U_1 \cup U_2$ with U_i open and path-connected, and $U_1 \cap U_2$ path-connected and simply connected, then the induced homomorphism $\Phi : \pi_1(U_1) * \pi_1(U_2) \longrightarrow \pi_1(X)$ is an isomorphism.*

Proof. Choose a basepoint $x_0 \in U_1 \cap U_2$. Use $[\gamma]_U$ to denote the class of γ in $\pi_1(U, x_0)$. Use $*$ as the free group multiplication.

$\boxed{\Phi \text{ is surjective:}}$ Let $[\gamma] \in \pi_1(X, x_0)$. Then we can find a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_i, t_{i+1}])$ is contained completely in U_1 or U_2 (it might be in $U_1 \cap U_2$). Then γ factors as a product of its restrictions γ_{i+1} to $[t_i, t_{i+1}]$, i.e.

$$[\gamma]_X = [\gamma_1 \gamma_2 \cdots \gamma_n]_X$$

But the γ_i are not loops, just paths. To make them into loops we must join the subdivision points $\gamma(t_i)$ to the basepoint, and we do this as follows: if $\gamma(t_i) \in U_1 \cap U_2$ then we choose a path η_i from x_0 to $\gamma(t_i)$ lying in $U_1 \cap U_2$; otherwise we choose such a path lying in whichever of U_1, U_2 contains $\gamma(t_i)$. This is why we need $U_i, U_1 \cap U_2$ to be path-connected.

[6]Coproducts in categories are the subject of a question in Assignment 6

Then define $\tilde{\gamma}_i = \eta_{i-1}\gamma_i\eta_i^{-1}$ and we obtain a factorization of loops

$$[\gamma]_X = [\tilde{\gamma}_1]_X \cdots [\tilde{\gamma}_n]_X.$$

We chose the η_i in just such a way that each loop in the right hand side lies either in U_1 or in U_2; hence we can choose $e_i \in \{1, 2\}$ so that $[\tilde{\gamma}_1]_{U_{e_1}} * \cdots * [\tilde{\gamma}_n]_{U_{e_n}}$ makes sense as a word in $\pi_1(U_1) * \pi_1(U_2)$, and hence we have $[\gamma]_X = \Phi([\tilde{\gamma}_1]_{U_{e_1}} * \cdots * [\tilde{\gamma}_n]_{U_{e_n}})$, showing surjectivity.

Φ is injective: take an arbitrary element of the free product $\gamma = [a_1]_{U_{e_1}} * \cdots * [a_k]_{U_{e_k}}$ (for $e_i \in \{1, 2\}$) , and suppose that $\Phi([a_1]_{U_{e_1}} * \cdots * [a_k]_{U_{e_k}}) = 1$. This means that $a_1 \cdots a_k$ is homotopically trivial in X. We wish to show that $\gamma = 1$ in the free product group.

Take the homotopy $H : I \times I \longrightarrow X$ taking $a_1 \cdots a_k$ to the constant path at x_0, and subdivide $I \times I$ into small squares $S_{ij} = [s_i, s_{i+1}] \times [t_i, t_{i+1}]$ so that each square is sent either into U_1 or U_2, and subdivide smaller if necessary to ensure that the endpoints of the domains of the loops a_i are part of the subdivision.

Set up the notation as follows: let v_{ij} be the grid point (s_i, t_i) and a_{ij} the path defined by H on the horizontal edge $v_{ij} \to v_{i+1,j}$, and b_{ij} the vertical path given by H on $v_{ij} \to v_{i,j+1}$. Then we can write $a_i = a_{p_{i-1}+1,0} \cdots a_{p_i,0}$ for some $\{p_i\}$, and we can factor each loop as a product of tiny paths:

$$\gamma = [a_1]_{U_{e_1}} * \cdots * [a_k]_{U_{e_k}} = [a_{0,0} \cdots a_{p_1,0}]_{U_{e_1}} * \cdots * [\cdots a_{p_k,0}]_{U_{e_k}}$$

Again, these paths a_{ij} (as well as the b_{ij}) are not loops, so, just as in the proof of surjectivity, choose paths h_{ij} from the basepoint to all the gridpoint images $H(v_{ij})$, staying within $U_1 \cap U_2$, U_1, or U_2 accordingly as $H(v_{ij})$. pre- and post-composing with the h_{ij}, we then obtain loops $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ lying in either U_1 or U_2.

In particular we can factor γ as a bunch of tiny loops, each remaining in U_1 or U_2:

$$\gamma = [\tilde{a}_{0,0}]_{U_{e_1}} * \cdots * [\tilde{a}_{p_1,0}]_{U_{e_1}} * \cdots * [\tilde{a}_{p_k,0}]_{U_{e_k}}$$

For each loop $\tilde{a}_{i,0}$, we may use H restricted to the square immediately above $a_{i,0}$ to define a homotopy $H_{i,0} : \tilde{a}_{i,0} \Rightarrow \tilde{b}_{i,0}\tilde{a}_{i,1}\tilde{b}_{i+1,0}^{-1}$: If $\tilde{a}_{i,0}$ is in U_1, say, and the homotopy $H_{i,0}$ occurs in U_1, then we may replace $[\tilde{a}_{i,0}]_{U_1}$ with $[\tilde{b}_{i,0}]_{U_1} * [\tilde{a}_{i,1}]_{U_1} * [\tilde{b}_{i+1,0}]_{U_1}^{-1}$ in the free product. If on the other hand $H_{i,0}$ occurs in U_2, then we observe that $\tilde{a}_{i,0}$ must lie in $U_1 \cap U_2$, and **since this is simply connected**, $[\tilde{a}_{1,0}]_{U_1} =$ empty word $= [\tilde{a}_{1,0}]_{U_2}$ in the free product, so it can be replaced with $[\tilde{b}_{i,0}]_{U_2} * [\tilde{a}_{i,1}]_{U_2} * [\tilde{b}_{i+1,0}]_{U_2}^{-1}$ in the free product. Doing this replacement for each square in the bottom row, the $[\tilde{b}_{i,0}]_{U_{e_i}}$ cancel, and we may repeat the replacement for the next row.

In this way we eventually reach the top row, which corresponds to a free product of constant paths at x_0, showing $\gamma = 1$ in the free product, as required. ☐

Let's give some examples of fundamental groups computed with the simple version of Van Kampen:

Example 1.21. *The wedge sum of pointed spaces (X, x), (Y, y) is $X \vee Y := X \sqcup Y/x \sim y$, and is the coproduct in the category of pointed spaces. If X, Y are topological manifolds, then let V_x, V_y be disc neighbourhoods of x, y so that $X \vee Y = U_1 \cup U_2$ with $U_1 = [X \sqcup V_y]$ and $U_2 = [V_x \sqcup Y]$. We conclude that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. For example, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$.*

At least for pointed manifolds, therefore, we can say that the π_1 functor preserves coproducts. Does this hold for all pointed spaces? No, but it does work when the point is a deformation retract of an open neighbourhood.

Example 1.22. *Let X, Y be connected manifolds of dimension n. Then their connected sum $X\sharp Y$ is naturally decomposed into two open sets $U \cup V$ with $U \cap V \cong I \times S^{n-1} \simeq S^{n-1}$. If $n > 2$ then $\pi_1(S^{n-1}) = 0$, and hence $\pi_1(X\sharp Y) = \pi_1(X) * \pi_1(Y)$.*

Example 1.23. *Using the classical $4g$-gon representation of the genus g orientable surface Σ_g, we showed that when punctured it is homotopic to $\vee_{2g} S^1$. Hence $\pi_1(\Sigma_g \backslash \{p\}) = F_{2g}$. What happens with non-orientable surfaces? What about puncturing manifolds of dimension > 2?*

The second version of Van Kampen will deal with cases where $U_1 \cap U_2$ is not simply-connected. By the inclusion diagram (1), we see that we have a canonical map from the fibered sum $\pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2)$ to $\pi_1(X)$: Van Kampen again states that this is an isomorphism. Recall that if $\iota_k : H \longrightarrow G_k$, $k = 1, 2$ are injections of groups, then the fibered product or "free product with amalgamation" may be constructed as a quotient of the free product, by additional relations generated by $(g_1\iota_1(h)) * g_2 \sim g_1 * (\iota_2(h)g_2)$ for $g_i \in G_i$ and $h \in H$. In other words,

$$G_1 *_H G_2 = (G_1 * G_2)/K,$$

where K is the normal subgroup generated by elements $\{d\iota_1(h)^{-1}\iota_2(h) \ : \ h \in H\}$.

Theorem 1.24 (Van Kampen, version 2[7])**.** *If $X = U_1 \cup U_2$ with U_i open and path-connected, and $U_1 \cap U_2$ path-connected, then the induced homomorphism $\Phi : \pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2) \longrightarrow \pi_1(X)$ is an isomorphism.*

Proof. Exercise! Slight modification of the given proof, need to understand the analogous condition to the one we used to show $[\tilde{a}]_{U_1}$ = empty word = $[\tilde{a}]_{U_2}$ in the free product. □

Example 1.25. *Express the 2-sphere as a union of two discs with intersection homotopic to S^1. By Van Kampen version 2, we have $\pi_1(S^2) = (0 * 0)/K = 0$.*

Example 1.26. *Take a genus g orientable surface Σ_g. Choose a point $p \in \Sigma_g$ and let $U_0 = \Sigma_g\backslash\{p\}$. Let U_p be a disc neighbourhood of p. Then we have $\Sigma_g = U_0 \cup U_p$, with intersection $U_0 \cap U_p \simeq S^1$. The inclusion map $S^1 \longrightarrow U_p$ is trivial in homotopy while $S^1 \longrightarrow U_0$ sends $1 \in \mathbb{Z}$ to $a_1b_1a_1^{-1}b^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$. Hence the amalgamation introduces a single relation:*

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots a_g, b_g \ \ | \ \ [a_1, b_1]\cdots[a_g, b_g]\rangle$$

Example 1.27. *Do the same as above, but with $\mathbb{R}P^2 = U_0 \cup U_p$, with $\pi_1(U_0) = \mathbb{Z} = \langle a\rangle$ and the inclusion of $U_0 \cap U_p \simeq S^1$ in U_0 sends $1 \mapsto a^2$, hence we obtain*

$$\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}.$$

Example 1.28 (Perverse computation of $\pi_1(S^3)$)**.** *Express S^3 as the union of two solid tori, glued along their boundary. Visualize it by simply looking at the interior and exterior of an embedded torus in $\mathbb{R}^3 \sqcup \infty$. Fatten the tori to open sets U_0, U_1 with $U_0 \cap U_1 \simeq T^2$, so that*

$$\pi_1(S^3) = \mathbb{Z} *_{\mathbb{Z}\times\mathbb{Z}} \mathbb{Z}.$$

The notation is not enough to determine the group: we need the maps $(\iota_i)_* : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}$ induced by the inclusions: by looking at generating loops, we get $\iota_0(1, 0) = 1, \iota_0(0, 1) = 0$ while $\iota_1(1, 0) = 0, \iota_1(0, 1) = 1$. Hence the amalgamation kills both generators, yielding the trivial group.

The proof of Van Kampen in Hatcher is slightly more general than this, as it allows arbitrarily many open sets U_α, with only the extra hypothesis that triple intersections be path-connected (in our proof, each vertex v_{ij} is joined to the basepoint by a path: since the vertex is surrounded by 4 squares, we would need quadruple intersections to be path-connected. This can be improved by using a hexagonal decomposition, or a brick configuration, where the vertices are surrounded by only 3 2-cells). The ultimate Van Kampen theorem does not refer to basepoints or put connectivity conditions on the intersection: it states that the fundamental groupoid of $U_1 \cup U_2$ is the fibered sum of $\Pi_1(U_1)$ and $\Pi_1(U_2)$ over $\Pi_1(U_1 \cap U_2)$. Viewing the topology of X as a category (where objects are open sets and arrows are inclusions), the Van Kampen theorem can be rephrased as follows:

Theorem 1.29 (Van Kampen, version 3)**.** Π_1 *is a functor from the topology of X to groupoids, which preserves fibered sum*[8]*.*

[7]See the proof in Hatcher

[8]See "Topology and Groupoids" by Ronald Brown.