2 Homology

We now turn to Homology, a functor which associates to a topological space X a sequence of abelian groups $H_k(X)$. We will investigate several important related ideas:

- Homology, relative homology, axioms for homology, Mayer-Vietoris
- Cohomology, coefficients, Poincaré Duality
- Relation to de Rham cohomology (de Rham theorem)
- Applications

The basic idea of homology is quite simple, but it is a bit difficult to come up with a proper definition. In the definition of the homotopy group, we considered loops in X, considering loops which could be "filled in" by a disc to be trivial.

In homology, we wish to generalize this, considering loops to be trivial if they can be "filled in" by any surface; this then generalizes to arbitrary dimension as follows (let X be a manifold for this informal discussion).

A k-dimensional chain is defined to be a k-dimensional submanifold with boundary $S \subset X$ with a chosen orientation σ on S. A chain is called a cycle when its boundary is empty. Then the k^{th} homology group is defined as the free abelian group generated by the k-cycles (where we identify (S, σ) with $-(S, -\sigma)$), modulo those k-cycles which are boundaries of $k+1$-chains. Whenever we take the boundary of an oriented manifold, we choose the boundary orientation given by the outward pointing normal vector.

Example 2.1. *Consider an oriented loop separating a genus 2 surface into two genus 1 punctured surfaces. This loop is nontrivial in the fundamental group, but is trivial in homology, i.e. it is* homologous *to zero.*

Example 2.2. *Consider two parallel oriented loops L_1, L_2 on T^2. Then we see that $L_1 - L_2 = 0$, i.e. L_1 is homologous to L_2.*

Example 2.3. *This definition of homology is not well-behaved: if we pick any embedded submanifold S in a manifold and slightly deform it to S' which still intersects S, then there may be no submanifold with $S \cup S'$ as its boundary. We want such deformations to be homologous, so we slightly relax our requirements: we allow the k-chains to be smooth maps $\iota : S \longrightarrow M$ which needn't be embeddings.*

This definition is still problematic: it's not clear what to do about non-smooth topological spaces, and also the definition seems to require knowledge of all possible manifolds mapping into M. We solve both problems by cutting S into triangles (i.e. simplices) and focusing only on maps of simplices into M.

Definition 11. An n-simplex $[v_0, \cdots, v_n]$ is the convex hull of $n+1$ ordered points (called *vertices*)in $\mathbb{R}^m$ for which $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

The standard n-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \ \forall i\},$$

and there is a canonical map $\Delta^n \longrightarrow [v_0, \cdots, v_n]$ via

$$(t_0, \ldots, t_n) \mapsto \sum_i t_i v_i,$$

called *barycentric coordinates* on $[v_0, \cdots, v_n]$. A *face* of $[v_0, \cdots, v_n]$ is defined as the simplex obtained by deleting one of the v_i, we denote it $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. The union of all faces is the *boundary* of the simplex, and its complement is called the *interior*, or the *open simplex*.

Definition 12. A Δ-complex decomposition of a topological space X is a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$ (n depending on α) such that σ_α is injective on the open simplex Δ^n_o, every point is in the image of exactly one $\sigma_\alpha|_{\Delta^n_o}$, and each restriction of σ_α to a face of $\Delta^{n(\alpha)}$ coincides with one of the maps σ_β, under the canonical identification of Δ^{n-1} with the face (which preserves ordering). We also require the topology to be compatible: $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in the simplex for each α.

It is easy to see that such a structure on X actually expresses it as a cell complex.

Example 2.4. *Give the standard decomposition of 2-dimensional compact manifolds.*

We may now define the simplicial homology of a Δ-complex X. We basically want to mod out cycles by boundaries, except now the chains will be made of linear combinations of the n-simplices which make up X.
Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices $e^n_\alpha = \sigma_\alpha(\Delta^n_o)$ of X. Elements $\sum_\alpha n_\alpha \sigma_\alpha \in \Delta_n(X)$ are called n-chains (finite sums).
Each n-simplex has a natural orientation based on its ordered vertices, and its boundary obtains a natural orientation from the outward-pointing normal vector field. Algebraically, this induced orientation is captured by the following formula (which captures the interior product by the outward normal vector to the i^{th} face):

$$\boxed{\partial[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n].}$$

This allows us to define the boundary homomorphism:

Definition 13. The boundary homomorphism $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ is determined by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}.$$

This definition of boundary is clearly a triangulated version of the usual boundary of manifolds, and satisfies $\partial \circ \partial = \emptyset$, i.e.

Lemma 2.5. *The composition $\partial_{n-1} \circ \partial_n = 0$.*

Proof.

$$\partial\partial[v_0 \cdots v_n] = \sum_{j<i} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n] + \sum_{j>i} (-1)^{i+j-1} [v_0, \cdots, \hat{v}_i, \cdots \hat{v}_j, \cdots, v_n]$$

the two displayed terms cancel. □

Now we have produced an algebraic object: a chain complex (just as we saw in the case of the de Rham complex). Let C_n be the abelian group $\Delta_n(X)$; then we get the simplicial chain complex:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

and the homology is defined as the simplicial homology

$$H^\Delta_n(X) := \frac{Z_n = \ker \partial_n}{B_n = \text{im } \partial_{n+1}}$$

Example 2.6. *The circle is a Δ-complex with one vertex and one 1-simplex. so $\Delta_0(S^1) = \Delta_1(S^1) = \mathbb{Z}$ and $\partial_1 = 0$ since $\partial e = v - v$. hence $H^\Delta_0(S^1) = \mathbb{Z} = H^\Delta_1(S^1)$ and $H^\Delta_k(S^1) = 0$ otherwise.*

Example 2.7. *For T^2 and Klein bottle: $\Delta_0 = \mathbb{Z}$, $\Delta_1 = \langle a, b, c\rangle$ and $\Delta_2 = \langle P, Q\rangle$. For $\mathbb{R}P^2$, same except $\Delta_0 = \mathbb{Z}^2$.*

Simplicial homology, while easy to calculate (at least by computer!), is not entirely satisfactory, mostly because it is so rigid - it is not clear, for example, that the groups do not depend on the triangulation. We therefore relax the definition and describe singular homology.

Definition 14. A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \longrightarrow X$. The free abelian group on the set of n-simplices is called $C_n(X)$, the group of n-chains.

There is a linear boundary homomorphism $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ given by

$$\partial_n \sigma = \sum_i (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]},$$

where $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ is canonically identified with Δ^{n-1}. The homology of the chain complex $(C_\bullet(X), \partial)$ is called the *singular homology* of X:

$$H_n(X) := \frac{\ker \partial : C_n(X) \longrightarrow C_{n-1}(X)}{\text{im}\partial : C_{n+1}(X) \longrightarrow C_n(X)}.$$