2 Homology

We now turn to Homology, a functor which associates to a topological space X a sequence of abelian groups $H_k(X)$. We will investigate several important related ideas:

- Homology, relative homology, axioms for homology, Mayer-Vietoris
- Cohomology, coefficients, Poincaré Duality
- Relation to de Rham cohomology (de Rham theorem)
- Applications

The basic idea of homology is quite simple, but it is a bit difficult to come up with a proper definition. In the definition of the homotopy group, we considered loops in X, considering loops which could be "filled in" by a disc to be trivial.

In homology, we wish to generalize this, considering loops to be trivial if they can be "filled in" by any surface; this then generalizes to arbitrary dimension as follows (let X be a manifold for this informal discussion).

A k-dimensional chain is defined to be a k-dimensional submanifold with boundary $S \subset X$ with a chosen orientation σ on S. A chain is called a cycle when its boundary is empty. Then the k^{th} homology group is defined as the free abelian group generated by the k-cycles (where we identify (S, σ) with $-(S, -\sigma)$), modulo those k-cycles which are boundaries of $k+1$-chains. Whenever we take the boundary of an oriented manifold, we choose the boundary orientation given by the outward pointing normal vector.

Example 2.1. *Consider an oriented loop separating a genus 2 surface into two genus 1 punctured surfaces. This loop is nontrivial in the fundamental group, but is trivial in homology, i.e. it is* homologous *to zero.*

Example 2.2. *Consider two parallel oriented loops L_1, L_2 on T^2. Then we see that $L_1 - L_2 = 0$, i.e. L_1 is homologous to L_2.*

Example 2.3. *This definition of homology is not well-behaved: if we pick any embedded submanifold S in a manifold and slightly deform it to S' which still intersects S, then there may be no submanifold with $S \cup S'$ as its boundary. We want such deformations to be homologous, so we slightly relax our requirements: we allow the k-chains to be smooth maps $\iota : S \longrightarrow M$ which needn't be embeddings.*

This definition is still problematic: it's not clear what to do about non-smooth topological spaces, and also the definition seems to require knowledge of all possible manifolds mapping into M. We solve both problems by cutting S into triangles (i.e. simplices) and focusing only on maps of simplices into M.

2.1 Simplicial homology

Definition 11. An n-simplex $[v_0, \cdots, v_n]$ is the convex hull of $n+1$ ordered points (called *vertices*)in $\mathbb{R}^m$ for which $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

The standard n-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \ \forall i\},$$

and there is a canonical map $\Delta^n \longrightarrow [v_0, \cdots, v_n]$ via

$$(t_0, \ldots, t_n) \mapsto \sum_i t_i v_i,$$

called *barycentric coordinates* on $[v_0, \cdots, v_n]$. A *face* of $[v_0, \cdots, v_n]$ is defined as the simplex obtained by deleting one of the v_i, we denote it $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. The union of all faces is the *boundary* of the simplex, and its complement is called the *interior*, or the *open simplex*.

Definition 12. A Δ-complex decomposition of a topological space X is a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$ (n depending on α) such that σ_α is injective on the open simplex Δ^n_o, every point is in the image of exactly one $\sigma_\alpha|_{\Delta^n_o}$, and each restriction of σ_α to a face of $\Delta^{n(\alpha)}$ coincides with one of the maps σ_β, under the canonical identification of Δ^{n-1} with the face (which preserves ordering). We also require the topology to be compatible: $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in the simplex for each α.

It is easy to see that such a structure on X actually expresses it as a cell complex.

Example 2.4. *Give the standard decomposition of 2-dimensional compact manifolds.*

We may now define the simplicial homology of a Δ-complex X. We basically want to mod out cycles by boundaries, except now the chains will be made of linear combinations of the n-simplices which make up X.

Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices $e^n_\alpha = \sigma_\alpha(\Delta^n_o)$ of X. Elements $\sum_\alpha n_\alpha \sigma_\alpha \in \Delta_n(X)$ are called n-chains (finite sums).

Each n-simplex has a natural orientation based on its ordered vertices, and its boundary obtains a natural orientation from the outward-pointing normal vector field. Algebraically, this induced orientation is captured by the following formula (which captures the interior product by the outward normal vector to the i^{th} face):

$$\boxed{\partial[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n].}$$

This allows us to define the boundary homomorphism:

Definition 13. The boundary homomorphism $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ is determined by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}.$$

This definition of boundary is clearly a triangulated version of the usual boundary of manifolds, and satisfies $\partial \circ \partial = \emptyset$, i.e.

Lemma 2.5. *The composition $\partial_{n-1} \circ \partial_n = 0$.*

Proof.

$$\partial\partial[v_0 \cdots v_n] = \sum_{j<i} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n] + \sum_{j>i} (-1)^{i+j-1} [v_0, \cdots, \hat{v}_i, \cdots \hat{v}_j, \cdots, v_n]$$

the two displayed terms cancel. ☐

Now we have produced an algebraic object: a chain complex (just as we saw in the case of the de Rham complex). Let C_n be the abelian group $\Delta_n(X)$; then we get the simplicial chain complex:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

and the homology is defined as the simplicial homology

$$H_n^\Delta(X) := \frac{Z_n = \ker \partial_n}{B_n = \operatorname{im} \partial_{n+1}}$$

Example 2.6. *The circle is a Δ-complex with one vertex and one 1-simplex. so $\Delta_0(S^1) = \Delta_1(S^1) = \mathbb{Z}$ and $\partial_1 = 0$ since $\partial e = v - v$. hence $H_0^\Delta(S^1) = \mathbb{Z} = H_1^\Delta(S^1)$ and $H_k^\Delta(S^1) = 0$ otherwise.*

Example 2.7. *For T^2 and Klein bottle: $\Delta_0 = \mathbb{Z}$, $\Delta_1 = \langle a, b, c\rangle$ and $\Delta_2 = \langle P, Q\rangle$. For $\mathbb{R}P^2$, same except $\Delta_0 = \mathbb{Z}^2$.*

2.2 Singular homology

Simplicial homology, while easy to calculate (at least by computer!), is not entirely satisfactory, mostly because it is so rigid - it is not clear, for example, that the groups do not depend on the triangulation. We therefore relax the definition and describe singular homology.

Definition 14. A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \longrightarrow X$. The free abelian group on the set of n-simplices is called $C_n(X)$, the group of n-chains.

There is a linear boundary homomorphism $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ given by

$$\partial_n \sigma = \sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]},$$

where $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ is canonically identified with Δ^{n-1}. The homology of the chain complex $(C_\bullet(X), \partial)$ is called the *singular homology* of X:

$$H_n(X) := \frac{\ker \partial : C_n(X) \longrightarrow C_{n-1}(X)}{\mathrm{im}\partial : C_{n+1}(X) \longrightarrow C_n(X)}.$$

We would like to justify the statement that the homology is a functor. In fact we would like to show that our assigning, to every space X, the complex of singular chains

$$X \mapsto (C_\bullet(X), \partial)$$

is actually a functor from topological spaces to the category of chain complexes of abelian groups, where the latter category has morphisms given by chain homomorphisms, just as in the case for the de Rham complex $(\Omega^\bullet(M), d)$. By actually taking homology, we then obtain a functor to abelian groups. We would actually like to show even more: that the functor $X \mapsto (C_\bullet(X), \partial)$ can be made into a 2-functor, sending homotopies of continuous maps to chain homotopies: this will allow us to show that $H_\bullet(X)$ is a homotopy invariant.

Given a singular n-simplex $\sigma : \Delta^n \longrightarrow X$ and a map $f : X \longrightarrow Y$, the composition $f \circ \sigma$ defines a simplex for the space Y. In this way we define

$$f_\sharp : C_n(X) \longrightarrow C_n(Y),$$

and we may verify that $f_\sharp \partial = \partial f_\sharp$, implying that $f_\sharp$ is a morphism of chain complexes, defining a functor since $(f \circ g)_\sharp = f_\sharp \circ g_\sharp$. As a consequence, this induces a homomorphism

$$f_* : H_n(X) \longrightarrow H_n(Y).$$

Now we see how $f_\sharp$ behaves for homotopic maps:

Theorem 2.8. *The chain maps $f_\sharp, g_\sharp$ induced by homotopic maps $f, g : X \longrightarrow Y$ are chain homotopic, i.e. there exists $P : C_n(X) \longrightarrow C_{n+1}(Y)$ such that*

$$g_\sharp - f_\sharp = P\partial + \partial P.$$

Hencce, $f_* = g_*$, i.e. the induced maps on homology are equal for homotopic maps.

Proof. The proof is completely analogous to the same result for the de Rham complex. Given a homotopy $F : X \times I \longrightarrow Y$ from f to g, define the *Prism operators* $P : C_n(X) \longrightarrow C_{n+1}(Y)$ as follows: for any n-simplex $\sigma : [v_0, \cdots, v_n] \longrightarrow X$, form the prism $[v_0, \cdots, v_n] \times I$, name the vertices $v_i = (v_i, 0)$ and $w_i = (v_i, 1)$, and decompose this prism in terms of $n+1$-simplices as follows:

$$[v_0, \cdots, v_n] \times I = \bigcup_{i=0}^{n} [v_0, \cdots, v_i, w_i, \cdots w_n].$$

Then we define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{Id})|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]} \quad \in C_{n+1}(Y)$$

Now we show that $\partial P = g_\sharp - f_\sharp - P\partial$, which expresses the fact that the boundary of the prism (left hand) consists of the top $\Delta^n \times 1$, bottom $\Delta^n \times 0$, and sides $\partial\Delta^n \times I$ of the prism.

$$\begin{aligned}\partial P(\sigma) = &\sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0 \cdots, \hat{v}_j, \cdots v_i, w_i, \cdots, w_n]} \\ &+ \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{Id})|_{[v_0 \cdots, v_i, w_i, \cdots \hat{w}_j, \cdots, w_n]}\end{aligned}$$

The terms with $i = j$ in the two lines cancel except for $i = j = 0$ and $i = j = n$, giving $g_\sharp(\sigma) - f_\sharp(\sigma)$. The terms with $i \neq j$ are $-P\partial(\sigma)$ by expressing it as a sum

$$\begin{aligned}P\partial(\sigma) = &\sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0 \cdots, v_i, w_i, \cdots \hat{w}_j, \cdots, w_n]} \\ &+ \sum_{i>j} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0 \cdots, \hat{v}_j, \cdots, v_i, w_i, \cdots, w_n]}\end{aligned}$$

□

Corollary 2.9. *$C_\bullet$ is a 2-functor and $H_\bullet$ is homotopy invariant.*

2.3 H_0 and H_1

Proposition 2.10. *If X has path components X_α, then $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$.*

Proof. A singular simplex always has path-connected image. Hence $C_n(X)$ is the direct sum of $C_n(X_\alpha)$. The boundary maps preserve this decomposition. So $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$. (Since chains are finite sums, we use the direct sum). □

Proposition 2.11. *If X is path-connected (and nonempty) then $H_0(X) \cong \mathbb{Z}$.*

Proof. Define $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$ via $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. This is surjective if X nonempty. We must show that $\ker\epsilon = \mathrm{im}\partial_1$.

For any singular 1-simplex $\sigma : \Delta^1 \longrightarrow X$, we have $\epsilon(\partial\sigma) = \epsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$. Hence $\mathrm{im}\partial_1 \subset \ker\epsilon$.

For the reverse inclusion: if $\sum_i n_i = 0$, we wish to show tha $\sum_i n_i\sigma_i$ is a boundary of a singular 1-simplex. Choose a path $\tau_i : I \longrightarrow X$ from a basepoint x_0 to $\sigma_i(v_0)$ and let σ_0 be the 0-simplex with image x_0. Then $\partial\tau_i = \sigma_i - \sigma_0$, viewing τ_i as a singular 1-simplex. Then $\partial\sum_i n_i\tau_i = \sum_i n_i\sigma_i$. □

Later, we will axiomatize homology as a functor from spaces to abelian groups: there are many different such functors, corresponding to different *homology theories*. To understand any homology theory it is fundamental to compute its value on the one-point space.

Proposition 2.12. *If $X = \{*\}$ then $H_n(X) = 0$ for $n > 0$ (and $H_0(X) = \mathbb{Z}$ by the above result).*

Proof. When the target is a single point, there can be only one singular n-simplex for each n, namely, the map sending Δ^n to the point $*$. Hence the chain groups are all $\mathbb{Z}$, generated by σ_n. The boundary map is $\partial\sigma_n = \sum_{i=0}^n (-1)^i\sigma_{n-1}$, which vanishes for n odd and is equal to σ_{n-1} for $n \neq 0$ and even. Hence the singular chain complex is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

which has homology

$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad 0$$

□

Note that the map $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$ defined above may be viewed as an extension of the singular chain complex (with $C_{-1}(X) = \mathbb{Z}$). The homology groups of this augmented chain complex are called the *reduced homology* of X, and denoted $\tilde{H}_n(X)$. Clearly $H_n(X) \cong \tilde{H}_n(X) \oplus \mathbb{Z}$ and $\tilde{H}_n(X) \cong H_n(X)$ for all $n > 0$.

Theorem 2.13 (Hurewicz isomorphism)**.** *The natural map $h : \pi_1(X, x_0) \longrightarrow H_1(X)$, given by regarding loops as singular 1-cycles, is a homomorphism. If X is path-connected, h induces an isomorphism $\pi_1(X)/[\pi_1(X), \pi_1(X)] \longrightarrow H_1(X)$, i.e. $H_1(X)$ is the abelianization of the fundamental group.*

In higher dimension, the Hurewicz theorem states that if the path-connected space X is $n-1$ connected for $n \geq 2$ (i.e. $\pi_k(X) = 0\ \forall k < n$), then $\pi_n(X)$ is isomorphic to $H_n(X)$.

Proof. First we describe some properties of the homology relation on paths $f \sim g \Leftrightarrow \exists\tau\ :\ f - g = \partial\tau$, as opposed to the homotopy of paths relation $f \simeq g$.

- if f is a constant path, then $f \sim 0$ since $H_1(*) = 0$.
- $f \simeq g \Rightarrow f \sim g$ since we can write the homotopy $I \times I \longrightarrow X$ as a singular 2-chain (with two singular 2-simplices – cut the square by the diagonal) with boundary $f - g + x_0 - x_1$, and since the constant paths x_0, x_1 are boundaries, so is $f - g$.
- $f \cdot g \sim f + g$, since we can define a singular 2-chain with boundary $f + g - f \cdot g$ by letting $\sigma : [v_0, v_1, v_2] \longrightarrow X$ be the composition of orthogonal projection onto $[v_0, v_2]$ followed by $f \cdot g : [v_0, v_2] \longrightarrow X$.

- $f^{-1} \sim -f$, since $f + f^{-1} \sim f \cdot f^{-1} \sim 0$.

Applying these properties to loops, we obtain that h is a homomorphism. Clearly $[\pi_1, \pi_1] \subset \ker h$, since H_1 is abelian. Hence h induces a homomorphism $\pi_1^{ab}(X) \longrightarrow H_1$.

A map in the opposite direction is given as follows: if f is a loop representative for a class in H_1, choose any path γ from x_0 to $f(0)$. Then $\psi : [f] \mapsto [\gamma f \gamma^{-1}]$ is well-defined when taking values in π_1^{ab}.

Furthermore, it vanishes on boundaries: check on a singular 2-simplex, and view the 2-simplex as a homotopy. It remains to show that $\psi \circ h = h \circ \psi = 1$. ☐

2.4 Relative homology and the excision property

It is natural to expect that the homology of a space X is related to the homology of one of its subspaces $A \subset X$; relative homology is a systematic way of analyzing this idea. Under some conditions on the pair (X, A), we will also investigate the relationship to the homology of X/A. This will also lead us to the Excision property and the Mayer-Vietoris sequence.

Definition 15. Let X be a space and $A \subset X$ a subspace. The *relative chains* $C_n(X, A)$ are chains in X modulo chains in A, i.e.

$$C_n(X, A) := \frac{C_n(X)}{C_n(A)}.$$

Since the boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it descends to a boundary map, also called $\partial : C_n(X, A) \longrightarrow C_{n-1}(X, A)$. We therefore get a chain complex

$$\cdots \longrightarrow C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \longrightarrow \cdots$$

whose cohomology gives the relative homology groups $H_n(X, A)$. Intuitively, relative homology is the homology of X modulo A.

It is clear that our previous functoriality results on $H_n(X)$ (sometimes called the *absolute* homology of X) carry over to the relative homology. For example:

Proposition 2.14. *if two maps of pairs $f, g : (X, A) \longrightarrow (Y, B)$ are homotopic through maps of pairs $(X, A) \longrightarrow (Y, B)$, then $f_* = g_*$ on relative cohomology.*

The first result about relative homology groups is an algebraic fact which follows directly from their definition. Since $C_n(X, A)$ is by definition the quotient of $C_n(X)$ by $C_n(A)$, let $i : C_n(A) \longrightarrow C_n(X)$ be the inclusion and j be the quotient map, so that we have the exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

We have this exact sequence for each n, and it also commutes with the boundary operator. Hence we get an exact sequence of *chain complexes*:

$$0 \longrightarrow (C_\bullet(A), \partial) \xrightarrow{i} (C_\bullet(X), \partial) \xrightarrow{j} (C_\bullet(X, A), \partial) \longrightarrow 0$$

Just as we saw for the de Rham complex, a short exact sequence of chain complexes gives a long exact sequence of homology groups. Since we are dealing with chain complexes, not cochain complexes, the connecting homomorphism δ coming from the boundary map ∂ is of degree -1. In this case, we obtain

Proposition 2.15 (Exactness). *Given $A \subset X$, we have the following exact sequence:*

$$\begin{tikzcd} H_\bullet(A) \arrow[rr, "i_*"] & & H_\bullet(X) \arrow[dl, "j_*"] \\ & H_\bullet(X, A) \arrow[ul, "\delta(-1)"] & \end{tikzcd}$$

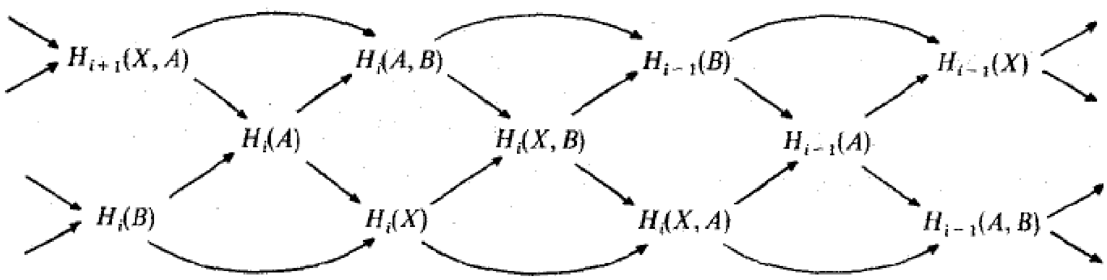

Figure 1: Braid diagram for triple

In fact, the boundary map δ has an obvious description in this application to relative homology: if $\alpha \in C_n(X, A)$ is a relative cycle, then $\delta[\alpha]$ is the $n-1$-homology class given by $[\partial\alpha] \in H_{n-1}(A)$.

Example 2.16. *Let $x_0 \in X$ and consider the pair (X, x_0). Then the long exact sequence in relative homology implies $H_n(X, x_0) \cong H_n(X)$ for all $n > 0$, while for $n = 0$ we have*

$$0 \longrightarrow H_0(x_0) \longrightarrow H_0(X) \longrightarrow H_0(X, x_0) \longrightarrow 0 \ ,$$

showing that $H_0(X, x_0) \cong \tilde{H}_0(X)$ and hence $H_n(X, x_0) \cong \tilde{H}_n(X)$ for all n.

Formal consequences of subspace inclusion for relative homology can be more complicated: for instance, suppose we have a triple (X, A, B) where $B \subset A \subset X$. Then we have short exact sequences

$$0 \longrightarrow C_n(A, B) \longrightarrow C_n(X, B) \longrightarrow C_n(X, A) \longrightarrow 0 \ ,$$

inducing the long exact sequence in homology:

$$\begin{tikzcd}
H_\bullet(A, B) \arrow[rr] & & H_\bullet(X, B) \arrow[dl] \\
& H_\bullet(X, A) \arrow[ul, "\delta(-1)"] &
\end{tikzcd}$$

In fact, this long exact sequence couples with the long exact sequences for each pair to form a braid diagram–see Fig. 2.4

The main result on relative homology is the *excision property*, which states that the homology of X relative to $A \subset X$ remains the same after deleting a subset Z whose closure sits in the interior of A. The property is so fundamental that it has been promoted to an axiom defining a homology theory, as we shall see.

Theorem 2.17 (Excision)**.** *Let $Z \subset A \subset X$, with $\overline{Z} \subset A^\circ$. Then the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_n(X \backslash Z, A \backslash Z) \longrightarrow H_n(X, A) \qquad \forall n.$$

An equivalent formulation is that if $A, B \subset X$ have interiors which cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \longrightarrow H_n(X, A)$ for all n (simply set $B = X \backslash Z$ or $Z = X \backslash B$).

Proof of Excision. Consider X as a union of A and B with interiors covering X. Then we have natural inclusion maps

$$\begin{tikzcd}
& C_\bullet(X) & \\
& C_\bullet(A) + C_\bullet(B) \arrow[u, "\iota"] & \\
C_\bullet(A) \arrow[ur] & & C_\bullet(B) \arrow[ul] \\
& C_\bullet(A \cap B) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

If the map ι were an isomorphism, then we would have $C_\bullet(X)/C_\bullet(A) = C_\bullet(B)/C_\bullet(A \cap B)$, giving the result. But the problem is that ι is not an isomorphism; there are "bad" simplices which can have nonempty intersection with $A - A \cap B$ and $B - A \cap B$. We would like to show that we can subdivide the bad simplices into smaller good ones via a chain map $\rho : C_\bullet(X) \longrightarrow C_\bullet(A) + C_\bullet(B)$, in such a way that it doesn't change the homology. In fact, we show that $C_\bullet(A) + C_\bullet(B)$ is a *deformation retract* of $C_\bullet(X)$, in the sense that $\rho \circ \iota = \mathrm{Id}$ and $\iota \circ \rho = \partial D + D\partial$ for some chain homotopy D. In fact we will choose D to preserve the subcomplexes $C_\bullet(A)$ and $C_\bullet(B)$, implying that we obtain a chain homotopy equivalence

$$C_\bullet(X)/C_\bullet(A) \longrightarrow C_\bullet(B)/C_\bullet(A \cap B),$$

yielding the proof of the theorem.

The map ρ will essentially be an iteration of the barycentric subdivision map S, which we now define (we will be a little sloppy to speed things up - see Hatcher for a full treatment).

Definition 16 (Subdivision operator)**.** If $w_0, \dots w_n$ are points in a vector space and b is any other point, then b can be added to a simplex, forming a cone: $b \cdot [w_0, \cdots, w_n] = [b, w_0, \cdots, w_n]$. Note that $\partial b = \mathrm{Id} - b\partial$, i.e. the boundary of a cone consists of the base together with the cone on the boundary. Given any simplex λ, let b_λ be the barycenter. Then we define inductively the *barycentric subdivision* $S\lambda = b_\lambda \cdot S(\partial\lambda)$, with the initial step $S[\emptyset] = [\emptyset]$ on the empty simplex. Note that the diameter of each simplex in the barycentric subdivision of $[v_0, \cdots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \cdots, v_n]$, so that they approach zero size as $n \to \infty$.

Now, given a singular n-simplex $\sigma : \Delta^n \longrightarrow X$, define $S\sigma = \sigma|_{S\Delta^n}$, in the sense that it is a signed sum of restrictions of σ to the n-simplices of the barycentric subdivision of Δ^n. $S : C_n(X) \longrightarrow C_n(X)$ is a chain map, since

$$\begin{aligned}
\partial S\lambda &= \partial(b_\lambda(S\partial\lambda)) \\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \text{ since } \partial b_\lambda + b_\lambda\partial = 1 \\
&= S\partial\lambda - b_\lambda(S\partial\partial\lambda) \text{ by induction} \\
&= S\partial\lambda.
\end{aligned}$$

This subdivision operator is chain homotopic to the identity, via the map $T : C_n(X) \longrightarrow C_{n+1}(X)$ given as follows: Subdivide $\Delta^n \times I$ into simplices inductively by joining all simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$ to the barycenter of $\Delta^n \times \{1\}$. Projecting $\Delta^n \times I \longrightarrow \Delta^n$, we may compose with any singular simplex $\sigma : \Delta^n \longrightarrow X$ to obtain a sum of $n+1$-simplices. Formalizing this, we have $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ and $T[\emptyset] = 0$. We may then check the formula $\partial T + T\partial = \mathrm{Id} - S$:

$$\begin{aligned}
\partial T\lambda &= \partial(b_\lambda(\lambda - T\partial\lambda)) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda)) \text{ using } \partial B_\lambda = \mathrm{Id} - b_\lambda\partial \\
&= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda) \text{ by induction} \\
&= \lambda - T\partial\lambda - S\lambda \text{ since } S\lambda = b_\lambda(S\partial\lambda)
\end{aligned}$$

Note that T also preserves $C_\bullet(A), C_\bullet(B)$.

For each singular n-simplex $\sigma : \Delta^n \longrightarrow X$, there exists a minimal $m(\sigma)$ such that $S^{m(\sigma)}(\sigma)$ lies in $C_n(A) + C_n(B)$. Note that the chain homotopy between S^m and Id is given by

$$D_m = T(1 + S + S^2 + \cdots + S^{m-1}).$$

We then define $D : C_n(X) \longrightarrow C_{n+1}(X)$ via $D\sigma = D_{m(\sigma)}\sigma$, and then we compute

$$\partial D\sigma + D\partial\sigma = \sigma - [S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)],$$

and finally we define $\rho(\sigma)$ to be the bracketed term. Claim: ρ maps $C_\bullet(X)$ to $C_\bullet(A)+C_\bullet(B)$. The first term $S^{m(\sigma)}$ clearly does, and since $m(\partial\sigma) \le m(\sigma)$, it follows that $(D_{m(\sigma)} - D)(\partial\sigma)$ consists of terms $TS^i(\partial\sigma)$ for $i \ge m(\partial\sigma)$, which all lie in $C_\bullet(A) + C_\bullet(B)$.

Finally, we have constructed ρ, D such that $\rho\iota = \mathrm{Id}$ (since m is zero) and $\partial D + D\partial = \mathrm{Id} - \iota\rho$, with D preserving the subcomplex $C_\bullet(A) + C_\bullet(B)$. As explained earlier, this proves the result.

□

Let $A \subset X$ be a nonempty closed subspace which is a deformation retract of some neighbourhood in X. We call such a pair (X, A) a *good pair* (CW pairs are automatically good pairs, see the Appendix in Hatcher).

Corollary 2.18. *If (X, A) is a good pair, then the quotient map $q : (X, A) \longrightarrow (X/A, A/A)$ induces isomorphisms*

$$q_* : H_n(X, A) \longrightarrow H_n(X/A, A/A) \cong \tilde{H}_n(X/A) \qquad \forall n.$$

Proof. Let V be a neighbourhood of A in X which deformation retracts onto A and let $\iota : A \hookrightarrow V$ be the inclusion. Then we have the diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\iota_*"] \arrow[d, "q_*"'] & H_n(X,V) \arrow[d, "q'_*"] \\ H_n(X/A, A/A) \arrow[r, "\iota'_*"'] & H_n(X/A, V/A) \end{tikzcd}$$

The map ι_* is an isomorphism, as follows: $H_n(V, A)$ are zero for all n, since the deformation retraction gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$ and $H_n(A, A) = 0$. Then using the long exact sequence for the triple (X, V, A) we see that ι_* is an iso.

ι'_* is also an iso, since the deformation retraction induces a deformation retraction of V/A onto A/A, so by the same argument we get ι'_* is an iso.

The groups on the right can be obtained by excision:

$$\begin{tikzcd} H_n(X,V) \arrow[d, "q'_*"'] & H_n(X\backslash A, V\backslash A) \arrow[l, "j_*"'] \arrow[d, "q''_*"] \\ H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "j'_*"'] \end{tikzcd}$$

The maps j_*, j'_* are iso by the excision property, and q''_* is an iso, since q restricted to the complement of A is a homeo. This implies q'_* is an iso, and hence q_* is an iso, as required. □

Corollary 2.19. *If (X, A) is a good pair, then the exact sequence for relative homology may be written as*

$$\begin{tikzcd} \tilde{H}_\bullet(A) \arrow[rr] & & \tilde{H}_\bullet(X) \arrow[dl] \\ & \tilde{H}_\bullet(X/A) \arrow[ul, "\delta(-1)"] & \end{tikzcd}$$

The above long exact sequence may be applied to the pair $(D^n, \partial D^n)$, where D^n is the closed unit n-ball; Note that $D^n \simeq *$ and hence $H_k(D^n) = 0\ \forall k$. Also note that $D^n/\partial D^n \simeq S^n$. Hence we have isomorphisms $\tilde{H}_i(S^n) \cong H_{i-1}(S^{n-1})$, implying that $\tilde{H}_k(S^n)$ vanishes for $k \neq n$ and is isomorphic to $\mathbb{Z}$ for $k = n$.

Corollary 2.20. *$H_k(S^n) \cong \mathbb{Z}$ for $k = 0, n$ and $H_i(S^n) = 0$ otherwise.*

We also get Brouwer's theorem from this:

Corollary 2.21. *∂D^n is not a retract of D^n, and hence every map $f : D^n \longrightarrow D^n$ has a fixed point.*

Proof. Let r be such a retraction, so that $ri = \mathrm{Id}$ for the inclusion $i : \partial D^n \hookrightarrow D^n$. Then the composition

$$H_{n-1}(\partial D^n) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(\partial D^n)$$

is the identity map on $H_{n-1}(\partial D^n) \cong \mathbb{Z}$. Of course this is absurd since $H_{n-1}(D^n) = 0$. □

Another easy consequence is the computation of $H_\bullet(X \wedge Y)$: if the wedge sum is formed at points $x \in X$ and $y \in Y$ such that $(x, X), (y, Y)$ are good pairs, then the inclusions $i : X \longrightarrow X \wedge Y$ and $j : Y \longrightarrow X \wedge Y$ induce isomorphisms

$$\tilde{H}_k(X) \oplus \tilde{H}_k(Y) \longrightarrow \tilde{H}_k(X \wedge Y).$$

This follows from the fact that $(X \sqcup Y, \{x, y\})$ is a good pair and $H(X \sqcup Y/\{x, y\}) \cong \tilde{H}(X \sqcup Y, \{x, y\}) = \tilde{H}(X) \oplus \tilde{H}(Y)$.

Yet another result which we may now prove easily: Brouwer's invariance of dimension.

Corollary 2.22. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic and nonempty, then $n = m$.*

This result is easily obtained with the definition of *local homology groups*

Definition 17. Let $x \in X$. Then the local homology groups of X at x are $H_n(X, X \backslash \{x\})$.

For any open neighbourhood U of x, excision gives isomorphisms

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\}),$$

hence the local homology groups only depend locally on x. For instance, a homeomorphism $f : X \longrightarrow Y$ must induce an isomorphism from the local homology of x to that of $f(x)$.

For topological n-manifolds, $H_k(X, X - \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \tilde{H}_{k-1}(\mathbb{R}^n - \{0\}) \cong \tilde{H}_{k-1}(S^{n-1})$ and hence it vanishes unless $k = n$, in which case it is isomorphic to $\mathbb{Z}$. Note that we obtain a fiber bundle over X with fiber above x given by $H_n(X, X - \{x\})$ and isomorphic to $\mathbb{Z}$. Is this a trivial fiber bundle?