1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made up by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We would like to make this precise.

1.1 Topological manifolds

Definition 1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

Note: "Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold, i.e. whose union is the whole space, is called an *atlas*.

We now give a bunch of examples of topological manifolds. The simplest is, technically, the empty set. More simple examples include a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.1 (Circle). *Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi i c}$ for a unique real number $0 \leq c < 1$, and define the map*

$$\nu_z : t \mapsto e^{2\pi i t}. \tag{1}$$

We note that ν_z maps the interval $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighbourhood of z given by $S^1 \backslash \{-z\}$, and it is a homeomorphism. Then $\varphi_z = \nu_z|_{I_c}^{-1}$ is a local coordinate chart near z.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.2 (n-torus). *$S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.*

Example 1.3 (open subsets). *Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M.*

For example, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{2}$$

known as the general linear group, which therefore forms a topological manifold.

Example 1.4 (Spheres). *The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:*

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the North pole and let $S = (-1, 0, \ldots, 0)$ be the South pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1} \vec{x}, \tag{3}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1} \vec{x}. \tag{4}$$

We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopic to S^n without *being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the (easy) classification of topological 1- and 2-manifolds.*

Example 1.5 (Projective spaces)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or even $\mathbb{H}$). Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.*

More precisely, let $X = \mathbb{K}^{n+1}\backslash\{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists\lambda \in \mathbb{K}^* = \mathbb{K}\backslash\{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an open *map, since if $U \subset X$ is open, then tU is also open $\forall t \in K^*$, implying that $\cup_{t\in\mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.*

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but not *quotients), we show that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the Hausdorff property for $\mathbb{K}P^n$). This graph is simply*

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x, y) = (x_i y_j - x_j y_i) \quad i \neq j.$$

(Does this work for $\mathbb{H}$? How can it be fixed?)

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{5}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.6 (Connected sum)**.** *Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.*

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\phi : B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \longrightarrow B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \tag{6}$$

$$x \mapsto \tfrac{2\epsilon^2}{|x|^2} x. \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the connected sum *$M\sharp N$, as the quotient $X/\sim$, where*

$$X = (M\backslash\overline{\varphi^{-1}(B(0, \epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0, \epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

Two important remarks concerning the connect sum: first, the connect sum of a sphere with itself is homeomorphic to the same sphere:

$$S^n\sharp S^n \cong S^n.$$

Second, by taking repeated connect sums of T^2 and $\mathbb{R}P^2$, we may obtain all compact 2-dimensional manifolds.

Example 1.7 (General gluing construction). *To construct a topological manifold "from scratch", we should be able to glue pieces of $\mathbb{R}^n$ together, as long as the gluing is consistent and by homeomorphisms. The following is a method for doing so, tailor-made so that all the requirements are satisfied.*

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} , \tag{8}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be homeomorphisms.

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij}\cap U_{jk}}$ for all i, j, k.

Second countability of the glued manifold will be guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that $\forall p \in \partial U_{ij} \subset U_i$ and $\forall q \in \partial U_{ji} \subset U_j$, there exist neighbourhoods $V_p \subset U_i$ and $V_q \subset U_j$ of p, q respectively with $\varphi_{ij}(V_p \cap U_{ij}) \cap V_q = \emptyset$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{9}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$. This space naturally comes with an atlas $\mathcal{A}$, where the charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

As an exercise, you may show that any topological manifold is homeomorphic to one constructed in this way.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, if we compare coordinates on the intersection $U_{ij} = U_i \cap U_j$, we see that the map

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

is a homeomorphism, simply because it is a composition of homeomorphisms. We can say this another way: topological manifolds are glued together by homeomorphisms.

This means that we may be able to differentiate a function in one coordinate chart but not in another, i.e. there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, which is simply a topological manifold where the gluing maps are required to be *smooth*.

First we recall the notion of a smooth map of finite-dimensional vector spaces.

Remark 1 (Aside on smooth maps of vector spaces). *Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. The function f is said to be differentiable at $p \in U$ if there exists a linear map $Df(p) : V \longrightarrow W$ such that*

$$||f(p + x) - f(p) - Df(p)(x)|| = o(||x||),$$

where $o : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous at 0 and $o(0) = 0$, and we choose any inner product on V, W, defining the norm $|| \cdot ||$. For infinite-dimensional vector spaces, the topology is highly sensitive to which norm is chosen, but we will work in finite dimensions.

Given linear coordinates $(x_1, \ldots, x_n)$ on V, and $(y_1, \ldots, y_m)$ on W, we may express f in terms of its m components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written as an $m \times n$ matrix, called the Jacobian matrix *of f at p.*

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{10}$$

We say that f is differentiable on U when it is differentiable at all $p \in U$ and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W)$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

The first derivative Df is also a map from U to a vector space ($\mathrm{Hom}(V, W)$), therefore if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)),$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and hence we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W).$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $S^2 V^* \otimes W$, since "mixed partials are equal".

After this aside, we can define a smooth manifold.

Definition 2. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, explained below.

Definition 3. An atlas $\mathcal{A} = \{U_i, \varphi_i\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Note: Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We may now verify that all the examples from section 1.1 are actually smooth manifolds:

Example 1.8 (Circle)**.** *For Example 1.1, only two charts, e.g. $\varphi_{\pm 1}$, suffice to define an atlas, and we have*

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t + 1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2}, \end{cases}$$

which is clearly C^∞. In fact all the charts φ_z are smoothly compatible. Hence the circle is a smooth manifold.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.2, is smooth. Example 1.3 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.9 (Spheres)**.** *The charts for the n-sphere given in Example 1.4 form a smooth atlas, since*

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0}\vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\vec{z},$$

which is smooth on $\mathbb{R}^n \backslash \{0\}$, as required.

Example 1.10 (Projective spaces)**.** *The charts for projective spaces given in Example 1.5 form a smooth atlas, since*

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1} z_2, \ldots, z_1^{-1} z_n), \tag{11}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.6 is clearly smooth since ϕ was chosen to be a smooth map, and any topological manifold from Example 1.7 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.

1.3 Manifolds with boundary

The concept of *manifold with boundary* is important for relating manifolds of different dimension. Our manifolds are defined intrinsically, meaning that they are not defined as subsets of another topological space; therefore, the notion of boundary will differ from the usual boundary of a subset.

To introduce boundaries in our manifolds, we need to change the local model which they are based on. For this reason, we introduce the half-space $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}$, equip it with the induced topology from $\mathbb{R}^n$, and model our spaces on this one.

Definition 4. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart. We shall see later that $M = \partial M \sqcup \operatorname{Int} M$.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, in the sense below.

Definition 5. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

For example, the function $f(x, y) = y$ is smooth on H^2 but $f(x, y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Note the important fact that if M is an n-manifold with boundary, $\operatorname{Int} M$ is a usual n-manifold, without boundary. Also, even more importantly, ∂M is an $n-1$-manifold without boundary, i.e. $\partial(\partial M) = \emptyset$. This is sometimes phrased as the equation

$$\partial^2 = 0.$$

Example 1.11 (Möbius strip). *The mobius strip E is a compact 2-manifold with boundary. As a topological space it is the quotient of $\mathbb{R} \times [0, 1]$ by the identification $(x, y) \sim (x + 1, 1 - y)$. The map $\pi : [(x, y)] \mapsto e^{2\pi i x}$ is a continuous surjective map to S^1, called a projection map. We may choose charts $[(x, y)] \mapsto e^{x + i\pi y}$ for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, and for any $\epsilon < \frac{1}{2}$.*

Note that ∂E is diffeomorphic to S^1. This actually provides us with our first example of a non-trivial fiber bundle, as we shall see. In this case, E is a bundle of intervals over a circle.

1.4 Cobordism

$(n + 1)$-Manifolds with boundary provide us with a natural equivalence relation on n-manifolds, called *cobordism.*

Definition 6. n-manifolds M_1, M_2 are *cobordant* when there exists a $n + 1$-manifold with boundary N such that ∂N is diffeomorphic to $M_1 \sqcup M_2$. The class of manifolds cobordant to M is called the *cobordism class* of M.

Note that while the Cartesian product of manifolds is a manifold, the Cartesian product of two manifolds with boundary is *not* a manifold with boundary. On the other hand, the Cartesian product of manifolds only one of which has boundary, is a manifold with boundary (why?)

Cobordism classes of manifolds inherit two natural operations, as follows: If $[M_1], [M_2]$ are cobordism classes, then the operation $[M_1] \cdot [M_2] = [M_1 \times M_2]$ is well-defined. Furthermore $[M_1] + [M_2] = [M_1 \sqcup M_2]$ is well-defined, and the two operations satisfy the axioms defining a commutative ring. The ring of cobordism classes of compact manifolds is called the *cobordism ring* and is denoted $\Omega^\bullet$. The subset of classes of k-dimensional manifolds is denoted $\Omega^k \subset \Omega^\bullet$.

Proposition 1. *The cobordism ring is 2-torsion, i.e. $x + x = 0 \ \ \forall x$.*

Proof. The zero element of the ring is $[\emptyset]$ and the multiplicative unit is $[*]$, the class of the one-point manifold. For any manifold M, the manifold with boundary $M\times[0,1]$ has boundary $M\sqcup M$. Hence $[M]+[M]=[\emptyset]=0$, as required. □

Example 1.12. *The n-sphere S^n is null-cobordant (i.e. cobordant to $\emptyset$), since $\partial\overline{B_{n+1}(0,1)}\cong S^n$, where $B_{n+1}(0,1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.*

Example 1.13. *Any oriented compact 2-manifold Σ_g is null-cobordant , since we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary given by Σ_g.*

We would like to state an amazing theorem of Thom, which is a complete characterization of the cobordism ring.

Theorem 1.14. *The cobordism ring is a (countably generated) polynomial ring over $\mathbb{F}_2$ with generators in every dimension $n\neq 2^k-1$, i.e.*

$$\Omega^\bullet=\mathbb{F}_2[x_2,x_4,x_5,x_6,x_8,\ldots].$$

This theorem implies that there are 3 cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2+x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives?

1.5 Smooth maps

For topological manifolds M,N of dimension m,n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous and associative, we obtain a category C^0**-Man** of topological manifolds and continuous maps. Recall that a category is simply a class of objects $\mathcal{C}$ (in our case, topological manifolds) and an associative class of arrows $\mathcal{A}$ (in our case, continuous maps) with source and target maps $\mathcal{A}\underset{t}{\overset{s}{\rightrightarrows}}\mathcal{C}$ and an identity arrow for each object, given by a map $\mathrm{Id}:\mathcal{C}\longrightarrow\mathcal{A}$ (in our case, the identity map of any manifold to itself). Conventionally we write the set of arrows $\{a\in\mathcal{A}\ :\ s(a)=x \text{ and } t(a)=y\}$ as $\mathrm{Hom}(x,y)$. Also note that the associative composition of arrows mentioned above then becomes a map

$$\mathrm{Hom}(x,y)\times\mathrm{Hom}(y,z)\longrightarrow\mathrm{Hom}(x,z).$$

If M,N are smooth manifolds, the right notion of morphism from M to N is that of a smooth map $f:M\longrightarrow N$.

Definition 7. A map $f:M\longrightarrow N$ is called smooth when for each chart (U,φ) for M and each chart (V,ψ) for N, the composition $\psi\circ f\circ\varphi^{-1}$ is a smooth map, i.e. $\psi\circ f\circ\varphi^{-1}\in C^\infty(\varphi(U),\mathbb{R}^n)$. The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M,N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

If $g:L\longrightarrow M$ and $f:M\longrightarrow N$ are smooth maps, then so is the composition $f\circ g$, since if charts φ,χ,ψ for L,M,N are chosen near $p\in L$, $g(p)\in M$, and $(fg)(p)\in N$, then $\psi\circ(f\circ g)\circ\varphi^{-1}=A\circ B$, for $A=\psi f\chi^{-1}$ and $B=\chi g\varphi^{-1}$ both smooth mappings $\mathbb{R}^n\longrightarrow\mathbb{R}^n$. By the chain rule, $A\circ B$ is differentiable at p, with derivative $D_p(A\circ B)=(D_{g(p)}A)(D_pB)$ (matrix multiplication).

Now we have a new category, which we may call C^∞**-Man**, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic. In fact, the definitions above carry over, word for word, to the setting of manifolds with boundary. Hence we have defined another category, C^∞-$\mathbf{Man}_\partial$, the category of smooth manifolds with boundary.

In defining the arrows for the category C^∞-$\mathbf{Man}_\partial$, we may choose to consider all smooth maps, or only those smooth maps $M\longrightarrow N$ such that ∂M is sent to ∂N, i.e. boundary-preserving maps. Call the resulting category in the latter case C^∞_∂-$\mathbf{Man}_\partial$.

Note that the boundary map, ∂, maps the objects of C^∞_∂-**Man**$_\partial$ to objects in C^∞-**Man**, and similarly for arrows, and such that the following square commutes:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \partial M & \xrightarrow[\psi|_{\partial M}]{} & \partial M' \end{array} \tag{12}$$

This is precisely what it means for ∂ to be a (covariant) *functor*, from the category of manifolds with boundary and boundary-preserving smooth maps, to the category of manifolds without boundary.

Fix a smooth manifold N and consider the class of pairs (M, φ) where M is a smooth manifold with boundary and φ is a smooth map $\varphi : M \longrightarrow N$. Define a category where these maps are the objects. How does the boundary operator act on this category?

Example 1.15. *We show that the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2. Consider the maps $f_+(x_0, x_1, x_2) = [1 + x_0 : x_1 + ix_2]$ and $f_-(x_0, x_1, x_2) = [x_1 - ix_2 : 1 - x_0]$. Since $f_\pm$ is continuous on $x_0 \neq \pm 1$, and since $f_- = f_+$ on $|x_0| < 1$, the pair (f_-, f_+) defines a continuous map $f : S^2 \longrightarrow \mathbb{C}P^1$. To check smoothness, we compute the compositions*

$$\varphi_0 \circ f_+ \circ \varphi_N^{-1} : (y_1, y_2) \mapsto y_1 + iy_2, \tag{13}$$

$$\varphi_1 \circ f_- \circ \varphi_S^{-1} : (y_1, y_2) \mapsto y_1 - iy_2, \tag{14}$$

both of which are obviously smooth maps.

Remark 2 (Exotic smooth structures)**.** *The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called* exotic *spheres.*

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a monoid *structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are* 28 *smooth spheres, which we will encounter later on.*

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called fake $\mathbb{R}^4$*s.*

Example 1.16 (Lie groups)**.** *A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.*

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a topological group. *If G is a given a smooth structure and m, ι are smooth maps, the result is a* Lie group.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called left multiplication *and* right multiplication. *Note that the group axioms imply that $R_g L_h = L_h R_g$.*

1.6 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ), (V, ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem*.

Theorem 1.17 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^m$ an open set and $f : U \longrightarrow \mathbb{R}^m$ a smooth map such that $Df(p)$ is an invertible linear operator. Then there is a neighbourhood $V \subset U$ of p such that $f(V)$ is open and $f : V \longrightarrow f(V)$ is a diffeomorphism. furthermore, $D(f^{-1})(f(p)) = (Df(p))^{-1}$.*

Proof. Without loss of generality, assume that U contains the origin, that $f(0) = 0$ and that $Df(p) = \mathrm{Id}$ (for this, replace f by $(Df(0))^{-1} \circ f$. We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define g so that $f(x) = x + g(x)$. Hence $g(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - g(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $\phi(y)$, and so $y - g(\phi(y)) = \phi(y)$, i.e. ϕ is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0, r)$ where $||Dh_y|| \le \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0, r)$,

$$||h_y(x) - h_y(x')|| \le \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \le ||h_y(x) - h_y(0)|| + ||h_y(0)|| \le \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0, r)}$. Let $\phi(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned} ||\phi(y) - \phi(y')|| &= ||h_y(\phi(y)) - h_{y'}(\phi(y'))|| \\ &\le ||g(\phi(y)) - g(\phi(y'))|| + ||y - y'|| \\ &\le \tfrac{1}{2}||\phi(y) - \phi(y')|| + ||y - y'||, \end{aligned}$$

so that we have $||\phi(y) - \phi(y')|| \le 2||y - y''||$, as required.

To see that ϕ is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = \phi(y)$ and $x' = \phi(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0, r)}$, which is not a problem.

$$\begin{aligned} ||\phi(y) - \phi(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))|| \\ &\le ||(Df(x))^{-1}||||(Df(x))(x - x') - (f(x) - f(x'))|| \\ &\le o(||x - x'||), \text{ using differentiability of } f \\ &\le o(||y - y'||), \text{ using continuity of } \phi. \end{aligned}$$

Now that we have shown ϕ is differentiable with derivative $(Df)^{-1}$, we use the fact that Df is C^∞ and inversion is C^∞, implying that $D\phi$ is C^∞ and hence ϕ also. □

This theorem immediately provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.