These are my teaching notes and they borrow without citation from many sources, including Bar-Natan, Godbillon, Guillemin-Pollack, Milnor, Sternberg, Lee, and Mrowka.

1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made up by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We would like to make this precise.

1.1 Topological manifolds

Definition 1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

Note: "Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold, i.e. whose union is the whole space, is called an *atlas*.

We now give a bunch of examples of topological manifolds. The simplest is, technically, the empty set. More simple examples include a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.1 (Circle). *Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi ic}$ for a unique real number $0 \leq c < 1$, and define the map*

$$\nu_z : t \mapsto e^{2\pi it}. \tag{1}$$

We note that ν_z maps the interval $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighbourhood of z given by $S^1 \backslash \{-z\}$, and it is a homeomorphism. Then $\varphi_z = \nu_z|_{I_c}^{-1}$ is a local coordinate chart near z.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.2 (n-torus). *$S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.*

Example 1.3 (open subsets). *Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M.*

For example, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{2}$$

known as the general linear group, which therefore forms a topological manifold.

Example 1.4 (Spheres). *The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:*

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the North pole and let $S = (-1, 0, \ldots, 0)$ be the South pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}, \tag{3}$$
$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}. \tag{4}$$

We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopic to S^n without *being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the (easy) classification of topological 1- and 2-manifolds.*

Example 1.5 (Projective spaces)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or even $\mathbb{H}$). Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.*

More precisely, let $X = \mathbb{K}^{n+1}\backslash\{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists\lambda \in \mathbb{K}^* = \mathbb{K}\backslash\{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an open *map, since if $U \subset X$ is open, then tU is also open $\forall t \in K^*$, implying that $\cup_{t\in\mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.*

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but not *quotients), we show that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the Hausdorff property for $\mathbb{K}P^n$). This graph is simply*

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x, y) = (x_i y_j - x_j y_i) \quad i \neq j.$$

(Does this work for $\mathbb{H}$? How can it be fixed?)

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{5}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.6 (Connected sum)**.** *Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.*

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\phi : B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \longrightarrow B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \tag{6}$$

$$x \mapsto \tfrac{2\epsilon^2}{|x|^2} x. \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the connected sum *$M\sharp N$, as the quotient $X/\sim$, where*

$$X = (M\backslash\overline{\varphi^{-1}(B(0, \epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0, \epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

Two important remarks concerning the connect sum: first, the connect sum of a sphere with itself is homeomorphic to the same sphere:

$$S^n \sharp S^n \cong S^n.$$

Second, by taking repeated connect sums of T^2 and $\mathbb{R}P^2$, we may obtain all compact 2-dimensional manifolds.

Example 1.7 (General gluing construction). *To construct a topological manifold "from scratch", we should be able to glue pieces of $\mathbb{R}^n$ together, as long as the gluing is consistent and by homeomorphisms. The following is a method for doing so, tailor-made so that all the requirements are satisfied.*

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \ , \tag{8}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be homeomorphisms.

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij}\cap U_{jk}}$ for all i, j, k.

Second countability of the glued manifold will be guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that $\forall p \in \partial U_{ij} \subset U_i$ and $\forall q \in \partial U_{ji} \subset U_j$, there exist neighbourhoods $V_p \subset U_i$ and $V_q \subset U_j$ of p, q respectively with $\varphi_{ij}(V_p \cap U_{ij}) \cap V_q = \emptyset$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{9}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$. This space naturally comes with an atlas $\mathcal{A}$, where the charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

As an exercise, you may show that any topological manifold is homeomorphic to one constructed in this way.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, if we compare coordinates on the intersection $U_{ij} = U_i \cap U_j$, we see that the map

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

is a homeomorphism, simply because it is a composition of homeomorphisms. We can say this another way: topological manifolds are glued together by homeomorphisms.

This means that we may be able to differentiate a function in one coordinate chart but not in another, i.e. there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, which is simply a topological manifold where the gluing maps are required to be *smooth*.

First we recall the notion of a smooth map of finite-dimensional vector spaces.

Remark 1 (Aside on smooth maps of vector spaces). *Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. The function f is said to be differentiable at $p \in U$ if there exists a linear map $Df(p) : V \longrightarrow W$ such that*

$$||f(p + x) - f(p) - Df(p)(x)|| = o(||x||),$$

where $o : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous at 0 and $\lim_{t\to 0} o(t)/t = 0$, and we choose any inner product on V, W, defining the norm $||\cdot||$. For infinite-dimensional vector spaces, the topology is highly sensitive to which norm is chosen, but we will work in finite dimensions.

Given linear coordinates $(x_1, \dots, x_n)$ on V, and $(y_1, \dots, y_m)$ on W, we may express f in terms of its m components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written as an $m \times n$ matrix, called the Jacobian matrix *of f at p.*

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{10}$$

We say that f is differentiable on U when it is differentiable at all $p \in U$ and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W)$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

The first derivative Df is also a map from U to a vector space ($\mathrm{Hom}(V, W)$), therefore if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)),$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and hence we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W).$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $S^2 V^* \otimes W$, since "mixed partials are equal".

After this aside, we can define a smooth manifold.

Definition 2. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, explained below.

Definition 3. An atlas $\mathcal{A} = \{U_i, \varphi_i\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Note: Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We may now verify that all the examples from section 1.1 are actually smooth manifolds:

Example 1.8 (Circle)**.** *For Example 1.1, only two charts, e.g. $\varphi_{\pm 1}$, suffice to define an atlas, and we have*

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t + 1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2}, \end{cases}$$

which is clearly C^∞. In fact all the charts φ_z are smoothly compatible. Hence the circle is a smooth manifold.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.2, is smooth. Example 1.3 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.9 (Spheres). *The charts for the n-sphere given in Example 1.4 form a smooth atlas, since*

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0}\vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\vec{z},$$

which is smooth on $\mathbb{R}^n \backslash \{0\}$, as required.

Example 1.10 (Projective spaces). *The charts for projective spaces given in Example 1.5 form a smooth atlas, since*

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1} z_2, \ldots, z_1^{-1} z_n), \tag{11}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.6 is clearly smooth since ϕ was chosen to be a smooth map, and any topological manifold from Example 1.7 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.

1.3 Manifolds with boundary

The concept of *manifold with boundary* is important for relating manifolds of different dimension. Our manifolds are defined intrinsically, meaning that they are not defined as subsets of another topological space; therefore, the notion of boundary will differ from the usual boundary of a subset.

To introduce boundaries in our manifolds, we need to change the local model which they are based on. For this reason, we introduce the half-space $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}$, equip it with the induced topology from $\mathbb{R}^n$, and model our spaces on this one.

Definition 4. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart. We shall see later that $M = \partial M \sqcup \operatorname{Int} M$.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, in the sense below.

Definition 5. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

For example, the function $f(x,y) = y$ is smooth on H^2 but $f(x,y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Note the important fact that if M is an n-manifold with boundary, $\operatorname{Int} M$ is a usual n-manifold, without boundary. Also, even more importantly, ∂M is an $n-1$-manifold without boundary, i.e. $\partial(\partial M) = \emptyset$. This is sometimes phrased as the equation

$$\partial^2 = 0.$$

Example 1.11 (Möbius strip)**.** *The mobius strip E is a compact 2-manifold with boundary. As a topological space it is the quotient of $\mathbb{R} \times [0,1]$ by the identification $(x,y) \sim (x+1, 1-y)$. The map $\pi : [(x,y)] \mapsto e^{2\pi i x}$ is a continuous surjective map to S^1, called a projection map. We may choose charts $[(x,y)] \mapsto e^{x + i\pi y}$ for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, and for any $\epsilon < \frac{1}{2}$.*

Note that ∂E is diffeomorphic to S^1. This actually provides us with our first example of a non-trivial fiber bundle, as we shall see. In this case, E is a bundle of intervals over a circle.

1.4 Cobordism

$(n+1)$-Manifolds with boundary provide us with a natural equivalence relation on n-manifolds, called *cobordism*.

Definition 6. n-manifolds M_1, M_2 are *cobordant* when there exists a $n+1$-manifold with boundary N such that ∂N is diffeomorphic to $M_1 \sqcup M_2$. The class of manifolds cobordant to M is called the *cobordism class* of M.

Note that while the Cartesian product of manifolds is a manifold, the Cartesian product of two manifolds with boundary is *not* a manifold with boundary. On the other hand, the Cartesian product of manifolds only one of which has boundary, is a manifold with boundary (why?)

Cobordism classes of manifolds inherit two natural operations, as follows: If $[M_1], [M_2]$ are cobordism classes, then the operation $[M_1] \cdot [M_2] = [M_1 \times M_2]$ is well-defined. Furthermore $[M_1] + [M_2] = [M_1 \sqcup M_2]$ is well-defined, and the two operations satisfy the axioms defining a commutative ring. The ring of cobordism classes of compact manifolds is called the *cobordism ring* and is denoted $\Omega^\bullet$. The subset of classes of k-dimensional manifolds is denoted $\Omega^k \subset \Omega^\bullet$.

Proposition 1.12. *The cobordism ring is 2-torsion, i.e. $x + x = 0 \ \ \forall x$.*

Proof. The zero element of the ring is $[\emptyset]$ and the multiplicative unit is $[*]$, the class of the one-point manifold. For any manifold M, the manifold with boundary $M\times[0,1]$ has boundary $M\sqcup M$. Hence $[M]+[M]=[\emptyset]=0$, as required. □

Example 1.13. *The n-sphere S^n is null-cobordant (i.e. cobordant to $\emptyset$), since $\partial\overline{B_{n+1}(0,1)}\cong S^n$, where $B_{n+1}(0,1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.*

Example 1.14. *Any oriented compact 2-manifold Σ_g is null-cobordant , since we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary given by Σ_g.*

We would like to state an amazing theorem of Thom, which is a complete characterization of the cobordism ring.

Theorem 1.15. *The cobordism ring is a (countably generated) polynomial ring over $\mathbb{F}_2$ with generators in every dimension $n\neq 2^k-1$, i.e.*

$$\Omega^\bullet=\mathbb{F}_2[x_2,x_4,x_5,x_6,x_8,\ldots].$$

This theorem implies that there are 3 cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2+x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives?

1.5 Smooth maps

For topological manifolds M,N of dimension m,n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous and associative, we obtain a category C^0**-Man** of topological manifolds and continuous maps. Recall that a category is simply a class of objects $\mathcal{C}$ (in our case, topological manifolds) and an associative class of arrows $\mathcal{A}$ (in our case, continuous maps) with source and target maps $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathcal{C} \end{tikzcd}$ and an identity arrow for each object, given by a map $\mathrm{Id}:\mathcal{C}\longrightarrow\mathcal{A}$ (in our case, the identity map of any manifold to itself). Conventionally we write the set of arrows $\{a\in\mathcal{A}\ :\ s(a)=x \text{ and } t(a)=y\}$ as $\mathrm{Hom}(x,y)$. Also note that the associative composition of arrows mentioned above then becomes a map

$$\mathrm{Hom}(x,y)\times\mathrm{Hom}(y,z)\longrightarrow\mathrm{Hom}(x,z).$$

If M,N are smooth manifolds, the right notion of morphism from M to N is that of a smooth map $f:M\longrightarrow N$.

Definition 7. A map $f:M\longrightarrow N$ is called smooth when for each chart (U,φ) for M and each chart (V,ψ) for N, the composition $\psi\circ f\circ\varphi^{-1}$ is a smooth map, i.e. $\psi\circ f\circ\varphi^{-1}\in C^\infty(\varphi(U),\mathbb{R}^n)$. The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M,N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

If $g:L\longrightarrow M$ and $f:M\longrightarrow N$ are smooth maps, then so is the composition $f\circ g$, since if charts φ,χ,ψ for L,M,N are chosen near $p\in L$, $g(p)\in M$, and $(fg)(p)\in N$, then $\psi\circ(f\circ g)\circ\varphi^{-1}=A\circ B$, for $A=\psi f\chi^{-1}$ and $B=\chi g\varphi^{-1}$ both smooth mappings $\mathbb{R}^n\longrightarrow\mathbb{R}^n$. By the chain rule, $A\circ B$ is differentiable at p, with derivative $D_p(A\circ B)=(D_{g(p)}A)(D_pB)$ (matrix multiplication).

Now we have a new category, which we may call C^∞**-Man**, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic. In fact, the definitions above carry over, word for word, to the setting of manifolds with boundary. Hence we have defined another category, C^∞-$\mathbf{Man}_\partial$, the category of smooth manifolds with boundary.

In defining the arrows for the category C^∞-$\mathbf{Man}_\partial$, we may choose to consider all smooth maps, or only those smooth maps $M\longrightarrow N$ such that ∂M is sent to ∂N, i.e. boundary-preserving maps. Call the resulting category in the latter case C^∞_∂-$\mathbf{Man}_\partial$.

Note that the boundary map, ∂, maps the objects of C^∞_∂**-Man**$_\partial$ to objects in C^∞**-Man**, and similarly for arrows, and such that the following square commutes:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\partial"'] & M' \arrow[d, "\partial"] \\ \partial M \arrow[r, "\psi|_{\partial M}"'] & \partial M' \end{tikzcd} \tag{12}$$

This is precisely what it means for ∂ to be a (covariant) *functor*, from the category of manifolds with boundary and boundary-preserving smooth maps, to the category of manifolds without boundary.

Fix a smooth manifold N and consider the class of pairs (M, φ) where M is a smooth manifold with boundary and φ is a smooth map $\varphi : M \longrightarrow N$. Define a category where these maps are the objects. How does the boundary operator act on this category?

Example 1.16. *We show that the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2. Consider the maps $f_+(x_0, x_1, x_2) = [1 + x_0 : x_1 + ix_2]$ and $f_-(x_0, x_1, x_2) = [x_1 - ix_2 : 1 - x_0]$. Since $f_\pm$ is continuous on $x_0 \neq \pm 1$, and since $f_- = f_+$ on $|x_0| < 1$, the pair (f_-, f_+) defines a continuous map $f : S^2 \longrightarrow \mathbb{C}P^1$. To check smoothness, we compute the compositions*

$$\varphi_0 \circ f_+ \circ \varphi_N^{-1} : (y_1, y_2) \mapsto y_1 + iy_2, \tag{13}$$

$$\varphi_1 \circ f_- \circ \varphi_S^{-1} : (y_1, y_2) \mapsto y_1 - iy_2, \tag{14}$$

both of which are obviously smooth maps.

Remark 2 (Exotic smooth structures)**.** *The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called* exotic *spheres.*

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a monoid *structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are* 28 *smooth spheres, which we will encounter later on.*

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called fake $\mathbb{R}^4$*s.*

Example 1.17 (Lie groups)**.** *A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.*

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a topological group. *If G is a given a smooth structure and m, ι are smooth maps, the result is a* Lie group.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called left multiplication *and* right multiplication. *Note that the group axioms imply that $R_g L_h = L_h R_g$.*

1.6 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ), (V, ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem*.

Theorem 1.18 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^m$ an open set and $f : U \longrightarrow \mathbb{R}^m$ a smooth map such that $Df(p)$ is an invertible linear operator. Then there is a neighbourhood $V \subset U$ of p such that $f(V)$ is open and $f : V \longrightarrow f(V)$ is a diffeomorphism. furthermore, $D(f^{-1})(f(p)) = (Df(p))^{-1}$.*

Proof. Without loss of generality, assume that U contains the origin, that $f(0) = 0$ and that $Df(p) = \mathrm{Id}$ (for this, replace f by $(Df(0))^{-1} \circ f$. We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define g so that $f(x) = x + g(x)$. Hence $g(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - g(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $\phi(y)$, and so $y - g(\phi(y)) = \phi(y)$, i.e. ϕ is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0, r)$ where $||Dh_y|| \leq \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0, r)$,

$$||h_y(x) - h_y(x')|| \leq \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \leq ||h_y(x) - h_y(0)|| + ||h_y(0)|| \leq \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0, r)}$. Let $\phi(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned}
||\phi(y) - \phi(y')|| &= ||h_y(\phi(y)) - h_{y'}(\phi(y'))|| \\
&\leq ||g(\phi(y)) - g(\phi(y'))|| + ||y - y'|| \\
&\leq \tfrac{1}{2}||\phi(y) - \phi(y')|| + ||y - y'||,
\end{aligned}$$

so that we have $||\phi(y) - \phi(y')|| \leq 2||y - y''||$, as required.

To see that ϕ is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = \phi(y)$ and $x' = \phi(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0, r)}$, which is not a problem.

$$\begin{aligned}
||\phi(y) - \phi(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))|| \\
&\leq ||(Df(x))^{-1}||||(Df(x))(x - x') - (f(x) - f(x'))|| \\
&\leq o(||x - x'||), \text{ using differentiability of } f \\
&\leq o(||y - y'||), \text{ using continuity of } \phi.
\end{aligned}$$

Now that we have shown ϕ is differentiable with derivative $(Df)^{-1}$, we use the fact that Df is C^∞ and inversion is C^∞, implying that $D\phi$ is C^∞ and hence ϕ also. □

This theorem immediately provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 1.19 (Constant rank theorem). *Let V, W be m, n-dimensional vector spaces and $U \subset V$ an open set. If $f : U \longrightarrow W$ is a smooth map such that Df has constant rank k in U, then for each point $p \in U$ there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0).$$

Proof. since rk $(f) = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \dots, x_k, y_1, \dots y_{m-k})$ on V and $(u_1, \dots u_k, v_1, \dots, v_{n-k})$ on W.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \dots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, \tilde{R} = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k \times k} & 0 \\ \frac{\partial \tilde{R}}{\partial x} & \frac{\partial \tilde{R}}{\partial y} \end{pmatrix}$$

we conclude that $\frac{\partial \tilde{R}}{\partial y} = 0$, meaning that

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)).$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - s(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0),$$

as required. □

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining *submanifolds*.

Definition 8. A *regular submanifold* of dimension k in an n-manifold M is a subspace $S \subset M$ such that $\forall s \in S$, there exists a chart (U, φ) for M, containing s, and with

$$S \cap U = \varphi^{-1}(x_{k+1} = \cdots = x_n = 0).$$

In other words, the inclusion $S \subset M$ is locally isomorphic to the vector space inclusion $\mathbb{R}^k \subset \mathbb{R}^n$.

Of course, the remaining coordinates $\{x_1, \dots, x_k\}$ define a smooth manifold structure on S itself, justifying the terminology.

Proposition 1.20. *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k regular submanifold. □

Example 1.21. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.9.*

The previous example leads to an observation of the following special case of the previous corollary.

Proposition 1.22. *If $f : M \longrightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to $\dim N$ along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Definition 9. If $f : M \longrightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \emptyset$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 10. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*

Proposition 1.23. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $F(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Then we can restrict the charts $(V', \psi|_{V'})$, $(U' = f^{-1}(V'), \varphi_{U'})$ so that we see the embedding. □

Having the constant rank theorem in hand, we may also apply it to study manifolds *with boundary*. The following two results illustrate how this may easily be done.

Proposition 1.24. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold of M. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \ldots, x_m) \mapsto x_m.$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. □

Example 1.25. *Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.*

Example 1.26. *Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?*

Proposition 1.27. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a regular submanifold with boundary (i.e. locally modeled on $\mathbb{R}^k \subset \mathbb{R}^n$ or the inclusion $H^k \subset H^n$ given by $(x_1, \ldots x_k) \mapsto (0, \ldots, 0, x_1, \ldots x_k)$.) Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a regular submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a regular submanifold of $\mathbb{R}^m$ of dimension $m - n$.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \ldots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 1.24, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) \ : \ x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. □

2 Transversality

We shall now continue to use the inverse and constant rank theorems to produce more manifolds, except now these shall be cut out only locally by functions. We shall ask when the *intersection* of two submanifolds yields a submanifold. You should think that intersecting a given submanifold with another is the local imposing of a certain number of constraints.

Two subspaces $K, L \subset V$ of a vector space V are called *transversal* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that

$$\dim V = \dim K + \dim L - \dim K \cap L.$$

We may apply this to submanifolds as follows:

Definition 11. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM.$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Note: at this point, we have not defined the tangent bundle of a manifold, but we may understand tangent spaces locally, in each chart. We may make sense of this as follows: Let $k : K \longrightarrow M$ and $l : L \longrightarrow M$ be the inclusion maps. Then we may consider T_pK, T_pL to be the images of the derivatives of k and l, in charts for K, L, M. Transversality then requires that these images span $\mathbb{R}^m$, where $m = \dim M$.

Proposition 2.1. *If $K, L \subset M$ are transverse regular submanifolds then $K \cap L$ is also a regular submanifold, of dimension $\dim K + \dim L - \dim M$.*

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^{m-k}$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^{m-l}$.

Then p must be a regular point for $(f, g) : L \cap M \cap U \longrightarrow \mathbb{R}^{2m-k-l}$ by the assumption on tangent spaces, and hence will be regular in a neighbourhood $\tilde{U}$ of p. Therefore $(f, g)|_{\tilde{U}}^{-1}(0, 0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap \tilde{U}$ is a regular submanifold. ☐

Example 2.2 (Exotic spheres)**.** *Consider the following intersections in $\mathbb{C}^5 \backslash 0$:*

$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\operatorname{Im}(Dk(a)) + \operatorname{Im}(Dl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 12. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $\operatorname{Im}(Df(a)) + \operatorname{Im}(Dg(b)) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 2.3. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K \times_M L = \{(a, b) \in K \times L \ : \ f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*

$$\begin{tikzcd}
K \times L \arrow[drr, "p_2"] \arrow[ddr, "p_1"'] & & \\
& K \times_M L \arrow[ul, "i"'] \arrow[r] \arrow[d] \arrow[dr, "f \cap g"] & L \arrow[d, "g"] \\
& K \arrow[r, "f"'] & M
\end{tikzcd}$$

where i is the inclusion and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

The manifold $K \times_M L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection product.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. Then we show that the following intersection of regular submanifolds is transverse:

$$\Gamma_{f\cap g} = (\Gamma_f \times \Gamma_g) \cap (K \times L \times \Delta_M),$$

where $\Delta_M = \{(p, p) \in M \times M \ : \ p \in M\}$ is the diagonal. To show this, let $f(k) = g(l) = m$ so that $x = (k, l, m, m) \in X$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))), \ v \in T_k K, \quad w \in T_l L\} \tag{15}$$

whereas we also have

$$T_x(K \times L \times \Delta_M) = \{((v, m), (w, m)) \ : \ v \in T_k K, \quad w \in T_l L, \quad m \in T_p M\} \tag{16}$$

By transversality of f, g, any tangent vector $m_i \in T_p M$ may be written as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)),$$

leading directly to the transversality of the spaces (15), (16). This shows that $\Gamma_{f\cap g}$ is a regular submanifold of $K \times L \times M \times M$. Actually since it sits inside $K \times L \times \Delta_M$, we may compose with the projection diffeomorphism to view it as a regular submanifold in $K \times L \times \Delta_M$. Then we observe that the restriction of the projection onto $K \times L$ to the submanifold $\Gamma_{f\cap g}$ is an embedding with image exactly X. Hence X is a smooth regular submanifold and $\Gamma_{f\cap g}$ may then be viewed as the graph of a smooth map $f \cap g : X \longrightarrow M$ which must make the diagram above commute by definition. $\square$

Example 2.4. *If $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$, we may view both K_i as "fibering" over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".*

Example 2.5. *Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \backslash \{0\}$ with the projection $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product*

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.

The following result is an exercise: just as we may take the product of a manifold with boundary K with a manifold without boundary L to obtain a manifold with boundary $K \times L$, we have a similar result for fiber products.

Proposition 2.6. *Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \longrightarrow M$ and $g : L \longrightarrow M$ are smooth maps such that both f and ∂f are transverse to g. Then the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.*

2.1 Stability

We wish to understand the intuitive notion that "transversality is a stable condition", which in some sense means that if true, it remains so under small perturbations (of the submanifolds or maps involved). After this, we will go much further using Sard's theorem, and show that not only is it *stable*, it is actually *generic*, meaning that even if it is not true, it can be made true by a small perturbation. In this sense, stability says that transversal maps form an open set, and genericity says that this open set is dense in the space of maps. To make this precise, we would introduce a topology on the space of maps, something which we leave for another course.

A property of a smooth map $f_0 : M \longrightarrow N$ is *stable* under perturbations when for any smooth homotopy f_t of f_0, i.e. a smooth map $f : [0,1] \times M \longrightarrow N$ with $f|_{\{0\}\times M} = f_0$, the property holds for all $f_t = f|_{\{t\}\times M}$ with $t < \epsilon$ for some $\epsilon > 0$.

Proposition 2.7. *Let M be a compact manifold and $f_0 : M \longrightarrow N$ a smooth map. Then the property of being an immersion or submersion are each stable under perturbations. If M' is compact, then the transversality of $f_0 : M \longrightarrow N$, $g_0 : M' \longrightarrow N$ is also stable under perturbations of f_0, g_0.*

As an exercise, show that local diffeomorphisms, diffeomorphisms, and embeddings are also stable.

Proof. Let $f_t, t \in [0,1]$ be a smooth homotopy of f_0, and suppose that f_0 is an immersion. This means that at each point $p \in M$, the jacobian of f_0 in some chart has a $m\times m$ submatrix with nonvanishing determinant, for $m = \dim M$. By continuity, this $m \times m$ submatrix must have nonvanishing determinant in a neighbourhood around $(0,p) \in [0,1] \times M$. $\{0\} \times M$ may be covered by a finite number of such neighbourhoods, since M is compact. Choose ϵ such that $[0,\epsilon) \times M$ is contained in the union of these intervals, giving the result.

The proof for submersions is identical. The condition that f_0 be transversal to g_0 is equivalent to the fact that $\Gamma_{f_0} \times \Gamma_{g_0}$ is transversal to $C = M \times Z \times \Delta_N$. Choosing coordinate charts adapted to C, we may express this locally as a submersion condition. Hence by the previous result we have stability. □

2.2 Genericity of transversality

The fundamental idea which allows us to prove that transversality is a generic condition is a the theorem of Sard showing that critical values of a smooth map $f : M \longrightarrow N$ (i.e. points $q \in N$ for which the map f and the inclusion $\iota : q \hookrightarrow N$ fail to be transverse maps) are *rare*. The following proof is taken from Milnor, based on Pontryagin.

The meaning of "rare" will be that the set of critical values is of *measure zero*, which means, in $\mathbb{R}^m$, that for any $\epsilon > 0$ we can find a sequence of balls in $\mathbb{R}^m$, containing $f(C)$ in their union, with total volume less than ϵ. Some easy facts about sets of measure zero: the countable union of measure zero sets is of measure zero, the complement of a set of measure zero is dense.

We begin with an elementary lemma describing the behaviour of measure-zero sets under differentiable maps.

Lemma 2.8. *Let $I^m = [0,1]^m$ be the unit cube, and $f : I^m \longrightarrow \mathbb{R}^n$ a C^1 map. If $m < n$ then $f(I^m)$ has measure zero. If $m = n$ and $A \subset I^m$ has measure zero, then $f(A)$ has measure zero.*

Proof. Since f is C^1, we have the mean value theorem stating for all $x, y \in I^m$

$$f(y) - f(x) = Df(z)(y - x)$$

for some z one the line from x to y. The derivative Df has an upper bound on the compact I^m and we conclude $|f(x) - f(y)| \le a|x - y|$ for some constant $a > 0$ depending only on I^m and f (this is called a Lipschitz constant). Then the image of a ball of radius r contained in K would be contained in a ball of radius at most ar, which would have volume proportional to r^n, $n \ge m$.

A is of measure zero, hence for each ϵ we have a countable covering of A by balls of radius r_k with total volume $c_n \sum_k r_k^m < \epsilon$. We deduce that $f(A_i)$ is covered by balls of radius ar_k with total volume $\le a^n c_n \sum_k r_k^n$ and since $n \ge m$ this is certainly arbitrarily small. We conclude that $f(A)$ is of measure zero.

If $m < n$ then f defines a C^1 map $I^m \times I^{n-m} \longrightarrow \mathbb{R}^n$ by pre-composing with the projection map to I^m. Since $I^m \times \{0\} \subset I^m \times I^{n-m}$ clearly has measure zero, its image must also. $\square$

Remark 3. *If we considered the case $n < m$, the resulting sum of volumes may be larger in $\mathbb{R}^n$. For example, the projection map $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $(x, y) \mapsto x$ clearly takes the set of measure zero $y = 0$ to one of positive measure.*

A subset $A \subset M$ of a manifold is said to have measure zero when its image in any coordinate chart has measure zero. Since manifolds are second countable and we may choose a countable basis V_i such that $\overline{V}_i \subset U_i$ are compact subsets of coordinate charts (any coordinate neighbourhood is a countable union of closed balls), it follows that a subset $A \subset M$ of measure zero may be expressed as a countable union of subsets $A_k \subset \overline{V}_i$ with $\varphi_i(A_k)$ satisfying the Lemma. We therefore obtain

Proposition 2.9. *Let $f : M \longrightarrow N$ be a C^1 map of manifolds where $\dim M = \dim N$. Then the image $f(A)$ of a set $A \subset M$ of measure zero also has measure zero.*

Corollary 2.10 (Baby Sard)**.** *Let $f : M \longrightarrow N$ be a C^1 of manifolds where $\dim M < \dim N$. Then $f(M)$ (i.e. the set of critical values) has measure zero in N.*

Now we investigate the measure of the critical values of a map $f : M \longrightarrow N$ where $\dim M = \dim N$. Of course the set of critical points need not have measure zero, but we shall see that because the values of f on the critical set do not vary much, the set of critical *values* will have measure zero.

Theorem 2.11 (Equidimensional Sard)**.** *Let $f : M \longrightarrow N$ be a C^1 map of n-manifolds, and let $C \subset M$ be the set of critical points. Then $f(C)$ has measure zero.*

Proof. It suffices to show result for the unit cube. Let $f : I^n \longrightarrow \mathbb{R}^n$ a C^1 map and let $C \subset I^n$ be the set of critical points.

Let a be the Lipschitz constant for f, I^n, obtained from the mean value equation

$$f(y) - f(x) = Df(z)(y - x), \tag{17}$$

and let T_x be the affine map approximating f at x, i.e.

$$T_x(y) = f(x) + Df(x)(y - x). \tag{18}$$

Then subtracting equations (17),(18), we obtain

$$f(y) - T_x(y) = (Df(z) - Df(x))(y - x). \tag{19}$$

Since Df is continuous, there is a positive function $b(\epsilon)$ with $b \to 0$ as $\epsilon \to 0$ such that

$$||f(y) - T_x(y)|| \leq b(|y - x|)||y - x||.$$

If x is a critical point, then T_x has vanishing determinant, meaning that it maps $\mathbb{R}^n$ into a hyperplane $P_x \subset \mathbb{R}^n$ (i.e. of dimension $n - 1$). If $||y - x|| < \epsilon$, then $||f(y) - f(x)|| < a\epsilon$, and by (19), the distance of $f(y)$ from P_x is less than $\epsilon b(\epsilon)$.

Therefore $f(y)$ lies in the cube centered at $f(x)$ of edge $a\epsilon$, but only $\epsilon b\epsilon$ in distance from the plane P_x. Choose the cube to have a face parallel to P_x, and we conclude $f(y)$ is in a region of volume $(a\epsilon)^{n-1}2\epsilon b(\epsilon)$.

Now partition I^n into h^n cubes each of edge h^{-1}. Any such cube containing a critical point x is certainly contained in a ball around x of radius $r = h^{-1}\sqrt{n}$. The image of this ball then has volume $\leq (ar)^{n-1}2rb(r) = Ar^n b(r)$ for $A = 2a^{n-1}$. The total volume of all the images is then less than

$$h^n A r^n b(r) = A n^{n/2} b(r).$$

Note that A and n are fixed, while $r = h^{-1}\sqrt{n}$ is determined by the number h of cubes. By increasing the number of cubes, we may decrease their radius arbitrarily, and hence the above total volume, as required. □

The argument above will not work for $\dim N < \dim M$; we need more control on the function f. In particular, one can find a C^1 function from $I^2 \longrightarrow \mathbb{R}$ which fails to have critical values of measure zero (hint: $C + C = [0, 2]$ where C is the Cantor set). As a result, Sard's theorem in general requires more differentiability of f.

Theorem 2.12 (Big Sard's theorem)**.** *Let $f : M \longrightarrow N$ be a C^k map of manifolds of dimension m, n, respectively. Let C be the set of critical points, i.e. points $x \in U$ with*

$$\text{rank } Df(x) < n.$$

Then $f(C)$ has measure zero if $k > \frac{m}{n} - 1$.

Proof. As before, it suffices to show for $f : I^m \longrightarrow \mathbb{R}^n$.

Define $C_1 \subset C$ to be the set of points x for which $Df(x) = 0$. Define $C_i \subset C_{i-1}$ to be the set of points x for which $D^j f(x) = 0$ for all $j \leq i$. So we have a descending sequence of closed sets:

$$C \supset C_1 \supset C_2 \supset \cdots \supset C_k.$$

We will show that $f(C)$ has measure zero by showing

1. $f(C_k)$ has measure zero,

2. each successive difference $f(C_i \backslash C_{i+1})$ has measure zero for $i \geq 1$,

3. $f(C \backslash C_1)$ has measure zero.

Step 1: For $x \in C_k$, Taylor's theorem gives the estimate

$$f(x+t) = f(x) + R(x,t), \quad \text{with} \quad ||R(x,t)|| \leq c||t||^{k+1},$$

where c depends only on I^m and f, and t sufficiently small.

If we now subdivide I^m into h^m cubes with edge h^{-1}, suppose that x sits in a specific cube I_1. Then any point in I_1 may be written as $x+t$ with $||t|| \leq h^{-1}\sqrt{m}$. As a result, $f(I_1)$ lies in a cube of edge $ah^{-(k+1)}$, where $a = 2cm^{(k+1)/2}$ is independent of the cube size. There are at most h^m such cubes, with total volume less than

$$h^m(ah^{-(k+1)})^n = a^n h^{m-(k+1)n}.$$

Assuming that $k > \frac{m}{n} - 1$, this tends to 0 as we increase the number of cubes.

Step 2: For each $x \in C_i \backslash C_{i+1}$, $i \geq 1$, there is a $i+1^{th}$ partial $\partial^{i+1} f_j / \partial x_{s_1} \cdots \partial x_{s_{i+1}}$ which is nonzero at x. Therefore the function

$$w(x) = \partial^k f_j / \partial x_{s_2} \cdots \partial x_{s_{i+1}}$$

vanishes at x but its partial derivative $\partial w / \partial x_{s_1}$ does not. WLOG suppose $s_1 = 1$, the first coordinate. Then the map

$$h(x) = (w(x), x_2, \ldots, x_m)$$

is a local diffeomorphism by the inverse function theorem (of class C^k) which sends a neighbourhood V of x to an open set V'. Note that $h(C_i \cap V) \subset \{0\} \times \mathbb{R}^{m-1}$. Now if we restrict $f \circ h^{-1}$ to $\{0\} \times \mathbb{R}^{m-1} \cap V'$, we obtain a map g whose critical points include $h(C_i \cap V)$. Hence we may prove by induction on m that $g(h(C_i \cap V)) = f(C_i \cap V)$ has measure zero. Cover by countably many such neighbourhoods V.

Step 3: Let $x \in C \backslash C_1$. Then there is some partial derivative, wlog $\partial f_1 / \partial x_1$, which is nonzero at x. the map

$$h(x) = (f_1(x), x_2, \ldots, x_m)$$

is a local diffeomorphism from a neighbourhood V of x to an open set V' (of class C^k). Then $g = f \circ h^{-1}$ has critical points $h(V \cap C)$, and has critical values $f(V \cap C)$. The map g sends hyperplanes $\{t\} \times \mathbb{R}^{m-1}$ to hyperplanes $\{t\} \times \mathbb{R}^{n-1}$, call the restriction map g_t. A point in $\{t\} \times \mathbb{R}^{m-1}$ is critical for g_t if and only if it is critical for g, since the Jacobian of g is

$$\begin{pmatrix} 1 & 0 \\ * & \frac{\partial g_t^i}{\partial x_j} \end{pmatrix}$$

By induction on m, the set of critical values for g_t has measure zero in $\{t\} \times \mathbb{R}^{n-1}$. By Fubini, the whole set $g(C')$ (which is measurable, since it is the countable union of compact subsets (critical values not necessarily closed, but critical points are closed and hence a countable union of compact subsets, which implies the same of the critical values.) is then measure zero. To show this consequence of Fubini directly, use the following argument:

First note that for any covering of $[a,b]$ by intervals, we may extract a finite subcovering of intervals whose total length is $\leq 2|b-a|$. Why? First choose a minimal subcovering $\{I_1, \ldots, I_p\}$, numbered according to their left endpoints. Then the total overlap is at most the length of $[a,b]$. Therefore the total length is at most $2|b-a|$.

Now let $B \subset \mathbb{R}^n$ be compact, so that we may assume $B \subset \mathbb{R}^{n-1} \times [a,b]$. We prove that if $B \cap P_c$ has measure zero in the hyperplane $P_c = \{x^n = c\}$, for any constant $c \in [a,b]$, then it has measure zero in $\mathbb{R}^n$.

If $B \cap P_c$ has measure zero, we can find a covering by open sets $R_c^i \subset P_c$ with total volume $< \epsilon$. For sufficiently small α_c, the sets $R_c^i \times [c - \alpha_c, c + \alpha_c]$ cover $B \cap \bigcup_{z \in [c-\alpha_c, c+\alpha_c]} P_z$ (since B is compact). As we vary c, the sets $[c-\alpha_c, c+\alpha_c]$ form a covering of $[a,b]$, and we extract a finite subcover $\{I_j\}$ of total length $\leq 2|b-a|$.

Let R^i_j be the set R^i_c for $I_j = [c-\alpha_c, c+\alpha_c]$. Then the sets $R^i_j \times I_j$ form a cover of B with total volume $\leq 2\epsilon|b-a|$. We can make this arbitrarily small, so that B has measure zero.

□

Corollary 2.13. *Let M be a compact manifold with boundary. There is no smooth map $f : M \longrightarrow \partial M$ leaving ∂M pointwise fixed. Such a map is called a smooth retraction of M onto its boundary.*

Proof. Such a map f must have a regular value by Sard's theorem, let this value be $y \in \partial M$. Then y is obviously a regular value for $f|_{\partial M} = \mathrm{Id}$ as well, so that $f^{-1}(y)$ must be a compact 1-manifold with boundary given by $f^{-1}(y) \cap \partial M$, which is simply the point y itself. Since there is no compact 1-manifold with a single boundary point, we have a contradiction. □

For example, this shows that the identity map $S^n \longrightarrow S^n$ may not be extended to a smooth map $f : \overline{B(0,1)} \longrightarrow S^n$.

Lemma 2.14. *Every smooth map of the closed n-ball to itself has a fixed point.*

Proof. Let $D^n = \overline{B(0,1)}$. If $g : D^n \longrightarrow D^n$ had no fixed points, then define the function $f : D^n \longrightarrow S^{n-1}$ as follows: let $f(x)$ be the point nearer to x on the line joining x and $g(x)$.

This map is smooth, since $f(x) = x + tu$, where

$$u = ||x-g(x)||^{-1}(x-g(x)),$$

and t is the positive solution to the quadratic equation $(x+tu)\cdot(x+tu) = 1$, which has positive discriminant $b^2 - 4ac = 4(1-|x|^2 + (x\cdot u)^2)$. Such a smooth map is therefore impossible by the previous corollary. □

Theorem 2.15 (Brouwer fixed point theorem)**.** *Any continuous self-map of D^n has a fixed point.*

Proof. The Weierstrass approximation theorem says that any continuous function on $[0,1]$ can be uniformly approximated by a polynomial function in the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. In other words, the polynomials are dense in the continuous functions with respect to the supremum norm. The Stone-Weierstrass is a generalization, stating that for any compact Hausdorff space X, if A is a subalgebra of $C^0(X,\mathbb{R})$ such that A separates points $(\forall x, y, \exists f \in A \;: f(x) \neq f(y))$ and contains a nonzero constant function, then A is dense in C^0.

Given this result, approximate a given continuous self-map g of D^n by a polynomial function p' so that $||p'-g||_\infty < \epsilon$ on D^n. To ensure p' sends D^n into itself, rescale it via

$$p = (1+\epsilon)^{-1}p'.$$

Then clearly p is a D^n self-map while $||p-g||_\infty < 2\epsilon$. If g had no fixed point, then $|g(x)-x|$ must have a minimum value μ on D^n, and by choosing $2\epsilon = \mu$ we guarantee that for each x,

$$|p(x)-x| \geq |g(x)-x| - |g(x)-p(x)| > \mu - \mu = 0.$$

Hence p has no fixed point. Such a smooth function can't exist and hence we obtain the result. □

We now proceed with the first step towards showing that transversality is generic.

Theorem 2.16 (Transversality theorem)**.** *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds where only X has boundary. Suppose that F and ∂F are transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ and ∂f_s are transverse to g.*

Proof. The fiber product $W = (X \times S) \times_Y Z$ is a regular submanifold (with boundary) of $X \times S \times Z$ and projects to S via the usual projection map π. We show that any $s \in S$ which is a regular value for both the projection map $\pi : W \longrightarrow S$ and its boundary map $\partial\pi$ gives rise to a f_s which is transverse to g. Then by Sard's theorem the s which fail to be regular in this way form a set of measure zero.

Suppose that $s \in S$ is a regular value for π. Suppose that $f_s(x) = g(z) = y$ and we now show that f_s is transverse to g there. Since $F(x, s) = g(z)$ and F is transverse to g, we know that

$$\mathrm{Im} DF_{(x,s)} + \mathrm{Im} Dg_z = T_y Y.$$

Therefore, for any $a \in T_y Y$, there exists $b = (w, e) \in T(X \times S)$ with $DF_{(x,s)}b - a$ in the image of Dg_z. But since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)}W$. Hence we observe that

$$(Df_s)(w - w') - a = DF_{(x,s)}[(w, e) - (w', e)] - a = (DF_{(x,s)}b - a) - DF_{(x,s)}(w', e),$$

where both terms on the right hand side lie in $\mathrm{Im} Dg_z$.

Precisely the same argument (with X replaced with ∂X and F replaced with ∂F) shows that if s is regular for $\partial\pi$ then ∂f_s is transverse to g. This gives the result. □

The previous result immediately shows that transversal maps to $\mathbb{R}^n$ are generic, since for any smooth map $f : M \longrightarrow \mathbb{R}^n$ we may produce a family of maps

$$F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

via $F(x, s) = f(x) + s$. This new map F is clearly a submersion and hence is transverse to any smooth map $g : Z \longrightarrow \mathbb{R}^n$. For arbitrary target manifolds, we will imitate this argument, but we will require a (weak) version of Whitney's embedding theorem for manifolds into $\mathbb{R}^n$.

2.3 Partitions of unity and Whitney embedding

In this section we develop the tool of partition of unity, which will allow us to *go from local to global*, i.e. to glue together objects which are defined locally, creating objects with global meaning. As a particular case of this, to define a global map to $\mathbb{R}^N$ which is an embedding, thereby proving Whitney's embedding theorem.

Definition 13. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x \in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 14. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 2.17. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0, 3)$ and $\{V_i = \varphi_i^{-1}(B(0, 1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P}_1$, and

$$K_{i+1} = \overline{P_1 \cup \cdots \cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1 \cup \cdots \cup P_r$.

Now note that M is the union of ring-shaped sets $K_i \backslash K_{i-1}^\circ$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+2} \backslash K_{i-1}^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p, α vary, these neighbourhoods cover the compact set $K_{i+1} \backslash K_i^\circ$ without leaving the band $K_{i+2} \backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement. □

Definition 15. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\mathrm{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R}\backslash\{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \quad \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\mathrm{supp} f_\alpha \subset U_i$ for some i.

Theorem 2.18. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and supp$f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support supp$g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots.$$

□

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2\dim M$ which Whitney showed (we shall reach $2\dim M + 1$ and possibly at the end of the course, show $N = 2\dim M$.)

Theorem 2.19 (Compact Whitney embedding in $\mathbb{R}^N$)**.** *Any* compact *manifold may be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 2.18 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M + 1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_x M$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). □

Theorem 2.20 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any* compact *n-manifold may be embedded in* $\mathbb{R}^{2n+1}$.

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n+1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M) \backslash \Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n+1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n+1$. □

Corollary 2.21. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in $\mathbb{R}^N$, then we can take $N = 2n$ instead of $2n+1$.*

Theorem 2.22 (noncompact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any smooth n-manifold may be embedded in $\mathbb{R}^{2n+1}$ (or immersed in $\mathbb{R}^{2n}$).*

Proof. We saw that any manifold may be written as a countable union of increasing compact sets $M = \cup K_i$, and that a regular covering $\{(U_{i,k} \supset V_{i,k}, \varphi_{i,k})\}$ of M can be chosen so that for fixed i, $\{V_{i,k}\}_k$ is a finite cover of $K_{i+1}\backslash K_i^\circ$ and each $U_{i,k}$ is contained in $K_{i+2}\backslash K_{i-1}^\circ$.

This means that we can express M as the union of 3 open sets W_0, W_1, W_2, where

$$W_j = \bigcup_{i\equiv j(\mathrm{mod}3)} (\cup_k U_{i,k}).$$

Each of the sets $R_i = \cup_k U_{i,k}$ may be injectively immersed in $\mathbb{R}^{2n+1}$ by the argument for compact manifolds, since they have a finite regular cover. Call these injective immersions $\Phi_i : R_i \longrightarrow \mathbb{R}^{2n+1}$. The image $\Phi_i(R_i)$ is bounded since all the charts are, by some radius r_i. The open sets R_i, $i \equiv j(\mathrm{mod}3)$ for fixed j are disjoint, and by translating each Φ_i, $i \equiv j(\mathrm{mod}3)$ by an appropriate constant, we can ensure that their images in $\mathbb{R}^{2n+1}$ are disjoint as well.

Let $\Phi_i' = \Phi_i + (2(r_{i-1} + r_{i-2} + \cdots) + r_i)\overrightarrow{e}_1$. Then $\Psi_j = \cup_{i\equiv j(\mathrm{mod}3)}\Phi_i' : W_j \longrightarrow \mathbb{R}^{2n+1}$ is an embedding.

Now that we have injective immersions Ψ_0, Ψ_1, Ψ_2 of W_0, W_1, W_2 in $\mathbb{R}^{2n+1}$, we may use the original argument for compact manifolds: Take the partition of unity subordinate to $U_{i,k}$ and resum it, obtaining a 3-element partition of unity $\{f_1, f_2, f_3\}$, with $f_j = \sum_{i\equiv j(\mathrm{mod}3)}\sum_k f_{i,k}$. Then the map

$$\Psi = (f_1\Psi_1, f_2\Psi_2, f_3\Psi_3, f_1, f_2, f_3)$$

is an injective immersion of M into $\mathbb{R}^{6n+3}$. To see that it is in fact an embedding, note that any closed set $C \subset M$ may be written as a union of closed sets $C = C_1 \cup C_2 \cup C_3$, where $C_j = \cup_{i\equiv j(\mathrm{mod}3)}(C \cap K_{i+1}\backslash K_i^\circ)$ is a disjoint union of compact sets. Ψ is injective, hence C_j is mapped to a disjoint union of compact sets, hence a closed set. Then $\Psi(C)$ is a union of 3 closed sets, hence closed, as required.

Using projection to hyperplanes we may again reduce to $\mathbb{R}^{2n+1}$, but if we exclude all hyperplanes perpendicular to $\mathrm{Span}((e_1, 0, 0, 0, 0, 0), (0, e_1, 0, 0, 0, 0), (0, 0, e_1, 0, 0, 0))$, we obtain an injective immersion Ψ' which is *proper*, meaning that inverse images of compact sets are compact. This space of forbidden planes has measure zero as long as $N - 1 > 3$, so that we may reduce to $2n + 1$ for $n > 1$. We leave as an exercise the $n = 1$ case (or see Bredon for a slightly different proof).

The fact that the resulting injective immersion Ψ' is proper implies that it is an embedding, by the closed map lemma, as follows. □

Lemma 2.23 (Closed map lemma for proper maps)**.** *Let $f : X \longrightarrow Y$ be a proper continuous map of topological manifolds. Then f is a closed map.*

Proof. Let $K \subset X$ be closed; we show that $f(K)$ contains all its limit points and hence is closed. Let $y \in Y$ be a limit point for $f(K)$. Choose a precompact neighbourhood U of y, so that y is also a limit point of $f(K) \cap \overline{U}$. Since f is proper, $f^{-1}(\overline{U})$ is compact, and hence $K \cap f^{-1}(\overline{U})$ is compact as well. But then by continuity, $f(K \cap f^{-1}(\overline{U})) = f(K) \cap \overline{U}$ is compact, implying it is closed. Hence $y \in f(K) \cap \overline{U} \subset f(K)$, as required. □

We now use Whitney embedding to extend our understanding of the genericity of transversality. First we need an understanding of the immediate neighbourhood of an embedded submanifold in $\mathbb{R}^N$. For this, we introduce a new manifold associated to an embedded submanifold: its normal bundle (for now we assume the manifold is embedded in $\mathbb{R}^N$).

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_yY = \{v \in \mathbb{R}^N : v \perp T_yY\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y, v) \in Y \times \mathbb{R}^N \ : \ v \in N_yY\}.$$

Proposition 2.24. *$NY \subset \mathbb{R}^N \times \mathbb{R}^N$ is an embedded submanifold of dimension N.*

Proof. Given $y \in Y$, choose coordinates $(u^1, \dots u^N)$ in a neighbourhood $U \subset \mathbb{R}^N$ of y so that $Y \cap U = \{u^{n+1} = \cdots = u^N = 0\}$. Define $\Phi : U \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n$ via

$$\Phi(x, v) = (u^{n+1}(x), \dots, u^N(x), \langle v, \tfrac{\partial}{\partial u^1}|_x\rangle, \dots, \langle v, \tfrac{\partial}{\partial u^n}|_x\rangle),$$

so that $\Phi^{-1}(0)$ is precisely $NY \cap (U \times \mathbb{R}^N)$. We then show that 0 is a regular value: observe that, writing v in terms of its components $v^j \frac{\partial}{\partial x^j}$ in the standard basis for $\mathbb{R}^N$,

$$\langle v, \tfrac{\partial}{\partial u^i}|_x\rangle = \langle v^j \tfrac{\partial}{\partial x^j}, \tfrac{\partial x^k}{\partial u^i}(u(x))\tfrac{\partial}{\partial x^k}|_x\rangle = \sum_{j=1}^{N} v^j \tfrac{\partial x^j}{\partial u^i}(u(x))$$

Therefore the Jacobian of Φ is the $((N-n)+n) \times (N+N)$ matrix

$$D\Phi(x) = \begin{pmatrix} \frac{\partial u^j}{\partial x^i}(x) & 0 \\ * & \frac{\partial x^j}{\partial u^i}(u(x)) \end{pmatrix}$$

The N rows of this matrix are linearly independent, proving Φ is a submersion. □

The normal bundle NY contains $Y \cong Y \times \{0\}$ as a regular submanifold, and is equipped with a smooth map $\pi : NY \longrightarrow Y$ sending $(y, v) \mapsto y$. The map π is a surjective submersion and is known as the bundle projection. The vector spaces $\pi^{-1}(y)$ for $y \in Y$ are called the fibers of the bundle and we shall see later that NY is an example of a vector bundle.

We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x, v) = x + v.$$

Definition 16. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y, v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N \ : \ \exists y \in Y \text{ with } |x - y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r \ : \ B(y, r) \subset U\},$$

which is continuous since $\forall \epsilon > 0, \exists x \in U$ for which $\epsilon(y) \le |x - y| + \epsilon$. For any other $y' \in Y$, this is $\le |y - y'| + |x - y'| + \epsilon$. Since $|x - y'| \le \epsilon(y')$, we have $|\epsilon(y) - \epsilon(y')| \le |y - y'| + \epsilon$.

Theorem 2.25 (Tubular neighbourhood theorem)**.** *Every regular submanifold of $\mathbb{R}^N$ has a tubular neighbourhood. Postpone proof briefly.*

Corollary 2.26. *Let X be a manifold with boundary and $f : X \longrightarrow Y$ be a smooth map to a manifold Y. Then there is an open ball $S = B(0, 1) \subset \mathbb{R}^N$ and a smooth map $F : X \times S \longrightarrow Y$ such that $F(x, 0) = f(x)$ and for fixed x, the map $f_x : s \mapsto F(x, s)$ is a submersion $S \longrightarrow Y$. In particular, F and ∂F are submersions.*

Proof. Embed Y in $\mathbb{R}^N$, and let $S = B(0, 1) \subset \mathbb{R}^N$. Then use the tubular neighbourhood to define

$$F(y, s) = (\pi \circ E^{-1})(f(y) + \epsilon(y)s),$$

□

The transversality theorem then guarantees that given any smooth $g : Z \longrightarrow Y$, for almost all $s \in S$ the maps $f_s, \partial f_s$ are transverse to g. We improve this slightly to show that f_s may be chosen to be *homotopic* to f.

Corollary 2.27 (Transversality homotopy theorem)**.** *Given any smooth maps $f : X \longrightarrow Y$, $g : Z \longrightarrow Y$, where only X has boundary, there exists a smooth map $f' : X \longrightarrow Y$ homotopic to f with $f', \partial f'$ both transverse to g.*

Proof. Let S, F be as in the previous corollary. Away from a set of measure zero in S, the functions $f_s, \partial f_s$ are transverse to g, by the transversality theorem. But these f_s are all homotopic to f via the homotopy $X \times [0,1] \longrightarrow Y$ given by

$$(x,t) \mapsto F(x,ts).$$

□

Proof, tubular neighbourhoood theorem. First we show that E is a local diffeomorphism near $y \in Y \subset NY$. if ι is the embedding of Y in $\mathbb{R}^N$, and $\iota' : Y \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$, hence we have $DE \circ D\iota' = D\iota$, showing that the image of $DE(y)$ contains T_yY. Now if ι is the embedding of N_yY in $\mathbb{R}^N$, and $\iota' : N_yY \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$. Hence we see that the image of $DE(y)$ contains N_yY, and hence the image is all of $T_y\mathbb{R}^N$. Hence E is a diffeomorphism on some neighbourhood

$$V_\delta(y) = \{(y', v') \in NY \ : \ |y' - y| < \delta, \ |v'| < \delta\}, \quad \delta > 0.$$

Now for $y \in Y$ let $r(y) = \sup\{\delta \ : \ E|_{V_\delta(y)}$ is a diffeomorphism$\}$ if this is ≤ 1 and let $r(y) = 1$ otherwise. The function $r(y)$ is continuous, since if $|y - y'| < r(y)$, then $V_\delta(y') \subset V_{r(y)}(y)$ for $\delta = r(y) - |y - y'|$. This means that $r(y') \geq \delta$, i.e. $r(y) - r(y') \leq |y - y'|$. Switching y and y', this remains true, hence $|r(y) - r(y')| \leq |y - y'|$, yielding continuity.

Finally, let $V = \{(y, v) \in NY \ : \ |v| < \frac{1}{2}r(y)\}$. We show that E is injective on V. Suppose $(y, v), (y', v') \in V$ are such that $E(y, v) = E(y', v')$, and suppose wlog $r(y') \leq r(y)$. Then since $y + v = y' + v'$, we have

$$|y - y'| = |v - v'| \leq |v| + |v'| \leq \tfrac{1}{2}r(y) + \tfrac{1}{2}r(y') \leq r(y).$$

Hence y, y' are in $V_{r(y)}(y)$, on which E is a diffeomorphism. The required tubular neighbourhood is then $U = E(V)$. □

The last theorem we shall prove concerning transversality is a very useful extension result which is essential for intersection theory:

Theorem 2.28 (Homotopic transverse extension of boundary map)**.** *Let X be a manifold with boundary and $f : X \longrightarrow Y$ a smooth map to a manifold Y. Suppose that ∂f is transverse to the closed map $g : Z \longrightarrow Y$. Then there exists a map $f' : X \longrightarrow Y$, homotopic to f and with $\partial f' = \partial f$, such that f' is transverse to g.*

Proof. First observe that since ∂f is transverse to g on ∂X, f is also transverse to g there, and furthermore since g is closed, f is transverse to g in a neighbourhood U of ∂X. (for example, if $x \in \partial X$ but x not in $f^{-1}(g(Z))$ then since the latter set is closed, we obtain a neighbourhood of x for which f is transverse to g.)

Now choose a smooth function $\gamma : X \longrightarrow [0,1]$ which is 1 outside U but 0 on a neighbourhood of ∂X. (why does γ exist? exercise.) Then set $\tau = \gamma^2$, so that $d\tau(x) = 0$ wherever $\tau(x) = 0$. Recall the map $F : X \times S \longrightarrow Y$ we used in proving the transversality homotopy theorem 2.27 and modify it via

$$F'(x, s) = F(x, \tau(x)s).$$

Then F' and $\partial F'$ are transverse to g, and we can pick s so that $f' : x \mapsto F'(x, s)$ and $\partial f'$ are transverse to g. Finally, if x is in the neighbourhood of ∂X for which $\tau = 0$, then $f'(x) = F(x, 0) = f(x)$. □

Corollary 2.29. *if $f : X \longrightarrow Y$ and $f' : X \longrightarrow Y$ are homotopic smooth maps of manifolds, each transverse to the closed map $g : Z \longrightarrow Y$, then the fiber products $W = X_f \times_g Z$ and $W' = X_{f'} \times_g Z$ are cobordant.*

Proof. if $F : X \times [0,1] \longrightarrow Y$ is the homotopy between $\{f, f'\}$, then by the previous theorem, we may find a (homotopic) homotopy $F' : X \times [0,1] \longrightarrow Y$ which is transverse to g. Hence the fiber product $U = (X \times [0,1])_{F'} \times_g Z$ is the cobordism with boundary $W \sqcup W'$. □

2.4 Intersection theory

The previous corollary allows us to make the following definition:

Definition 17. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X compact, g closed, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \sharp(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g, and where we assume the fiber product to consist of a finite number of points (this is always guaranteed, e.g. if g is proper, or if g is a closed embedding).

Example 2.30. *If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.*

Example 2.31. *If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle\cos(\theta/2)e_1 + \sin(\theta/2)e_2\rangle$ and $\iota_2 : \theta \mapsto \langle\cos(\theta/2)e_2 + \sin(\theta/2)e_3\rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2\rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points.*

Example 2.32. *Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2 \dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.*

Example 2.33. *Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.*

Example 2.34. *By the previous argument, any compact manifold is not contractible.*

Example 2.35. *Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.*

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 2.36. *Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-homotopic, we cannot deform the planes to remove all intersection.*

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 18. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

Example 2.37. *Let $f : S^1 \longrightarrow S^1$ be given by $z \mapsto z^k$. Then $\deg_2(f) = k \pmod 2$.*

Example 2.38. *If $p : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a polynomial of degree k, then as a map $S^2 \longrightarrow S^2$ we have* $\deg_2(p) = k \pmod 2$*, and hence any odd polynomial has at least one root. To get the fundamental theorem of algebra, we must consider* oriented cobordism

Even if submanifolds C, C' do not intersect, it may be that there are more sophisticated geometrical invariants which cause them to be "intertwined" in some way. One example of this is linking number.

Definition 19. Suppose that $M, N \subset \mathbb{R}^{k+1}$ are compact embedded submanifolds with $\dim M + \dim N = k$, and let us assume they are transverse, meaning they do not intersect at all.

Then define $\lambda : M \times N \longrightarrow S^k$ via

$$(x, y) \mapsto \frac{x - y}{|x - y|}.$$

Then we define the $\pmod 2$ linking number of M, N to be $\deg_2(\lambda)$.

Example 2.39. *Consider the standard Hopf link in $\mathbb{R}^3$. Then it is easy to calculate that* $\deg_2(\lambda) = 1$*. On the other hand, the standard embedding of disjoint circles (differing by a translation, say) has* $\deg_2(\lambda) = 0$*. Hence it is impossible to deform the circles through embeddings of $S^1 \sqcup S^1 \longrightarrow \mathbb{R}^3$, so that they are unlinked. Why must we stay within the space of embeddings, and not allow the circles to intersect?*

3 The tangent bundle and vector bundles

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M, which is made up of the disjoint union of all tangent spaces to all points in M. If M is embedded in $\mathbb{R}^N$, then TM is a regular submanifold of $\mathbb{R}^N \times \mathbb{R}^N$, but we define it intrinsically, without reference to an embedding.

As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

Definition 20. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM, and forms a real vector space of dimension $\dim M$.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_pM,$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 3.1. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n,$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

A more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \;\Leftrightarrow\; y = \varphi_j \circ \varphi_i^{-1}(x) \text{ and } v = D(\varphi_j \circ \varphi_i^{-1})_x u,$$

and verify the conditions of the general gluing construction 1.7. Then show that a different atlas gives a canonically diffeomorphic manifold, i.e. that the result is independent of atlas.

A description of the tangent bundle is not complete without defining the derivative of a general smooth map of manifolds $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \longrightarrow \psi_\alpha(V_\alpha)$ which satisfy

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1}).$$

Differentiating, we obtain

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1}),$$

and hence we obtain a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Df : TM \longrightarrow TN$. Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Just as $\pi^{-1}(x) = T_xM \subset TM$ is a vector space for all x, making TM into a "bundle of vector spaces", the map $Df : T_xM \longrightarrow T_{f(x)}N$ is a linear map and hence Df is a "bundle of linear maps".

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Df \circ Dg = Dh$. As a result, we obtain the following category-theoretic statement.

Proposition 3.2. *The map T which takes a manifold M to its tangent bundle TM, and which takes maps $f : M \longrightarrow N$ to the derivative $Df : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself.*

For this reason, the derivative map Df is sometimes called the "tangent mapping" Tf.

Example 3.3. *If $\iota : M \longrightarrow N$ is an embedding of M into N, then $D\iota : TM \longrightarrow TN$ is also an embedding, and hence $D^k\iota : T^kM \longrightarrow T^kN$ are all embeddings.*

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart (U_i, φ_i), we would say that a vector field X_i is simply a vector-valued function on U_i, i.e. a function $X_i : \varphi(U_i) \longrightarrow \mathbb{R}^n$. Of course if we had another vector field X_j on (U_j, φ_j), then the two would agree as vector fields on the overlap $U_i \cap U_j$ when $D(\varphi_j \circ \varphi_i^{-1}) : X_i \mapsto X_j$. So, if we specify a collection $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue on overlaps, this would define a global vector field. This leads precisely to the following definition.

Definition 21. A smooth vector field on the manifold M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X : M \longrightarrow M$ is the identity. Essentially it is a smooth assignment of a unique tangent vector to each point in M.

Such maps X are also called *cross-sections* or simply *sections* of the tangent bundle TM, and the set of all such sections is denoted $C^\infty(M, TM)$ or sometimes $\Gamma^\infty(M, TM)$, to distinguish them from simply smooth maps $M \longrightarrow TM$.

Example 3.4. *From a computational point of view, given an atlas $(\tilde{U}_i, \varphi_i)$ for M, let $U_i = \varphi_i(\tilde{U}_i) \subset \mathbb{R}^n$ and let $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Then a global vector field $X \in \Gamma^\infty(M, TM)$ is specified by a collection of vector-valued functions $X_i : U_i \longrightarrow \mathbb{R}^n$ such that $D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x))$ for all $x \in \varphi_i(\tilde{U}_i \cap \tilde{U}_j)$.*

For example, if $S^1 = U_0 \sqcup U_1 / \sim$, with $U_0 = \mathbb{R}$ and $U_1 = \mathbb{R}$, with $x \in U_0 \backslash \{0\} \sim y \in U_1 \backslash \{0\}$ whenever $y = x^{-1}$, then $\varphi_{01} : x \mapsto x^{-1}$ and $D\varphi_{01}(x) : v \mapsto -x^{-2}v$. Then if we define (letting x be the standard coordinate along $\mathbb{R}$)

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x} \\ X_1 &= -y^2 \frac{\partial}{\partial y}, \end{aligned}$$

we see that this defines a global vector field, which does not vanish in U_0 but vanishes to order 2 at a single point in U_1. Find the local expression in these charts for the rotational vector field on S^1 given in polar coordinates by $\frac{\partial}{\partial \theta}$.

3.1 Properties of vector fields

The space $C^\infty(M, \mathbb{R})$ of smooth functions on M is not only a vector space but also a ring, with multiplication $(fg)(p) := f(p)g(p)$. That this defines a smooth function is clear from the fact that it is a composition of the form

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R} \ .$$

Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$, given by $f \mapsto f \circ \varphi$. This is called the pullback of functions, and defines a homomorphism of rings since $\Delta \circ \varphi = (\varphi \times \varphi) \circ \Delta$.

The association $M \mapsto C^\infty(M, \mathbb{R})$ and $\varphi \mapsto \varphi^*$ is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M, \mathbb{R})$, but actually all automorphisms are of this form (Exercise!).

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 22. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$.

In the following, we show that derivations of the algebra of functions actually correspond to vector fields.

The vector fields $\Gamma^\infty(M, TM)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless $\dim M = 0$). They also form a module over the ring of smooth functions $C^\infty(M, \mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$, we claim that $fX : x \mapsto f(x)X(x)$ defines a smooth vector field: this is clear from local considerations: if $\{X_i\}$ is a local description of X and $\{f_i\}$ is a local description of f with respect to a cover, then

$$D\varphi_{ij}(f_i(x)X_i(x)) = f_i(x) D\varphi_{ij} X_i(x) = f_j(\varphi_{ij}(x)) X_j(\varphi_{ij}(x)).$$

The important property of vector fields which we are interested in is that they act as $\mathbb{R}$-derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i (\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M, \mathbb{R})$.

Lemma 3.5. *Let f be a smooth function on $U \subset \mathbb{R}^n$, and $X : U \longrightarrow \mathbb{R}^n$ a vector field. Then $Df : TU \longrightarrow T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ and let $Df_2 : TM \longrightarrow \mathbb{R}$ be the composition of Df with the projection to the fiber $T\mathbb{R} \longrightarrow \mathbb{R}$. Then*

$$X(f) = Df_2(X).$$

Proof. In local coordinates, we have $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$ whereas $Df : X(x) \mapsto (f(x), \sum_i \frac{\partial f}{\partial x^i} a^i)$, so that we obtain the result by projection. □

Proposition 3.6. *Local partial differentiation extends to an injective map $\Gamma^\infty(M, TM) \longrightarrow$ Der$(C^\infty(M, \mathbb{R}))$.*

Proof. Globally, we verify that

$$\begin{aligned} X_j(f_j) &= X_j(f_i \circ \varphi_{ij}^{-1}) = ((\varphi_{ij})_* X_i)(f_i \circ \varphi_{ij}^{-1}) && (20)\\ &= D(f_i \circ \varphi_{ij}^{-1})_2((\varphi_{ij})_* X_i) && (21)\\ &= (Df_i)_2(X_i) = X_i(f_i). && (22) \end{aligned}$$

□

In fact, vector fields provide all possible derivations of the algebra $A = C^\infty(M, \mathbb{R})$:

Theorem 3.7. *The map $\Gamma^\infty(M, TM) \longrightarrow$ Der$(C^\infty(M, \mathbb{R}))$ is an isomorphism.*

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y)) g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i) g_i(z) - \sum_i (x^i(z) - x^i(y)) Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i the associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p) D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside U_i; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. □

Since vector fields are derivations, we have a natural source of examples, coming from infinitesimal automorphisms of M:

Example 3.8. *Let φ_t : be a smooth family of diffeomorphisms of M with $\varphi_0 = \mathrm{Id}$. That is, let $\varphi : (-\epsilon, \epsilon) \times M \longrightarrow M$ be a smooth map and $\varphi_t : M \longrightarrow M$ a diffeomorphism for each t. Then $X(f)(p) = \frac{d}{dt}|_{t=0}(\varphi_t^* f)(p)$ defines a smooth vector field. A better way of seeing that it is smooth is to rewrite it as follows: Let $\frac{\partial}{\partial t}$ be the coordinate vector field on $(-\epsilon, \epsilon)$ and observe $X(f)(p) = \frac{\partial}{\partial t}(\varphi^* f)(0, p)$.*

In many cases, a smooth vector field may be expressed as above, i.e. as an infinitesimal automorphism of M, but this is not always the case. In general, it gives rise to a "local 1-parameter group of diffeomorphisms", as follows:

Definition 23. A local 1-parameter group of diffeomorphisms is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map

$$\begin{aligned} \Phi : & U \longrightarrow M \\ & (t, x) \mapsto \varphi_t(x) \end{aligned}$$

such that $\mathbb{R} \times \{x\} \cap U$ is connected, $\varphi_0(x) = x$ for all x and if $(t, x), (t + t', x), (t', \varphi_t(x))$ are all in U then $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$.

Then the local existence and uniqueness of solutions to systems of ODE implies that every smooth vector field $X \in \Gamma^\infty(M, TM)$ gives rise to a local 1-parameter group of diffeomorphisms (U, Φ) such that the curve $\gamma_x : t \mapsto \varphi_t(x)$ is such that $(\gamma_x)_*(\frac{d}{dt}) = X(\gamma_x(t))$ (this means that γ_x is an integral curve or "trajectory" of the "dynamical system" defined by X). Furthermore, if (U', Φ') are another such data, then $\Phi = \Phi'$ on $U \cap U'$.

Definition 24. A vector field $X \in \Gamma^\infty(M, TM)$ is called *complete* when its local 1-parameter group of diffeomorphisms has $U = \mathbb{R} \times M$.

Theorem 3.9. *If M is compact, then every smooth vector field is complete.*

Example 3.10. *The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on $\{1 - tx > 0\}$.*

3.2 Vector bundles

Definition 25. A smooth real vector bundle of rank k over the base manifold M is a manifold E (called the total space), together with a smooth surjection $\pi : E \longrightarrow M$ (called the bundle projection), such that

- $\forall p \in M$, $\pi^{-1}(p) = E_p$ has the structure of k-dimensional vector space,
- Each $p \in M$ has a neighbourhood U and a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ (called a local trivialization of E over U) such that $\pi_1(\Phi(\pi^{-1}(x))) = x$, where $\pi_1 : U \times \mathbb{R}^k \longrightarrow U$ is the first projection, and also that $\Phi : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^k$ is a linear map, for all $x \in M$.

Given two local trivializations $\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ and $\Phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\Phi_j \circ \Phi_i^{-1} : U_{ij} \times \mathbb{R}^k \longrightarrow U_{ij} \times \mathbb{R}^k$, where $U_{ij} = U_i \cap U_j$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R}).$$

These are called transition functions of the bundle, and since they come from $\Phi_j \circ \Phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$ as well as the "cocycle condition"

$$g_{ij} g_{jk} g_{ki} = \mathrm{Id}|_{U_i \cap U_j \cap U_k}.$$

Example 3.11. *To build a vector bundle, choose an open cover $\{U_i\}$ and form the pieces $\{U_i \times \mathbb{R}^k\}$ Then glue these together on double overlaps $\{U_{ij}\}$ via functions $g_{ij} : U_{ij} \longrightarrow GL(k, \mathbb{R})$. As long as g_{ij} satisfy $g_{ij} = g_{ji}^{-1}$ as well as the cocycle condition, the resulting space has a vector bundle structure.*

Example 3.12. *Let $S^2 = U_0 \sqcup U_1$ for $U_i = \mathbb{R}^2$, as before. Then on $U_{01} = \mathbb{R}^2 \backslash \{0\} = \mathbb{C} \backslash \{0\}$, define*

$$g_{01}(z) = [z^k], \quad k \in \mathbb{Z}.$$

In real coordinates $z = re^{i\theta}$, $g_{01}(r, \theta) = r^k \begin{pmatrix} cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}$. This defines a vector bundle $E_k \longrightarrow S^2$ of rank 2 for each $k \in \mathbb{Z}$ (or a complex vector bundle of rank 1, since $g_{01} : U_{01} \longrightarrow GL(1, \mathbb{C})$). Actually, since the map g_{01} is actually holomorphic as a function of z, we have defined holomorphic *vector bundles on $\mathbb{C}P^1$.*

Example 3.13 (The tangent bundle)**.** *The tangent bundle TM is indeed a vector bundle, of rank* $\dim M$*. For any chart (U, φ) of M, there is an associated local trivialization $(\pi^{-1}(U), \Phi)$ of TM, and the transition function $g_{ji} : U_{ij} \longrightarrow GL(n, \mathbb{R})$ between two trivializations obtained from $(U_i, \varphi_i), (U_j, \varphi_j)$ is simply the Jacobian matrix*

$$g_{ji} : p \mapsto D(\varphi_j \circ \varphi_i^{-1})(p).$$

Just as for the tangent bundle, we can define the analog of a vector-valued function, where the function has values in a vector bundle:

Definition 26. A smooth section of the vector bundle $E \xrightarrow{\pi} M$ is a smooth map $s : M \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_M$. The set of all smooth sections, denoted $\Gamma^\infty(M, E)$, is an infinite-dimensional real vector space, and is also a module over the ring $C^\infty(M, \mathbb{R})$.

Having introduced vector bundles, we must define the notion of *morphism* between vector bundles, so as to form a category.

Definition 27. A smooth bundle map between the bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a pair (f, F) of smooth maps $f : M \longrightarrow M'$ and $F : E \longrightarrow E'$ such that $\pi' \circ F = f \circ \pi$ and such that $F : E_p \longrightarrow E'_{f(p)}$ is a linear map for all p.

Example 3.14. *I claim that the bundles $E_k \xrightarrow{\pi} S^2$ are all non-isomorphic, except that E_k is isomorphic to E_{-k} over the antipodal map $S^2 \longrightarrow S^2$.*

Example 3.15. *Suppose $f : M \longrightarrow N$ is a smooth map. Then $f_* : TM \longrightarrow TN$ is a bundle map covering f, i.e. (f_*, f) defines a bundle map.*

Example 3.16 (Pullback bundle)**.** *if $f : M \longrightarrow N$ is a smooth map and $E \xrightarrow{\pi} N$ is a vector bundle over N, then we may form the fiber product $M_f\times_\pi E$, which then is a bundle over M with local trivializations $(f^{-1}(U_i), f^* g_{ij})$, where (U_i, g_{ij}) is the local transition data for E over N. This bundle is called the pullback bundle and is denoted by f^*E. The natural projection to E defines a vector bundle map back to E:*

$$\begin{tikzcd} f^*E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

There is also a natural pullback map on sections: given a section $s \in \Gamma^\infty(N, E)$, the composition $s \circ f$ gives a map $M \longrightarrow E$. This then determines a smooth map $f^*s : M \longrightarrow f^*E$ by the universal property of the fiber product. We therefore obtain a pullback map

$$f^* : \Gamma^\infty(N, E) \longrightarrow \Gamma^\infty(M, f^*E).$$

Example 3.17. *If $f : M \longrightarrow N$ is an embedding, then so is the bundle map $f_* : TM \longrightarrow TN$. By the universal property of the fiber product we obtain a bundle map, also denoted f_*, from TM to f^*TN. This is a vector bundle inclusion and $f^*TN/f_*TM = NM$ is a vector bundle over M called the normal bundle of M.* Note: we haven't covered subbundles and quotient bundles in detail. I'll leave this as an exercise.

3.3 Associated bundles

We now describe a functorial construction of vector bundles, using functors from vector spaces. Consider the category $\mathbf{Vect}_\mathbb{R}$ of finite-dimensional real vector spaces and linear maps. We will describe several functors from $\mathbf{Vect}_\mathbb{R}$ to itself.

Example 3.18. *If $V \in \mathbf{Vect}_\mathbb{R}$, then $V^* \in \mathbf{Vect}_\mathbb{R}$, and if $f : V \longrightarrow W$ then $f^* : W^* \longrightarrow V^*$. Since the composition of duals is the dual of the composition, duality defines a contravariant functor $* : \mathbf{Vect}_\mathbb{R} \longrightarrow \mathbf{Vect}_\mathbb{R}$.*

Example 3.19. *If $V, W \in \mathbf{Vect}_\mathbb{R}$, then $V \oplus W \in \mathbf{Vect}_\mathbb{R}$, and this defines a covariant functor $\mathbf{Vect}_\mathbb{R} \times \mathbf{Vect}_\mathbb{R} \longrightarrow \mathbf{Vect}_\mathbb{R}$.*

Example 3.20. *If $V, W \in \mathbf{Vect}_\mathbb{R}$, then $V \otimes W \in \mathbf{Vect}_\mathbb{R}$ and this again defines a covariant functor $\mathbf{Vect}_\mathbb{R} \times \mathbf{Vect}_\mathbb{R} \longrightarrow \mathbf{Vect}_\mathbb{R}$.*

Example 3.21. *If $V \in \mathbf{Vect}_\mathbb{R}$, then*

$$\otimes^\bullet V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus (\otimes^k V) \oplus \cdots$$

is an infinite-dimensional vector space, with a product $a \otimes b$. Quotienting by the double-sided ideal $I = \langle v \otimes v \ : \ v \in V\rangle$, we obtain the exterior algebra

$$\wedge^\bullet V = \mathbb{R} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V,$$

with $n = \dim V$. The product is customarily denoted $(a, b) \mapsto a \wedge b$. The direct sum decompositions above, where $\wedge^k V$ or $\otimes^k V$ is labeled by the integer k, are called $\mathbb{Z}$-gradings, and since the product takes $\wedge^k \times \wedge^l \longrightarrow \wedge^{k+l}$, these algebras are called Z-graded algebras.

If $(v_1, \dots v_n)$ is a basis for V, then $v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $i_1 < \cdots < i_k$ form a basis for $\wedge^k V$. This space then has dimension $\binom{n}{k}$, hence the algebra $\wedge^\bullet V$ has dimension 2^n.

Note in particular that $\wedge^n V$ has dimension 1, is also called the determinant line $\det V$*, and a choice of nonzero element in* $\det V$ *is called an "orientation" on the vector space V.*

Recall that if $f : V \longrightarrow W$ is a linear map, then $\wedge^k f : \wedge^k V \longrightarrow \wedge^k W$ is defined on monomials via

$$\wedge^k f(a_1 \wedge \cdots \wedge a_k) = f(a_1) \wedge \cdots \wedge f(a_k).$$

In particular, if $A : V \longrightarrow V$ is a linear map, then for $n = \dim V$, the top exterior power $\wedge^n A : \wedge^n V \longrightarrow \wedge^n V$ is a linear map of a 1-dimensional space onto itself, and is hence given by a number, called $\det A$*, the determinant of A.*

We may now apply any of these functors to vector *bundles*. The main observation is that if F is a vector space functor as above, we may apply it to any vector bundle $E \xrightarrow{\pi} M$ to obtain a new vector bundle

$$F(E) = \sqcup_{p \in M} F(E_p).$$

If (U_i) is an atlas for M and E has local trivializations $(U_i \times \mathbb{R}^k)$, glued together via $g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R})$, then $F(E)$ may be given the local trivialization $(U_i \times F(\mathbb{R}^k))$, glued together via $F(g_{ji})$. This new vector bundle $F(E)$ is called the "associated" vector bundle to E, given by the functor F.

Example 3.22. *If $E \longrightarrow M$ is a vector bundle, then $E^* \longrightarrow M$ is the dual vector bundle. If E, F are vector bundles then $E \oplus F$ is called the direct or "Whitney" sum, and has rank* $\mathrm{rk}\ E + \mathrm{rk}\ F$*. $E \otimes F$ is the tensor product bundle, which has rank* $\mathrm{rk}\ E \cdot \mathrm{rk}\ F$*.*

Example 3.23. *If $E \longrightarrow M$ is a vector bundle of rank n, then $\otimes^k E$ and $\wedge^k E$ are its tensor power bundles, of rank n^k and $\binom{n}{k}$, respectively. The top exterior power $\wedge^n E$ has rank 1, and is hence a line bundle. If this line bundle is trivial (i.e. isomorphic to $M \times \mathbb{R}$) then E is said to be an orientable bundle.*

Example 3.24. *Starting with the tangent bundle $TM \longrightarrow M$, we may form the cotangent bundle T^*M, the bundle of tensors of type (r, s), $\otimes^r TM \otimes \otimes^s T^*M$.*

We may also form the bundle of multivectors $\wedge^k TM$, which has sections $\Gamma^\infty(M, \wedge^k TM)$ called multivector fields.

Finally, we may form the bundle of k-forms, $\wedge^k T^*M$, whose sections $\Gamma^\infty(M, \wedge^k T^*M) = \Omega^k(M)$ are called differential k-forms, and will occupy us for some time.

We have now produced several vector bundles by applying functors to the tangent bundle. We are familiar with vector fields, which are sections of TM, and we know that a vector field is written locally in coordinates $(x^1, \dots, x^n)$ as

$$X = \sum_i a^i \tfrac{\partial}{\partial x^i},$$

with coefficients a^i smooth functions.

There is an easy way to produce examples of 1-forms in $\Omega^1(M)$, using smooth functions f. We note that the action $X \mapsto X(f)$ defines a dual vector at each point of M, since $(X(f))_p$ depends only on the vector X_p and not the behaviour of X away from p. Recall that $X(f) = Df_2(X)$.

Definition 28. The exterior derivative of a function f, denoted *df*, is the section of T^*M given by the fiber projection Df_2.

Since $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, we see that $(dx^1, \dots, dx^n)$ is the dual basis to $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$. Therefore, a section of T^*M has local expression

$$\xi = \sum_i \xi_i dx^i,$$

for ξ_i smooth functions, given by $\xi_i = \xi(\frac{\partial}{\partial x^i})$. In particular, the exterior derivative of a function df can be written

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

A section of the tensor bundle $\otimes^r TM \otimes \otimes^s T^*M$ can be written as

$$\Theta = \sum_{\substack{i_1,\cdots,i_r \\ j_1,\cdots,j_s}} a^{i_1\cdots i_r}_{j_1\cdots j_s} \tfrac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \tfrac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

where $a^{i_1\cdots i_r}_{j_1\cdots j_s}$ are n^{r+s} smooth functions.

A general differential form $\rho \in \Omega^k(M)$ can be written

$$\rho = \sum_{i_1<\cdots<i_k} \rho_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

3.4 Differential forms

There are several properties of differential forms which make them indispensible: first, the k-forms are intended to give a notion of k-dimensional volume (this is why they are multilinear and skew-symmetric, like the determinant) and in a way compatible with the boundary map (this leads to the exterior derivative, which we define below). Second, they behave well functorially, as we see now.

Given a smooth map $f : M \longrightarrow N$, we obtain bundle maps $f_* : TM \longrightarrow TN$ and hence $f^* := \wedge^k(f_*)^* : \wedge^k T^*N \longrightarrow \wedge^k T^*M$. Hence we have the diagram

$$\begin{tikzcd} \wedge^k T^*M \arrow[d, "\pi_M"'] & \wedge^k T^*N \arrow[l, "f^*"'] \arrow[d, "\pi_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

The interesting thing is that if $\rho \in \Omega^k(N)$ is a differential form on N, then it is a section of π_N. Composing with f, f^*, we obtain a section $f^*\rho := f^* \circ \rho \circ f$ of π_M. Hence we obtain a natural map

$$\Omega^k(N) \xrightarrow{f^*} \Omega^k(M).$$

Such a natural map does not exist (in either direction) for multivector fields, for instance.

Suppose that $\rho \in \Omega^k(N)$ is given in a coordinate chart by $\rho = \sum \rho_{i_1\cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$. Now choose a coordinate chart for M with coordinates $x^1, \ldots x^m$. What is the local expression for $f^*\rho$? We need only compute f^*dy_i. We use a notation where f^k denotes the k^{th} component of f in the coordinates $(y^1, \ldots y^n)$, i.e. $f^k = y^k \circ f$.

$$f^*dy_i(\tfrac{\partial}{\partial x^j}) = dy_i(f_* \tfrac{\partial}{\partial x^j}) \tag{23}$$
$$= dy_i(\sum_k \tfrac{\partial f^k}{\partial x^j} \tfrac{\partial}{\partial y_k}) \tag{24}$$
$$= \tfrac{\partial f^i}{\partial x^j}. \tag{25}$$

Hence we conclude that

$$f^*dy_i = \sum_j \tfrac{\partial f^i}{\partial x^j} dx^j.$$

Finally we compute

$$f^*\rho = \sum_{i_1<\cdots<i_k} f^*\rho_{i_1\cdots i_k} f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}) \tag{26}$$
$$= \sum_{i_1<\cdots<i_k} (\rho_{i_1\cdots i_k} \circ f) \sum_{j_1} \cdots \sum_{j_k} \tfrac{\partial f^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial f^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge \cdots dx^{j_k}. \tag{27}$$