1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made up by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We would like to make this precise.

1.1 Topological manifolds

Definition 1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

Note: "Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold, i.e. whose union is the whole space, is called an *atlas*.

We now give a bunch of examples of topological manifolds. The simplest is, technically, the empty set. More simple examples include a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.1 (Circle). *Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi i c}$ for a unique real number $0 \le c < 1$, and define the map*

$$\nu_z : t \mapsto e^{2\pi i t}. \tag{1}$$

We note that ν_z maps the interval $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighbourhood of z given by $S^1 \backslash \{-z\}$, and it is a homeomorphism. Then $\varphi_z = \nu_z|_{I_c}^{-1}$ is a local coordinate chart near z.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.2 (n-torus). *$S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.*

Example 1.3 (open subsets). *Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M.*

For example, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \ne 0\}, \tag{2}$$

known as the general linear group, which therefore forms a topological manifold.

Example 1.4 (Spheres). *The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:*

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the North pole and let $S = (-1, 0, \ldots, 0)$ be the South pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}, \tag{3}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}. \tag{4}$$

We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopic to S^n without being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the (easy) classification of topological 1- and 2-manifolds.

Example 1.5 (Projective spaces). *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or even $\mathbb{H}$). Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.*

More precisely, let $X = \mathbb{K}^{n+1}\backslash\{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists\lambda \in \mathbb{K}^* = \mathbb{K}\backslash\{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an open *map, since if $U \subset X$ is open, then tU is also open $\forall t \in K^*$, implying that $\cup_{t\in\mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.*

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but not *quotients), we show that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the Hausdorff property for $\mathbb{K}P^n$). This graph is simply*

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x, y) = (x_i y_j - x_j y_i) \ \ i \neq j.$$

(Does this work for $\mathbb{H}$? How can it be fixed?)

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{5}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.6 (Connected sum). *Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.*

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\phi : B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \longrightarrow B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \tag{6}$$

$$x \mapsto \frac{2\epsilon^2}{|x|^2}x. \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the connected sum $M\sharp N$, *as the quotient $X/\sim$, where*

$$X = (M\backslash\overline{\varphi^{-1}(B(0, \epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0, \epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

Two important remarks concerning the connect sum: first, the connect sum of a sphere with itself is homeomorphic to the same sphere:

$$S^n\sharp S^n \cong S^n.$$

Second, by taking repeated connect sums of T^2 and $\mathbb{R}P^2$, we may obtain all compact 2-dimensional manifolds.

Example 1.7 (General gluing construction). *To construct a topological manifold "from scratch", we should be able to glue pieces of $\mathbb{R}^n$ together, as long as the gluing is consistent and by homeomorphisms. The following is a method for doing so, tailor-made so that all the requirements are satisfied.*

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \;, \tag{8}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be homeomorphisms.

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij} \cap U_{jk}}$ for all i, j, k.

Second countability of the glued manifold will be guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that $\forall p \in \partial U_{ij} \subset U_i$ and $\forall q \in \partial U_{ji} \subset U_j$, there exist neighbourhoods $V_p \subset U_i$ and $V_q \subset U_j$ of p, q respectively with $\varphi_{ij}(V_p \cap U_{ij}) \cap V_q = \emptyset$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{9}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$. This space naturally comes with an atlas $\mathcal{A}$, where the charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

As an exercise, you may show that any topological manifold is homeomorphic to one constructed in this way.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, if we compare coordinates on the intersection $U_{ij} = U_i \cap U_j$, we see that the map

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

is a homeomorphism, simply because it is a composition of homeomorphisms. We can say this another way: topological manifolds are glued together by homeomorphisms.

This means that we may be able to differentiate a function in one coordinate chart but not in another, i.e. there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, which is simply a topological manifold where the gluing maps are required to be *smooth*.

First we recall the notion of a smooth map of finite-dimensional vector spaces.

Remark 1 (Aside on smooth maps of vector spaces). *Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. The function f is said to be differentiable at $p \in U$ if there exists a linear map $Df(p) : V \longrightarrow W$ such that*

$$||f(p + x) - f(p) - Df(p)(x)|| = o(||x||),$$

where $o : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous at 0 and $\lim_{t \to 0} o(t)/t = 0$, and we choose any inner product on V, W, defining the norm $||\cdot||$. For infinite-dimensional vector spaces, the topology is highly sensitive to which norm is chosen, but we will work in finite dimensions.

Given linear coordinates $(x_1, \ldots, x_n)$ on V, and $(y_1, \ldots, y_m)$ on W, we may express f in terms of its m components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written as an $m \times n$ matrix, called the Jacobian matrix of f at p.

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{10}$$

We say that f is differentiable on U when it is differentiable at all $p \in U$ and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W)$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

The first derivative Df is also a map from U to a vector space ($\mathrm{Hom}(V, W)$), therefore if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)),$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and hence we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W).$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $S^2 V^* \otimes W$, since "mixed partials are equal".

After this aside, we can define a smooth manifold.

Definition 2. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, explained below.

Definition 3. An atlas $\mathcal{A} = \{U_i, \varphi_i\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Note: Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We may now verify that all the examples from section 1.1 are actually smooth manifolds:

Example 1.8 (Circle)**.** *For Example 1.1, only two charts, e.g. $\varphi_{\pm 1}$, suffice to define an atlas, and we have*

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t + 1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2}, \end{cases}$$

which is clearly C^∞. In fact all the charts φ_z are smoothly compatible. Hence the circle is a smooth manifold.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.2, is smooth. Example 1.3 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.9 (Spheres)**.** *The charts for the n-sphere given in Example 1.4 form a smooth atlas, since*

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0} \vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2} \vec{z} = |\vec{z}|^{-2} \vec{z},$$

which is smooth on $\mathbb{R}^n \backslash \{0\}$, as required.

Example 1.10 (Projective spaces)**.** *The charts for projective spaces given in Example 1.5 form a smooth atlas, since*

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1} z_2, \ldots, z_1^{-1} z_n), \tag{11}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.6 is clearly smooth since ϕ was chosen to be a smooth map, and any topological manifold from Example 1.7 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.