1.3 Manifolds with boundary

The concept of *manifold with boundary* is important for relating manifolds of different dimension. Our manifolds are defined intrinsically, meaning that they are not defined as subsets of another topological space; therefore, the notion of boundary will differ from the usual boundary of a subset.

To introduce boundaries in our manifolds, we need to change the local model which they are based on. For this reason, we introduce the half-space $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}$, equip it with the induced topology from $\mathbb{R}^n$, and model our spaces on this one.

Definition 4. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart. We shall see later that $M = \partial M \sqcup \operatorname{Int} M$.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, in the sense below.

Definition 5. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

For example, the function $f(x, y) = y$ is smooth on H^2 but $f(x, y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Note the important fact that if M is an n-manifold with boundary, $\operatorname{Int} M$ is a usual n-manifold, without boundary. Also, even more importantly, ∂M is an $n-1$-manifold without boundary, i.e. $\partial(\partial M) = \emptyset$. This is sometimes phrased as the equation

$$\partial^2 = 0.$$

Example 1.11 (Möbius strip)**.** *The mobius strip E is a compact 2-manifold with boundary. As a topological space it is the quotient of $\mathbb{R} \times [0, 1]$ by the identification $(x, y) \sim (x + 1, 1 - y)$. The map $\pi : [(x, y)] \mapsto e^{2\pi i x}$ is a continuous surjective map to S^1, called a projection map. We may choose charts $[(x, y)] \mapsto e^{x+i\pi y}$ for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, and for any $\epsilon < \frac{1}{2}$.*

Note that ∂E is diffeomorphic to S^1. This actually provides us with our first example of a non-trivial fiber bundle, as we shall see. In this case, E is a bundle of intervals over a circle.

1.4 Cobordism

$(n + 1)$-Manifolds with boundary provide us with a natural equivalence relation on n-manifolds, called *cobordism.*

Definition 6. n-manifolds M_1, M_2 are *cobordant* when there exists a $n+1$-manifold with boundary N such that ∂N is diffeomorphic to $M_1 \sqcup M_2$. The class of manifolds cobordant to M is called the *cobordism class* of M.

Note that while the Cartesian product of manifolds is a manifold, the Cartesian product of two manifolds with boundary is *not* a manifold with boundary. On the other hand, the Cartesian product of manifolds only one of which has boundary, is a manifold with boundary (why?)

Cobordism classes of manifolds inherit two natural operations, as follows: If $[M_1], [M_2]$ are cobordism classes, then the operation $[M_1] \cdot [M_2] = [M_1 \times M_2]$ is well-defined. Furthermore $[M_1] + [M_2] = [M_1 \sqcup M_2]$ is well-defined, and the two operations satisfy the axioms defining a commutative ring. The ring of cobordism classes of compact manifolds is called the *cobordism ring* and is denoted $\Omega^\bullet$. The subset of classes of k-dimensional manifolds is denoted $\Omega^k \subset \Omega^\bullet$.

Proposition 1.12. *The cobordism ring is 2-torsion, i.e. $x + x = 0 \ \ \forall x$.*

Proof. The zero element of the ring is $[\emptyset]$ and the multiplicative unit is $[*]$, the class of the one-point manifold. For any manifold M, the manifold with boundary $M \times [0,1]$ has boundary $M \sqcup M$. Hence $[M]+[M] = [\emptyset] = 0$, as required. □

Example 1.13. *The n-sphere S^n is null-cobordant (i.e. cobordant to $\emptyset$), since $\partial\overline{B_{n+1}(0,1)} \cong S^n$, where $B_{n+1}(0,1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.*

Example 1.14. *Any oriented compact 2-manifold Σ_g is null-cobordant , since we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary given by Σ_g.*

We would like to state an amazing theorem of Thom, which is a complete characterization of the cobordism ring.

Theorem 1.15. *The cobordism ring is a (countably generated) polynomial ring over $\mathbb{F}_2$ with generators in every dimension $n \neq 2^k - 1$, i.e.*

$$\Omega^\bullet = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots].$$

This theorem implies that there are 3 cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2 + x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives?

1.5 Smooth maps

For topological manifolds M, N of dimension m, n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous and associative, we obtain a category C^0**-Man** of topological manifolds and continuous maps. Recall that a category is simply a class of objects $\mathcal{C}$ (in our case, topological manifolds) and an associative class of arrows $\mathcal{A}$ (in our case, continuous maps) with source and target maps $\mathcal{A} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C}$ and an identity arrow for each object, given by a map $\mathrm{Id} : \mathcal{C} \longrightarrow \mathcal{A}$ (in our case, the identity map of any manifold to itself). Conventionally we write the set of arrows $\{a \in \mathcal{A} \ : \ s(a) = x \text{ and } t(a) = y\}$ as $\mathrm{Hom}(x, y)$. Also note that the associative composition of arrows mentioned above then becomes a map

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \longrightarrow \mathrm{Hom}(x, z).$$

If M, N are smooth manifolds, the right notion of morphism from M to N is that of a smooth map $f : M \longrightarrow N$.

Definition 7. A map $f : M \longrightarrow N$ is called smooth when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth map, i.e. $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M, N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

If $g : L \longrightarrow M$ and $f : M \longrightarrow N$ are smooth maps, then so is the composition $f \circ g$, since if charts φ, χ, ψ for L, M, N are chosen near $p \in L$, $g(p) \in M$, and $(fg)(p) \in N$, then $\psi \circ (f \circ g) \circ \varphi^{-1} = A \circ B$, for $A = \psi f \chi^{-1}$ and $B = \chi g \varphi^{-1}$ both smooth mappings $\mathbb{R}^n \longrightarrow \mathbb{R}^n$. By the chain rule, $A \circ B$ is differentiable at p, with derivative $D_p(A \circ B) = (D_{g(p)}A)(D_pB)$ (matrix multiplication).

Now we have a new category, which we may call C^∞**-Man**, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic. In fact, the definitions above carry over, word for word, to the setting of manifolds with boundary. Hence we have defined another category, C^∞**-Man**$_\partial$, the category of smooth manifolds with boundary.

In defining the arrows for the category C^∞**-Man**$_\partial$, we may choose to consider all smooth maps, or only those smooth maps $M \longrightarrow N$ such that ∂M is sent to ∂N, i.e. boundary-preserving maps. Call the resulting category in the latter case C^∞_∂**-Man**$_\partial$.

Note that the boundary map, ∂, maps the objects of C^∞_∂**-Man**$_\partial$ to objects in C^∞**-Man**, and similarly for arrows, and such that the following square commutes:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M' \\ {\scriptstyle\partial}\big\downarrow & & \big\downarrow{\scriptstyle\partial} \\ \partial M & \xrightarrow[\psi|_{\partial M}]{} & \partial M' \end{array} \tag{12}$$

This is precisely what it means for ∂ to be a (covariant) *functor*, from the category of manifolds with boundary and boundary-preserving smooth maps, to the category of manifolds without boundary.

Fix a smooth manifold N and consider the class of pairs (M, φ) where M is a smooth manifold with boundary and φ is a smooth map $\varphi : M \longrightarrow N$. Define a category where these maps are the objects. How does the boundary operator act on this category?

Example 1.16. *We show that the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2. Consider the maps $f_+(x_0, x_1, x_2) = [1 + x_0 : x_1 + ix_2]$ and $f_-(x_0, x_1, x_2) = [x_1 - ix_2 : 1 - x_0]$. Since $f_\pm$ is continuous on $x_0 \neq \pm 1$, and since $f_- = f_+$ on $|x_0| < 1$, the pair (f_-, f_+) defines a continuous map $f : S^2 \longrightarrow \mathbb{C}P^1$. To check smoothness, we compute the compositions*

$$\varphi_0 \circ f_+ \circ \varphi_N^{-1} : (y_1, y_2) \mapsto y_1 + iy_2, \tag{13}$$
$$\varphi_1 \circ f_- \circ \varphi_S^{-1} : (y_1, y_2) \mapsto y_1 - iy_2, \tag{14}$$

both of which are obviously smooth maps.

Remark 2 (Exotic smooth structures)**.** *The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called* exotic *spheres.*

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a monoid *structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are* 28 *smooth spheres, which we will encounter later on.*

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called fake $\mathbb{R}^4$*s.*

Example 1.17 (Lie groups)**.** *A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.*

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a topological group. *If G is a given a smooth structure and m, ι are smooth maps, the result is a* Lie group.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called left multiplication *and* right multiplication. *Note that the group axioms imply that $R_g L_h = L_h R_g$.*

1.6 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U,φ), (V,ψ) for the smooth manifolds M,N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi\circ f\circ\varphi^{-1} \in C^\infty(\varphi(U),\mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem*.

Theorem 1.18 (Inverse function theorem). *Let $U\subset\mathbb{R}^m$ an open set and $f : U \longrightarrow \mathbb{R}^m$ a smooth map such that $Df(p)$ is an invertible linear operator. Then there is a neighbourhood $V\subset U$ of p such that $f(V)$ is open and $f : V \longrightarrow f(V)$ is a diffeomorphism. furthermore, $D(f^{-1})(f(p)) = (Df(p))^{-1}$.*

Proof. Without loss of generality, assume that U contains the origin, that $f(0) = 0$ and that $Df(p) = \mathrm{Id}$ (for this, replace f by $(Df(0))^{-1}\circ f$. We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define g so that $f(x) = x + g(x)$. Hence $g(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y-g(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $\phi(y)$, and so $y-g(\phi(y)) = \phi(y)$, i.e. ϕ is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0,r)$ where $||Dh_y|| \le \frac{1}{2}$. This then implies (mean value theorem) that for $x,x'\in B(0,r)$,

$$||h_y(x) - h_y(x')|| \le \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \le ||h_y(x) - h_y(0)|| + ||h_y(0)|| \le \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y\in B(0,\frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0,r)}$. Let $\phi(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned}||\phi(y) - \phi(y')|| &= ||h_y(\phi(y)) - h_{y'}(\phi(y'))||\\ &\le ||g(\phi(y)) - g(\phi(y'))|| + ||y - y'||\\ &\le \tfrac{1}{2}||\phi(y) - \phi(y')|| + ||y - y'||,\end{aligned}$$

so that we have $||\phi(y) - \phi(y')|| \le 2||y - y''||$, as required.

To see that ϕ is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = \phi(y)$ and $x' = \phi(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0,r)}$, which is not a problem.

$$\begin{aligned}||\phi(y) - \phi(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))||\\ &\le ||(Df(x))^{-1}||||(Df(x))(x - x') - (f(x) - f(x'))||\\ &\le o(||x - x'||),\ \text{using differentiability of } f\\ &\le o(||y - y'||),\ \text{using continuity of } \phi.\end{aligned}$$

Now that we have shown ϕ is differentiable with derivative $(Df)^{-1}$, we use the fact that Df is C^∞ and inversion is C^∞, implying that $D\phi$ is C^∞ and hence ϕ also. □

This theorem immediately provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.