2.4 Intersection theory

The previous corollary allows us to make the following definition:

Definition 17. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X compact, g closed, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \sharp(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g, and where we assume the fiber product to consist of a finite number of points (this is always guaranteed, e.g. if g is proper, or if g is a closed embedding).

Example 2.30. *If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.*

Example 2.31. *If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle\cos(\theta/2)e_1 + \sin(\theta/2)e_2\rangle$ and $\iota_2 : \theta \mapsto \langle\cos(\theta/2)e_2 + \sin(\theta/2)e_3\rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2\rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points.*

Example 2.32. *Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2\dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.*

Example 2.33. *Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.*

Example 2.34. *By the previous argument, any compact manifold is not contractible.*

Example 2.35. *Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.*

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 2.36. *Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-homotopic, we cannot deform the planes to remove all intersection.*

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 18. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

Example 2.37. *Let $f : S^1 \longrightarrow S^1$ be given by $z \mapsto z^k$. Then $\deg_2(f) = k \pmod 2$.*

Example 2.38. *If* $p : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ *is a polynomial of degree* k*, then as a map* $S^2 \longrightarrow S^2$ *we have* $\deg_2(p) = k \pmod 2$*, and hence any odd polynomial has at least one root. To get the fundamental theorem of algebra, we must consider* oriented cobordism

Even if submanifolds C, C' do not intersect, it may be that there are more sophisticated geometrical invariants which cause them to be "intertwined" in some way. One example of this is linking number.

Definition 19. Suppose that $M, N \subset \mathbb{R}^{k+1}$ are compact embedded submanifolds with $\dim M + \dim N = k$, and let us assume they are transverse, meaning they do not intersect at all.

Then define $\lambda : M \times N \longrightarrow S^k$ via

$$(x, y) \mapsto \frac{x - y}{|x - y|}.$$

Then we define the $\pmod 2$ linking number of M, N to be $\deg_2(\lambda)$.

Example 2.39. *Consider the standard Hopf link in* $\mathbb{R}^3$*. Then it is easy to calculate that* $\deg_2(\lambda) = 1$*. On the other hand, the standard embedding of disjoint circles (differing by a translation, say) has* $\deg_2(\lambda) = 0$*. Hence it is impossible to deform the circles through embeddings of* $S^1 \sqcup S^1 \longrightarrow \mathbb{R}^3$*, so that they are unlinked. Why must we stay within the space of embeddings, and not allow the circles to intersect?*

3 The tangent bundle and vector bundles

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M, which is made up of the disjoint union of all tangent spaces to all points in M. If M is embedded in $\mathbb{R}^N$, then TM is a regular submanifold of $\mathbb{R}^N \times \mathbb{R}^N$, but we define it intrinsically, without reference to an embedding.

As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

Definition 20. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM, and forms a real vector space of dimension $\dim M$.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_pM,$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 3.1. *For an* n*-manifold* M*, the set* TM *has a natural topology and smooth structure which make it a* $2n$*-manifold, and make* $\pi : TM \longrightarrow M$ *a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n,$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

A more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \;\Leftrightarrow\; y = \varphi_j \circ \varphi_i^{-1}(x) \;\text{ and }\; v = D(\varphi_j \circ \varphi_i^{-1})_x u,$$

and verify the conditions of the general gluing construction 1.7. Then show that a different atlas gives a canonically diffeomorphic manifold, i.e. that the result is independent of atlas.

A description of the tangent bundle is not complete without defining the derivative of a general smooth map of manifolds $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \longrightarrow \psi_\alpha(V_\alpha)$ which satisfy

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1}).$$

Differentiating, we obtain

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1}),$$

and hence we obtain a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Df : TM \longrightarrow TN$. Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{Df} & TN \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

Just as $\pi^{-1}(x) = T_xM \subset TM$ is a vector space for all x, making TM into a "bundle of vector spaces", the map $Df : T_xM \longrightarrow T_{f(x)}N$ is a linear map and hence Df is a "bundle of linear maps".

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Df \circ Dg = Dh$. As a result, we obtain the following category-theoretic statement.

Proposition 3.2. *The map T which takes a manifold M to its tangent bundle TM, and which takes maps $f : M \longrightarrow N$ to the derivative $Df : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself.*

For this reason, the derivative map Df is sometimes called the "tangent mapping" Tf.

Example 3.3. *If $\iota : M \longrightarrow N$ is an embedding of M into N, then $D\iota : TM \longrightarrow TN$ is also an embedding, and hence $D^k\iota : T^kM \longrightarrow T^kN$ are all embeddings.*

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart (U_i, φ_i), we would say that a vector field X_i is simply a vector-valued function on U_i, i.e. a function $X_i : \varphi(U_i) \longrightarrow \mathbb{R}^n$. Of course if we had another vector field X_j on (U_j, φ_j), then the two would agree as vector fields on the overlap $U_i \cap U_j$ when $D(\varphi_j \circ \varphi_i^{-1}) : X_i \mapsto X_j$. So, if we specify a collection $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue on overlaps, this would define a global vector field. This leads precisely to the following definition.

Definition 21. A smooth vector field on the manifold M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X : M \longrightarrow M$ is the identity. Essentially it is a smooth assignment of a unique tangent vector to each point in M.

Such maps X are also called *cross-sections* or simply *sections* of the tangent bundle TM, and the set of all such sections is denoted $C^\infty(M, TM)$ or sometimes $\Gamma^\infty(M, TM)$, to distinguish them from simply smooth maps $M \longrightarrow TM$.