Exercise 1. Show that every Morse function on a compact odd-dimensional manifold must have an even number of critical points. If M is a 3-dimensional homology sphere (i.e. $H_*(M) = H_*(S^3)$) but with $\pi_1(M) \neq \{1\}$), show that any Morse function must have at least six critical points. Hint: $H_1(M)$ is the abelianization of $\pi_1(M)$.

Exercise 2. Let S^2 be the standard sphere $\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$ equipped with the Euclidean metric, let $f = z$ be the height function on S^2, and let N be the North pole.

- Write the gradient of f explicitly in coordinates on S^2.
- Write the negative gradient flow $\frac{d}{dt}p = -\mathrm{grad}(f)(p)$ explicitly as a system of ODE.
- Solve the ODE system for all initial conditions.
- The above describes explicitly a family Φ_t of diffeomorphisms of S^2, i.e. the flow of the gradient vector field. Let v_0 be the Euclidean volume form on S^2 and let $v_t = \Phi_t^* v_0$. Show that
$$\int_{S^2} v_t = \int_{S^2} v_0.$$

Exercise 3. Draw a picture of the negative gradient flow lines for the standard Morse function on $\mathbb{R}P^2$ which we saw in class. Describe precisely the set of integral curves $\{u : \mathbb{R} \longrightarrow \mathbb{R}P^2 \ : \ \frac{d}{dt}u = -\mathrm{grad}(f)(u)\}$ going between each pair (p_-, p_+) of critical points (this simply means $\lim_{t\to\pm\infty} u(t) = p_\pm$). Informally, describe the possible ways that a family of integral curves can "break" into a number of other integral curves.

Exercise 4. Do a Morse theory analysis of a simple Flag variety. For example, take the variety $\mathrm{Fl}_{1,2}$ consisting of pairs (p, ℓ) where $\ell \subset \mathbb{P}^2$ is a line and $p \in \ell$. Viewing this as a subspace of $\mathbb{P}^2 \times (\mathbb{P}^2)^*$ (here $(\mathbb{P}^2)^*$ means the space of lines in $\mathbb{P}^2$, or in other words $\mathrm{Gr}_2\mathbb{C}^3$, which is non-canonically isomorphic to $\mathrm{Gr}_1\mathbb{C}^3 = \mathbb{P}^2$), find a perfect Morse function and determine the Poincaré polynomial.

Bonus: Describe the ascending and descending manifolds as precisely as possible. Does the Morse function degenerate to a Bott-Morse function? What is the Bott-Morse polynomial?