1.5.3 Handle cancellation

We can isotope the attaching maps and framings without changing the diffeomorphism type of the resulting manifold. There are also two elementary observations which help to relate any two handlebody decompositions:

Proposition 1.14. *If $M' = (M \cup \mathbb{H}_b) \cup \mathbb{H}_a$ with $a \leq b$, then M' can be obtained by first attaching $\mathbb{H}_a$ and then $\mathbb{H}_b$. Therefore we can always attach handles in a weakly increasing order of index.*

Proof. This is simply because the attaching sphere S^{a-1} can be deformed to miss the belt sphere of $\mathbb{H}_b$ and moved off the handle. □

Proposition 1.15 (Handle cancellation)**.** *If $M = (M_0 \cup \mathbb{H}^\lambda) \cup \mathbb{H}^{\lambda+1}$, where the attaching sphere of $\mathbb{H}^{\lambda+1}$* **intersects the belt sphere of $\mathbb{H}^\lambda$ transversely in one point**, *then M is diffeomorphic to M_0.*

Proof. Let $S^{\lambda-1}, S^\lambda$ be the attaching spheres for the handles in question. The fact that we can attach $\mathbb{H}^{\lambda+1}$ to $M_0 \cup \mathbb{H}^\lambda$ in such a way that the attaching S^λ intersects the belt sphere transversely in a single point tells us something very special about the embedding of the first attaching sphere $S^{\lambda-1}$ in ∂M_0 – that it is the boundary of a λ-disc embedded in ∂M_0.

Write M_0 as $M_0 \#_b \mathbb{D}^m$, and since all embeddings of discs are equivalent[10], we can move the above λ-disc so that it is half a great λ-sphere on the boundary of $\mathbb{D}^m$ whose boundary is the standard $S^{\lambda-1}$. Attaching $\mathbb{H}^\lambda$ along this sphere is then just a joining of two handles, giving a disc bundle $\mathbb{D}^m \cup \mathbb{H}^\lambda \cong S^\lambda \times \mathbb{D}^{m-\lambda}$, where now the attaching sphere for the $\mathbb{H}^{\lambda+1}$ is a section of the bundle.

Finally we observe that a $\mathbb{H}^{\lambda+1}$ attachment along a section of $S^\lambda \times \mathbb{D}^{m-\lambda}$ is diffeomorphic to $\mathbb{D}^m$ (Why? Not a difficult exercise.)

This gives the result, as

$$M \cong M_0 \#_b ((\mathbb{D}^m \cup \mathbb{H}^\lambda) \cup \mathbb{H}^{\lambda+1}) \cong M_0 \#_b \mathbb{D}^m \cong M_0.$$

□

Theorem 1.16. *(Kirby) Any two smooth handlebody decompositions of M are related by an isotopy of the attaching maps "handle slides" and creation or cancellation of handle pairs.*

Proof. Kirby's proof of this theorem relies on Cerf theory, which says that we can interpolate between any two Morse functions, if we allow finitely many times where critical points collide to produce degenerate critical points. It is when we pass these non-Morse times where we obtain a cancellation or creation of handle pairs. □

Example 1.17. *Give examples of Handle attachment in dimension 2 and 3, sliding a 1-handle off a 2-handle in dimension 3. Look at handle slide for a 1-dimensional boundary. Attach two nonorientable handles to a disk, do a handle slide. Explain attachments of 1-handles and 2-handles on $\mathbb{D}^3$ where the new boundary is not a sphere, so that no* 3*-handle can be added.*

1.5.4 Kirby calculus

Kirby calculus is the study of handlebody decompositions of 4-manifolds. Because of visual limitations, we don't draw the 4-dimensional handles, but rather only consider the boundary of the initial 0-handle $\mathbb{D}^4$, namely $S^3 = \mathbb{R}^3 \cup \infty$, and draw the image of the attaching maps in $\mathbb{R}^3$.

Introduction to handle birth/death and handle slides in dimension 4. Describe birth of $1-2$ pair and $2-3$ pair (3-handle not shown of course). Also describe sliding 2-handle across another 2-handle (band connected sum with a framed push-off).

[10] By Palais' disc theorem (see Kosinski's text), any embedding of discs in a connected manifold are ambient isotopic (if the embedding is equidimensional then one requires the orientations to agree)

1.6 Handle presentation theorem

Theorem 1.18. *If c is a critical value of f contaning a single critical point p of Morse index λ, then for sufficiently small $\epsilon > 0$, $M^{c+\epsilon}$ is diffeomorphic to $M^{c-\epsilon}$ with a λ-handle attached. In particular, $M^{c+\epsilon}$ is homotopic to $M^{c-\epsilon}$ with a λ-cell attached. If we rearrange[11] the order of handle attachment to be in non-decreasing index, then we get a CW complex decomposition of M.*

Remark 8. *Sometimes this theorem is stated as follows: Let a, b be regular values of f. Then $M_{a,b} = f^{-1}([a, b])$ is a manifold with boundary, which may be viewed as a cobordism between $f^{-1}(a)$ and $f^{-1}(b)$. If there are no critical points in $M_{a,b}$, then it is a trivial cobordism. If there is exactly one critical point of index λ, then $M_{a,b}$ is an "elementary cobordism" of index λ, namely, an attachment of a λ-handle to the $f^{-1}(a) \times \{1\}$ boundary of $f^{-1}(a) \times [0, 1]$.*

Proof. Suppose we choose Morse coordinates (x_λ, x_μ) in a neighbourhood U of the critical point p and assume $f(p) = 0$ for simpler notation. Choose $\epsilon > 0$ such that $f^{-1}([-\epsilon, \epsilon])$ contains only the critical point p. The difference between M^ϵ and $M^{-\epsilon}$ is a manifold which may extend beyond U. **First we will deform** M^ϵ to an intermediate manifold $\tilde{M}^\epsilon$, contained in and diffeomorphic to M^ϵ, such that the difference between $M^{-\epsilon}$ and $\tilde{M}^\epsilon$ is contained entirely inside the coordinate neighbourhood U.

Construction of $\tilde{M}^\epsilon$: Pick a non-negative function ϕ with $\phi(0) > \epsilon$, $\phi(t) = 0$ for $t \geq 2\epsilon$ and $-1 < \phi'(t) \leq 0$. Then set

$$F(x) = \begin{cases} f(x) - \phi(x_\lambda^2 + 2x_\mu^2) & \text{for } x \in U, \\ f(x) & \text{elsewhere} \end{cases}$$

Then we have the following properties of F:

- $F \leq \epsilon$ agrees with $f \leq \epsilon$, since $F \leq f$ and if $F(x) \leq \epsilon$ with $\phi(x_\lambda^2 + 2x_\mu^2) \geq 0$, then we have immediately $f(x) = -x_\lambda^2 + x_\mu^2 \leq \frac{1}{2}x_\lambda^2 + x_\mu^2 < \epsilon$).
- F has the same critical points as f: Let $u_+ = x_\lambda^2$ and $u_- = x_\mu^2$. Then $F = f - \lambda(u_- + 2u_+) = -u_- + u_+ - \lambda(u_- + 2u_+)$ and $dF = -(1 + \lambda')du_- + (1 - 2\lambda')du_+$. The condition on λ' implies that F is critical whenever $du_- = du_+ = 0$, but this is exactly at the critical point of f.
- $F(p) < -\epsilon$, so that there are no critical points in $F^{-1}[-\epsilon, \epsilon]$, showing that
$$\tilde{M}^\epsilon := F^{-1}(-\infty, -\epsilon]$$
is diffeomorphic to M^ϵ, while it differs from $M^{-\epsilon}$ only inside U.

We will now use the coordinate system in U to construct the diffeomorphism explicitly. First we observe the following fact

structure of $\tilde{M}^\epsilon \cap x_\lambda = q$: The intersection of $\tilde{M}^\epsilon$ with the μ-plane $x_\lambda = q$ is diffeomorphic to a disc of radius $r(q) > 0$, where $r(q)$ is smooth and $r(q) = \sqrt{q^2 - \epsilon}$ for $q^2 > 2\epsilon$.

We must show that $\tilde{M}^\epsilon$ is diffeomorphic to a handle attachment. The handle attachment in question will be attached along $S_\epsilon^{\lambda-1} \subset \partial M^{-\epsilon}$, via an attachment map $h : T(\epsilon) \longrightarrow T'_{S_\epsilon^{\lambda-1}} = \{x \in M^{-\epsilon} \mid x_\lambda^2 < 2\epsilon\}$, given by

$$h(x_\lambda, x_\mu) = \sqrt{2\epsilon}(\frac{x_\lambda}{|x_\lambda|}\sqrt{\tfrac{3}{2} - x_\lambda^2}, x_\mu).$$

Then the diffeomorphism $g : M^{-\epsilon} \cup_h \mathbb{H}^\lambda \longrightarrow \tilde{M}^\epsilon$ is given by

$$g(x) = \begin{cases} \sigma(x_\lambda, x_\mu) = (x_\lambda, x_\mu \frac{r(x_\lambda)}{\sqrt{x_\lambda^2 - \epsilon}}) & \text{if } x \in M^{-\epsilon} - S, \\ \tau(x_\lambda, x_\mu) = (x_\lambda\sqrt{2\epsilon}, x_\mu \frac{r(x_\lambda\sqrt{2\epsilon})}{\sqrt{1 - x_\lambda^2}}) & \text{if } x \in \mathbb{D}^n - S^{\lambda-1}. \end{cases}$$

since $\sigma h \alpha = \tau$, g is well-defined on the handle attachment, and it is a diffeomorphism. ☐

[11] Such a rearrangement is always possible, but we can show the stronger result that we can always choose a Morse function to have critical points in order of nondecreasing index.