1.7 Examples of handle decompositions

Example 1.19. *Consider S^2 embedded in $\mathbb{R}^3$ as an irregular ellipsoid $x^2+2y^2+3z^2=1$. The "height function" $f=r^2$ is Morse, and invariant under the antipodal map $A:(x,y,z)\mapsto(-x,-y,-z)$. Hence it descends to a Morse function with three critical points on $\mathbb{R}P^2$, with indices $0,1,2$. In this case we can see explicitly how the handle decomposition works; after the 1-handle is attached to the 0-handle, the manifold is non-orientable, so that it has one boundary component and so is ready for the single 2-handle.*

Example 1.20 (Planar equilateral pentagons)**.** *A planar equilateral pentagon can be described by 4 unit complex numbers $(z_1,z_2,z_3,z_4)=(e^{i\theta_1},e^{i\theta_2},e^{i\theta_3},e^{i\theta_4})$ with $\operatorname{Im}\sum z_i=0$ and $\operatorname{Re}\sum z_i=1$. We will analyze this space by viewing it as the -1 level set of the function $f=-\operatorname{Re}\sum z_i=\sum\cos\theta_i$. Note that -1 is a regular value since $df=[\sin\theta_i]$ only vanishes when $\theta_i\in\pi\mathbb{Z}$, so that $\cos\theta_i=\pm1$, so that $\operatorname{Re}\sum z_i=0 \pmod 2$.*

The space $f<0$ is indeed a smooth 3-dimensional manifold embedded in T^4, and we use f as a Morse function on it. The minimum value is at -4, where there is a single critical point of index 0. Then there is another critical value at $f=-2$, with the following critical points $(-1,-1,-1,1)$,$(-1,-1,1,-1)$, $(-1,1,-1,-1)$ and $(1,-1,-1,-1)$. These are all of index 1, since "the function can only be decreased in one direction". There are no other critical values in $(-\infty,1]$. Hence we see that the 3-manifold may be constructed by 4 1-handle attachments on a 3-ball. This means the boundary must be a genus 4 orientable (it is boundary of an orientable 3-manifold) Riemann surface.

1.8 Morse inequalities

In this section we compare the Morse polynomial of a Morse function f on the manifold M

$$\mathcal{M}_t(f)=\sum_{p\in\operatorname{Crit}(f)}t^{\lambda_p}=\sum_{\lambda}\mu_f(\lambda)t^{\lambda},$$

with the Poincaré polynomial $P_t(M)=\sum b_i t^i$, where $b_i=\dim_k H_i(M,k)$ are the Betti numbers of M with respect to some coefficient field k.

To analyze the difference between these polynomials, let us do so "inductively" on the sublevel sets $M^a=f^{-1}(-\infty,a]$. Let the Morse polynomial for M^a be $\mathcal{M}_t(f)^a$.

If there are no critical points in $[a,b]$, then clearly $\mathcal{M}_t(f)^a=\mathcal{M}_t(f)^b$ by definition and $P_t(M^a)=P_t(M^b)$, by theorem A.

If there is a single critical point of index λ in $[a,b]$, then by definition

$$\mathcal{M}_t(f)^b-\mathcal{M}_t(f)^a=t^{\lambda}.$$

What happens to the Poincaré polynomial? We use Theorem B, which states that that M^b is a λ-handle attached to M^a.

Consider the attaching sphere $S^{\lambda-1}\subset M^a$. This cycle either bounds a chain in M^a or not. This is a global criterion in M^a.

Completable case: if the attaching sphere bounds a chain in M^a: then this chain, together with the new λ-handle, forms a new nontrivial cycle of dimension λ. Then $\Delta P_t=t^{\lambda}$, and therefore

$$\Delta(\mathcal{M}_t-P_t)=0.$$

If the attaching sphere is a nontrivial cycle in M^a: then the new λ-handle kills this cycle and we get $\Delta P_t=t^{\lambda-1}$, and so

$$\Delta(\mathcal{M}_t-P_t)=t^{\lambda}+t^{\lambda-1}=t^{\lambda-1}(1+t).$$

By induction, therefore, we have

Theorem 1.21 (Morse inequalities)**.**

$$\mathcal{M}_t(f) - P_t(M) = (1+t)Q_t(f),$$

where $Q_t(f)$ is a polynomial with nonnegative integer coefficients. In particular we have

- $\mu_f(\lambda) \geq b_\lambda$ *(Weak Morse inequalities),*
- $\chi(M) = \sum(-1)^i\mu(i)$*,*
- *for each $k \geq 0$,*
$$b_k - b_{k-1} + \cdots + (-1)^k b_0 \leq \mu_k - \mu_{k-1} + \cdots + (-1)^k\mu_0.$$

Proof. The weak Morse inequalities are obtained by simply truncating the equation $\mathcal{M}_t - P_t = (1+t)Q_t$. To give a rigorous proof of the main statement, we will use the exact sequence in relative homology for the inclusion $M^a \subset M^b$. First we determine $H_k(M^b, M^a, k)$ using excision:

Excision says that if $Z \subset A \subset X$ with $\overline{Z} \subset A^{int}$ then the inclusion $(X - Z, A - Z) \subset (X, A)$ induces an isomorphism on homology. For (M^b, M^a), we can take Z to be the complement in M^a of a small tubular neighbourhood of the attaching sphere. Then $H_n(M^b, M^a; k) = H_n(D^\lambda, S^{\lambda-1}; k) = \tilde{H}_n(S^\lambda; k)$, which is k for $n = \lambda$ and zero otherwise. By the long exact sequence in relative homology therefore we have

$$0 \longrightarrow H_\lambda(M^a) \longrightarrow H_\lambda(M^b) \longrightarrow \mathbb{K} \xrightarrow{\delta} H_{\lambda-1}(M^a) \longrightarrow H_{\lambda-1}(M^b) \longrightarrow 0$$

Whether δ is 0 or rank 1 gives the two alternatives: if $\delta = 0$ then $\Delta P = t^\lambda$ and we are in the completable case. □

Corollary 1.22. $\chi(M) = 0$ *for M an odd-dimensional compact manifold.*

Proof. If f is Morse, so that $\chi(M) = \mathcal{M}_{-1}(f)$, then $-f$ is Morse also, so $\chi(M) = \mathcal{M}_{-1}(-f) = -\mathcal{M}_{-1}(f) = -\chi(M)$. □

Definition 10. f is a perfect Morse function for the coefficient field k if $\mathcal{M}_t(f) = P_t(M, k)$.

Example 1.23. *Note that $\mathbb{R}P^n$ has $H_k(\mathbb{R}P^n, \mathbb{R})$ vanishes except for $k = 0$ and, when n odd, $k = n$. Applying the Morse inequalities we get that there is at least one critical point. However, over $\mathbb{K} = \mathbb{Z}_2$, we have $H_k(\mathbb{R}P^n) = \mathbb{Z}_2$ for all $0 \leq k \leq n$. Hence the Morse inequalities yield at least $n + 1$ critical points. This bound is achieved by the function (a generalization of the above ellipsoid)*

$$f = \sum_{i=1}^{n+1} i|x_i|^2$$

Corollary 1.24. *Let M be a compact manifold. If the gap condition $|\lambda(p) - \lambda(q)| \neq 1$ holds for all $p, q \in \mathrm{Crit}(f)$, then f is a perfect Morse function (for any field).*

Proof. Under the gap assumption we wish to show that the connecting homomorphism $\delta : H_\lambda(M^b, M^a) \longrightarrow H_{\lambda-1}(M^a)$ is always zero. Assuming that $H_\lambda(M^b, M^a)$ is not itself zero, this means by the gap that $\lambda - 1$ is not a morse index for the manifold. Assuming inductively that this means $H_{\lambda-1}(M^a) = 0$, this implies that δ must vanish. But then we obtain the exact sequence

$$0 \longrightarrow H_k(M^a) \longrightarrow H_k(M^b) \longrightarrow H_k(M^b, M^a) \longrightarrow 0$$

so that if k is not a morse index, then $H_k(M^a) = 0$ implies $H_k(M^b) = 0$ as well, establishing the induction. □

Example 1.25. *The height function on the sphere has Morse polynomial $1 + t^n$, which satisfies the gap condition for $n > 1$, and so gives the Betti numbers of S^n. For $|m - n| \geq 2$, the gap condition is satisfied for the sum of height functions on $S^m \times S^n$, so that $1 + t^m + t^n + t^{m+n}$ gives the Betti numbers for $S^m \times S^n$. Finally, the Morse function defined earlier on $\mathbb{C}P^n$ had all even indices. Hence it satisfies the gap condition and is perfect.*