1.9 The Grassmannian

The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ is the set of complex k-dimensional linear subspaces of $\mathbb{C}^n$. It is a compact complex manifold of dimension $k(n-k)$ and it is a homogeneous space of the unitary group, given by $U(n)/(U(k)\times U(n-k))$. The Grassmannian is a particularly good example of many aspects of Morse theory (this is true of the unitary group itself as well as its other homogeneous spaces such as the flag varieties) and we will investigate it a few times.

Complex manifold structure: The usual way to exhibit the manifold structure is as follows. We represent a k-plane V as an $n\times k$ matrix where the columns span the subspace. Two matrices A, A' represent the same element of the Grassmannian iff $\exists h\in GL(k,\mathbb{C})\ :\ A'=Ah$. Standard coordinate charts are as follows: Choose $I=\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}$ and let V_{I° be the $n-k$-plane spanned by $\{e_j\ :\ j\notin I\}$. Then

$$U_I=\{\Lambda\in \mathrm{Gr}_k(\mathbb{C}^n)\ :\ \Lambda\cap V_{I^\circ}=\{0\}\}.$$

The coordinates of a point $\Lambda\in U_I$ consist of the unique matrix representing Λ such that the I^{th} $k\times k$ minor is the identity matrix. The remaining $k(n-k)$ entries are free. This gives a chart $\varphi_I: U_I\longrightarrow \mathbb{C}^{k(n-k)}$. For two multiindices I,J we can see quite easily that $\varphi_J\circ\varphi_I^{-1}$ is smooth (in fact holomorphic) as follows: If $p\in U_I\cap U_J$ then $\varphi_I(p)$ has nonsingular J^{th} $k\times k$-minor Λ_{IJ}. This means that the transition function $\varphi_J\circ\varphi_I^{-1}$ is given by $A\mapsto A\,\Lambda_{IJ}^{-1}$, which is smooth (holomorphic).

It's an adjoint orbit: $\mathbb{C}^n$ has a natural Hermitian inner product $\langle x,y\rangle = x\cdot\overline{y}=\sum x_i\overline{y}_i$, and any k-plane $V\subset\mathbb{C}^n$ has an orthogonal complement $V^\perp$. The decomposition $\mathbb{C}^n=V\oplus V^\perp$ may be represented as a projection operator $P_V:\mathbb{C}^n\longrightarrow\mathbb{C}^n$: If $x=x_V+x_{V^\perp}$ then $P_Vx=x_V$. The operator P_V is in fact characterized by its being a rank k self-adjoint ($P^*=P$) operator satisfying $P_V^2=P_V$. It is convenient to multiply it by i, to get a skew-adjoint operator iP_V, which is then a skew-adjoint matrix, i.e. an element of the real Lie algebra $\mathfrak{u}(n)$ of the unitary group.

$$\begin{aligned}\mathrm{Gr}_k(\mathbb{C}^n)\hookrightarrow \mathfrak{u}(n)&=\{A\in\mathrm{End}(\mathbb{C}^n)\ :\ A^*+A=0\}\\ V&\mapsto iP_V\end{aligned}$$

From this embedding, we can see the adjoint action of $U(n)$ on $\mathfrak{u}(n)$, i.e. $g\cdot A=gAg^{-1}$, acts transitively on $\mathrm{Gr}_k(\mathbb{C}^n)$ with stabilizer $U(k)\times U(n-k)$. In other words, we see that **the Grassmannian may be viewed as an adjoint orbit for** $U(n)$; in fact it is the orbit of the element

$$\begin{pmatrix} i_{11} & & & & & \\ & \ddots & & & & \\ & & i_{kk} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0_{nn}\end{pmatrix}$$

To understand the Lie algebra it is useful to consider a "Cartan subalgebra", which is a choice of a maximal abelian subalgebra. For $\mathfrak{u}(n)$ this is particularly easy as we can take the diagonal matrices $\mathfrak{h}=\mathrm{diag}(ia_1,\ldots,ia_n)$, which all commute. In general, every adjoint orbit intersects $\mathfrak{h}$ in a nonempty finite set – this is familiar to us since any skew-adjoint matrix is diagonalizable and so it is conjugate to an element in $\mathfrak{h}$ – in fact possibly to many elements in $\mathfrak{h}$, since the ordering of the eigenvalues can be changed, yielding different points in $\mathfrak{h}$. The intersection of the orbit with $\mathfrak{h}$ is always an orbit of the Weyl group $W=N(T)/T$, which in the case of $U(n)$ is just S_n, permuting the n coordinate axes.

There are different orbit types, depending on whether some eigenvalues are repeated. Matrices with distinct eigenvalues are called *regular* elements, the others are *singular*. The set of singular elements in $\mathfrak{h}$ is always a

union of hyperplanes, in our case given by $a_i = a_j$. The diagram[12] of $\mathfrak{h}$, together with these singular hyperplanes, is called the *diagram of the group*, and it is a picture which is invariant under the Weyl group. The particular adjoint orbit we are interested in, $\mathrm{Gr}_k(\mathbb{C}^n)$, is the "deepest" nontrivial orbit type, and it intersects $\mathfrak{h}$ in $\binom{n}{k}$ points, i.e. all diagonal matrices with k entries equal to i and $n-k$ entries equal to zero.

Morse function: The real lie algebra $\mathfrak{u}(n)$ has a Euclidean inner product [13] $\langle A, B\rangle = \mathrm{Tr}(AB^*) = -\mathrm{Tr}(AB)$, which is clearly invariant under conjugation. The function we choose is a linear function on $\mathfrak{u}(n)$ given by inner product with a regular element in $\mathfrak{h}$. Let $a_1 > \cdots > a_n \geq 0$ be real numbers and let $Z = \mathrm{diag}(ia_1, \ldots, ia_n)$ so that $X \in \mathfrak{h} \subset \mathfrak{u}(n)$ is regular. Then define

$$\boxed{f(V) = \langle iP_V, Z\rangle.}$$

Example 1.26. *For $k = 1$, $\mathrm{Gr}_k(\mathbb{C}^n) = \mathbb{C}P^{n-1}$ and a point V can be represented by its homogeneous coordinates $V = [z] \in \mathbb{C}P^{n-1}$ Then the projection operator is $P_V = \frac{1}{|z|^2} zz^*$, and*

$$f(V) = -\mathrm{Tr}(iP_V Z) = \tfrac{1}{|z|^2} \sum_{i=1}^{n} a_i |z_i|^2,$$

which is the Morse function we used earlier.

Critical points: To find the critical points, it is simpler to find the zeros of the gradient vector field, since this is just the projection of Z to the tangent space of our orbit. For this it is instructive to think about projecting a vector in $\mathbb{R}^3 \cong \mathfrak{so}(3)$ to the adjoint orbit given by the unit 2-sphere – recall that the Lie bracket is just the vector cross product.

Lemma 1.27. *Let $\mathcal{O}_{X_0} = \mathrm{Ad}_{U(n)}(X_0)$ be the adjoint orbit through $X_0 = \mathrm{diag}(i, \ldots, i, 0 \ldots, 0)$. Then we have that:*

- $T_X\mathcal{O}_{X_0} = \mathrm{ad}_{\mathfrak{u}(n)}(X) = \{[Y, X] \ : \ Y \in \mathfrak{u}(n)\}$.
- $J : Y \mapsto [Y, X]$ *defines a complex structure on $T_X\mathcal{O}_{X_0}$, i.e. $J^2 = -1$.*
- *The orthogonal projection of $A \in \mathfrak{u}(n)$ to $T_X\mathcal{O}_{X_0}$ is given by $A \mapsto -[[A, X], X] = -J[A, X]$.*

Theorem 1.28. *The critical points of f on $\mathcal{O}_{X_0}$ are exactly those points in the intersection with $\mathfrak{h}$, namely they are the $\binom{n}{k}$ subspaces of $\mathbb{C}^n$ spanned by k axes.*

Proof. The points $X \in \mathcal{O}_{X_0}$ on which the orthogonal projection of Z vanishes are precisely those for which $-J[Z, X] = 0$, i.e. $[Z, X] = 0$. Since Z is diagonal with distinct eigenvalues, this forces X to be diagonal, and hence it lies in $\mathfrak{h}$. □

Focal point analysis: We now determine the Morse indices of the critical points, using the most important method for doing this. First we recognize f as the Euclidean distance squared function, then we use a general method for determining Morse indices of the norm-squared function.

By the polarization identity, and the fact that $||X||^2$ is constant on an orbit $\mathcal{O}_{X_0}$, we have

$$f(X) = \langle X, Z\rangle = -\tfrac{1}{2}(||X - Z||^2 + C) \quad X \in \mathcal{O}_{X_0}$$

for some constant C. So we see that f has the same critical points as the distance-squared function $g(X) = ||X - Z||^2$, and if $p \in \mathrm{Crit}(f) = \mathrm{Crit}(g)$, then $\lambda_g(p) = 2k(n-k) - \lambda_f(p)$, because of the sign change.

Now we need some general facts about Morse theory of the distance-squared function on manifolds $M \subset \mathbb{R}^N$.

[12] Note that this is not the diagram where you draw the weights of the adjoint representation: the weights are elements in the dual of the complexified Cartan subalgebra, here we are dealing with a real Cartan subalgebra, and so we see the real loci of the kernel of the roots.

[13] Note that $\mathfrak{u}(n)$ is not semisimple and this inner product is *not* the Killing form, which for $\mathfrak{u}(n)$ is given by $\kappa(A, B) = \mathrm{Tr}(\mathrm{ad}_A \mathrm{ad}_B) = -2n\mathrm{Tr}(AB) + 2\mathrm{Tr}(A)\mathrm{Tr}(B)$ – it is degenerate. The semisimple subalgebra $\mathfrak{su}(n)$, however, has $\kappa(A, B) = -2n\mathrm{Tr}(AB)$.

Lemma 1.29. *Let $Z \in \mathbb{R}^N \backslash M$ and $g(X) = ||Z - X||^2$. Then $p \in M$ is a critical point of g if and only if Zp is normal to M.*

Remark 9. *This gives an alternative proof that all critical points lie in $\mathfrak{h}$: if X is critical then the line $X + t(Z - X)$ is perpendicular to the orbit through X, i.e. $[Z - X, X] = 0$, i.e. $[Z, X] = 0$. But then $[Z - X, Z] = 0$ as well, i.e. the same line is perpendicular to the orbit through Z. But Z is regular, so this means the line must be in $\mathfrak{h}$.*

Such a critical point may be degenerate; this is measured by the extent to which nearby segments $(p + \delta p)Z$ are normal to M:

Definition 11. Let $E : NM \longrightarrow \mathbb{R}^N$ be the natural map $(p, v) \mapsto p + v$, which is a diffeomorphism in a neighbourhood of the zero section. Let $(p, v) \in NM$ such that $E(p, v) = e \in \mathbb{R}^N$. Then e is called a *focus* of M when $E_*|_{(p,v)}$ is degenerate. The focus e is said to have multiplicity $\mu = \dim \ker E_*|_{(p,v)}$.

Lemma 1.30. *$E(p, v) = e$ is a focus if and only if p is a degenerate critical point of $p \mapsto ||p - e||^2$, and the kernel of the Hessian at p has dimension given by the multiplicity of the focus.*

Lemma 1.31. *If $Z \in \mathfrak{h}_{reg}$, then the critical points of f are all nondegenerate, and f is Morse; If Z is not regular, then f has degenerate critical points (and is Morse-Bott). If Z lies on a single hyperplane in the singular set, then the critical points of f are degenerate, with multiplicity 2.*

Proof. The tangent space to $p \in \mathcal{O}_{X_0}$ is spanned by $[Y, p]$, and similarly the tangent space to $(p, v) \in N\mathcal{O}_{X_0}$ is spanned by $\{H_Y = ([Y, p], [Y, v]), W = (0, W) \ : \ [W, p] = 0\}$, where $([Y, p], [Y, v])$ are the "horizontal vectors" and $(0, W)$ are the "vertical vectors".

The tangent mapping of E sends

$$E_* : \begin{cases} H_Y \mapsto [Y, p] + [Y, v] = [Y, p + v] \\ W \mapsto W \end{cases}$$

Therefore we see that the kernel of E_* is spanned by H_Y such that $[Y, Z = p + v] = 0$. If Z is regular, then $[Y, Z] = 0$ implies $[Y, p] = 0$ as well, i.e. E_* has no kernel. If Z lies on exactly one singular hyperplane not containing p, then there is a real 2-dimensional space of matrices in $T_p\mathcal{O}$ which commute with Z. The key calculation is as follows: Take

$$Y = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, p = \begin{pmatrix} i1_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}$$

so that $[Y, p]$ spans $T_p\mathcal{O}$. Then there is a 2-d space of matrices of the form Y which commutes with Z when Z has a repeated eigenvalue not lying in the top left $k \times k$ block.

□

Lemma 1.32. *For $Z \in \mathfrak{h}_{reg}$, the index of a critical point $p \in \mathcal{O}_{X_0}$ is the number, counted with multiplicity, of foci along the line segment from p to Z. That is, the index is twice the humber of singular hyperplanes encountered along the path from p to Z.*

Remark 10. *Why do we count only those foci between Z and p, rather than* all *foci in the half ray $\{t(Z - p) \ : \ t > 0\}$? Think about the ellipsoid; foci which are "beneath" Z should not be counted because they represent directions where the curvature is less than the sphere of radius $||Z - p||$, and hence represent* ascending*, not descending directions.*

Corollary 1.33. *Consider the adjoint orbits of $U(3)$ in $\mathfrak{u}(3) \cong \mathbb{R}^9$: we can draw the diagram of the group in 3-space or just take the $\mathfrak{su}(3)$ slice; show that Poincaré polynomials for the orbits are*

$$1 + t^2 + t^4 \quad \text{for } \mathbb{C}P^2,$$

and

$$1 + 2t^2 + 2t^4 + t^6 \quad \text{for } FL(1, 2).$$

The same method determines the homology of $\mathrm{Gr}_2\mathbb{C}^4$*: There are 6 critical points given by all diagonal matrices with two is and two* 0*s; Starting from* $(i,i,0,0)$ *and going down to* $(0,0,i,i)$ *using consecutive transpositions i.e.* $(m,m+1)$ *in* S_4 *(this represents traveling from one connected component of* $\mathfrak{h}_{reg}$ *to another crossing a hyperplane), we see that after one transposition we get to* $(i,0,i,0)$ *(therefore it has index 2); another transposition gives us* $(i,0,0,i)$ *and* $(0,i,i,0)$ *(both of index 4); another transposition gives only* $(0,i,0,i)$ *(index 6) and finally one last transposition for* $(0,0,i,i)$*. Hence*

$$P_t(\mathrm{Gr}_2(\mathbb{C}^4)) = 1 + t^2 + 2t^4 + t^6 + t^8.$$

Morse-Smale dynamics: We will now employ a (not very general?) trick to figure out the morse indices of f by explicitly determining the global structure of the negative gradient flow. We will explain how to do it without this trick later.

Lemma 1.34. *The gradient flow of the Morse function, i.e. the equation* $\frac{d}{dt}\gamma(t) = -\mathrm{grad}(f)(\gamma(t))$ *has solution curve through* V *given by*

$$\gamma(t) = e^{-itZ}V = \mathrm{diag}(e^{-a_1 t}, \ldots, e^{-a_n t})V.$$

Definition 12 (Morse-Smale dynamics)**.** We will show that the $\pm\infty$ limits of the gradient flow of a Morse function on a compact manifold are critical points. The *stable manifold* $W^+(p)$ *of* $p \in \mathrm{Crit}(f)$ consists of all points flowing to p as $t \to \infty$. The *unstable manifold* $W^-(p)$ *of* $p \in \mathrm{Crit}(f)$ consists of all points limiting to p as $t \to -\infty$. We will show that these are manifolds diffeomorphic to $\mathbb{R}^{n-\lambda(p)}$, $\mathbb{R}^{\lambda(p)}$, respectively.

In the case of the height function on the Grassmannian, we can determine the (un)stable manifolds globally, instead of doing a local analysis near the critical points.

Observe for $\mathrm{Gr}_1(\mathbb{C}^n)$ that

$$\lim_{t\to\infty} \mathrm{diag}(e^{-a_1 t}, \ldots, e^{-a_n t}) \begin{bmatrix} z_1 \\ \vdots \\ z_{k-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \sim \lim_{t\to\infty} \begin{bmatrix} e^{-a_1-a_k} z_1 \\ \vdots \\ e^{-a_{k-1}-a_k} z_{k-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V_k$$

This shows that the stable manifold for the $k^t h$ critical point is a copy of $\mathbb{C}^{k-1}$, and the associated unstable manifold is a $\mathbb{C}^{n-k}$. Hence the index of V_k is $2(n-k)$, as we calculated before. This yields a perfect Morse function on $\mathbb{C}P^{n-1}$.

For general Grassmannian, we have a similar observation. Any $n \times k$ matrix of rank k can be brought into echelon form and we have similarly

$$\lim_{t\to\infty} e^{-tD} \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & & \vdots \\ 1 & & \vdots \\ & \ddots & * \\ & & 1 \\ & & \end{bmatrix} \sim \begin{bmatrix} & \cdots & \\ \vdots & & \vdots \\ & & \vdots \\ 1 & & \vdots \\ & \ddots & \\ & & 1 \\ & & \end{bmatrix}$$

The echelon form is not a good coordinate on the stable manifold since we can reduce each row with a leading 1 to zeros. Therefore after this reduction we get

$$\dim_{\mathbb{C}} W^+(V_I) = (i_1 - 1) + (i_2 - 2) + \cdots + (i_k - k) = \sum_{j=1}^{k} i_j - \tfrac{1}{2}k(k+1)$$

This tells us the index of V_I, i.e. $\lambda(V_I) = 2(k(n-k) - \sum_{j=1}^{k} i_j - \frac{1}{2}k(k+1))$. A direct result of this calculation is that the Morse polynomial equals the Poincaré polynomial and is given by

$$\frac{\Pi_{i=1}^{n}(1-t^{2i})}{\Pi_{i=1}^{k}(1-t^{2i})\Pi_{i=1}^{n-k}(1-t^{2i})}$$

Furthermore because the Morse function has only even indices we can conclude that the integral homology has no torsion.

Remark 11. *The stable or "ascending manifolds" we described above are called "Schubert cells" and their closure in* $\mathrm{Gr}_k(\mathbb{C}^n)$ *are singular algebraic varieties called "Schubert varieties".*

2 Bott-Morse theory

It often happens that a manifold of interest comes equipped with a function which is not Morse. In many cases, for example if we want to maintain a symmetry, we don't want to perturb the function. Bott developed a version of Morse theory to deal with such situations.

Definition 13. A compact, connected submanifold $S \subset M$ is called a *nondegenerate critical submanifold* of $f \in C^\infty(M, \mathbb{R})$ if $S \subset \mathrm{Crit}(f)$ and $\ker \mathrm{Hess}_p(f) = T_pS$ for each point $p \in S$. In other words, f is nondegenerate in the normal directions to S. If $\mathrm{Crit}(f)$ consists of nondegenerate critical submanifolds, then f is called a *Morse-Bott function*.

Consider the exact sequence describing the tangent bundle of S:

$$0 \longrightarrow TS \longrightarrow TM|_S \longrightarrow NS \longrightarrow 0\ . \tag{5}$$

The Hessian of f is in $\mathrm{Sym}^2 T_p^* M$ and vanishes upon restriction to T_pS. Hence it lies in $\mathrm{Sym}^2 N_p^* S$ and hence defines a nondegenerate bilinear form on the normal bundle of S

$$Q_f = d_S^2 f \in \Gamma(S, \mathrm{Sym}^2 N^* S).$$

We may therefore view S as a family of usual Morse critical points.

The fact that NS has a nondegenerate bilinear form allows us to associate an invariant to S analogous to the Morse index λ. As before, we may assign to S an integer $\lambda(f, S)$ counting the number of negative 'eigenvalues' of Q_f. But there is the additional fact that NS may be decomposed[14] as a direct sum of subbundles $NS = \nu_- \oplus \nu_+$ where Q_f is $\pm$-definite on $\nu_\pm$ and $Q_f(\nu_-, \nu_+) = 0$. In particular, the rank of ν_- is $\lambda(f, S)$.

Using exactly the same proof (using the Moser-Palais trick) as for the original Morse lemma, we obtain the following Morse-Bott lemma, which says that f is diffeomorphic to its quadratic approximation in a tubular neighbourhood of S:

Theorem 2.1. *Let S be a nondegenerate critical submanifold of the Morse-Bott function f. Then there is a neighbourhood U of the zero section $S \subset NS$ and an open embedding $\Phi : U \longrightarrow M$ such that $\Phi|_S = \mathrm{Id}_S$ and*

$$\Phi^* f = f(S) + Q_f.$$

In particular, if we choose $NS = \nu_- \oplus \nu_+$ then $\Phi^* f = f(S) - u_- + u_+$, where $u_\pm = \pm Q_f|_{\nu_\pm}$.

There is now the natural question of how to describe the change in topology of a sublevel set $M^{c-\epsilon}$ when passing a critical level corresponding to S. The answer is that we must do a family of $\lambda(f, S)$-handle attachments parametrized by S. In other words we attach a thickening of the disk bundle $D\nu_- \longrightarrow S$ to $M^{c-\epsilon}$, along a framed embedding of the total space of the boundary sphere bundle $S\nu_-$ into $\partial M^{c-\epsilon}$.

To be more precise: a Morse-Bott handle is modeled[15] on $D\nu_- \oplus D\nu_+$, a bundle over S with fiber isomorphic to $D^\lambda \times D^{n-\lambda}$. We attach this handle to $M^{c-\epsilon}$ by giving an embedding

$$\iota : S\nu_- \longrightarrow \partial M^{c-\epsilon},$$

giving a *bundle* of attaching spheres. We also need a framing, which in this case is an identification of the normal bundle of the attaching bundle $S\nu_-$ in $\partial M^{c-\epsilon}$ with its normal bundle in the standard model $\nu_- \oplus \nu_+$, that is, with $\pi^*\nu_+$, where $\pi : S\nu_- \longrightarrow S$ is the bundle projection.

$$\text{Framing data: } N(\iota(S\nu_-)) \xrightarrow{\cong} \pi^*\nu_+.$$

[14] One way of doing this is to choose a positive-definite metric on NS; then Q_f may be viewed as a symmetric automorphism $Q_f : NS \longrightarrow NS$. Then $\nu_\pm$ are the $\pm$-eigenbundles of this operator.

[15] Here we want to "smooth the corners" as we did for the usual handle attachment.

Example 2.2. *If $S = S^1$, show a handle attachment on a 3-manifold with boundary, in which there is a framed embedded S^0-bundle over S^1. There are two interesting cases: when ν_- is trivial and when it is nontrivial. Explain the example of a Morse-Bott handle attachment yielding $\mathbb{R}P^2$ as the union of a disc and a Möbius disc bundle.*

The change in topology is described as in the old case; we have the isomorphism

$$H_\bullet(M^{c+\epsilon}, M^{c-\epsilon}; \mathbb{K}) = H_\bullet(D\nu_-, \partial D\nu_-; \mathbb{K})$$

We are used to the idea that $H_n(D^k, \partial D^k; \mathbb{K})$ is $\mathbb{K}$ for $n = k$, since $D^k/\partial D^k$ is homeomorphic to the k-dimensional sphere. This is not always true for bundles of spheres when they are nontrivial. We say that a rank r bundle $E \longrightarrow S$ is $\mathbb{K}$-orientable when there is a class

$$\tau \in H^r(DE, \partial DE; \mathbb{K})$$

called the *Thom class*, which restricts to each fiber to give a generator of $H^r(DE_x, \partial DE_x; \mathbb{K})$. Note that every vector bundle is $\mathbb{Z}_2$-orientable but not $\mathbb{R}$-orientable.

If our bundle $\nu_- \longrightarrow S$ is $\mathbb{K}$-orientable, then we have a Thom class $\tau \in H^\lambda(D\nu_-, \partial D\nu_-; \mathbb{K})$, but there will be a lot more homology in the sphere bundle, coming from the topology of S. Indeed, we have the Thom isomorphism theorem:

Theorem 2.3 (Thom isomorphism theorem)**.** *Let $\pi : E \longrightarrow S$ be $\mathbb{K}$-orientable, and τ_E be a Thom class for E. Then for all k the map*

$$H_{k+r}(DE, \partial D(E); \mathbb{K}) \longrightarrow H_k(S, \mathbb{K}) \tag{6}$$
$$c \mapsto \pi_*(c \cap \tau_E) \tag{7}$$

is an isomorphism.

By this result we see that if ν_- is $\mathbb{K}$-orientable, then the relative cohomology group is given by $H_\bullet(S, \mathbb{K})$, but with a degree shift by the Morse index. In other words, the polynomial describing the potential contribution to the Betti numbers is

$$t^{\lambda(f,S)} P_t(S, \mathbb{K}).$$

Definition 14. The $\mathbb{K}$-Morse-Bott polynomial of a Morse-Bott function $f : M \longrightarrow \mathbb{R}$ on a compact manifold M is

$$M_t(f; \mathbb{K}) = \sum_S t^{\lambda(f,S)} P_t(S; \mathbb{K})$$

As a result of the above reasoning, we have the following Morse-Bott inequalities:

Theorem 2.4 (Morse-Bott inequalities)**.**

$$M_t(f; \mathbb{K}) - P_t(M, \mathbb{K}) = (1 + t)Q_t(f),$$

where $Q_t(f)$ has positive coefficients.

Remark 12. *Just as in the old case, the discrepancy between M_t and P_t occurs when the embedding of the attaching sphere (now a sphere bundle) induces a nonzero map on homology. If these maps are all zero maps, then we are in the $\mathbb{K}$-completable case and (assuming all critical submanifolds have ν_- $\mathbb{K}$-orientable) we get equality of the Morse and Betti polynomials over $\mathbb{K}$.*

Remark 13. *The lacunary principle of Morse also has an extension to the Morse-Bott case: here we need not only the $\lambda(f, S)$ to be even for all S, but also that S has only even Betti numbers being nonzero.*

Example 2.5. *Recall that on $\mathbb{C}P^n$ we took the Morse function*

$$f([z_0,\dots,z_n]) = \frac{\sum c_i|z_i|^2}{\sum |z_i|^2},$$

for $c_0 < \cdots < c_n$. What if we take an extreme case $c_0 = 0$ and $c_1 = \cdots = c_n = 1$. Then this is a function with an isolated global minimum (index 0) at $[1,0,\dots,0]$ and a critical submanifold of maxima $S = \{[0,*,\dots,*]\} \cong \mathbb{C}P^{n-1}$. S has index $2 = \dim \mathbb{C}P^n - \dim \mathbb{C}P^{n-1}$. Therefore we have

$$M_t(f) = 1 + t^2(1 + t^2 + \cdots + t^{2n-2}).$$

Example 2.6. *If a torus lies down flat, the height function z has two nondegenerate critical submanifolds diffeomorphic to S^1. The indices are 0,1 and we get immediately*

$$M_t(z) = (1+t) + t(1+t) = 1 + 2t + t^2,$$

showing that the height function is a perfect Morse function in this case.

Remark 14. *Another nice feature of Morse-Bott functions is that they behave well under submersions: if $\pi : M \longrightarrow N$ is a submersion, and $f \in C^\infty(N,\mathbb{R})$ is Morse-Bott, then $\pi^* f$ is also Morse-Bott, with preserved indices in the sense $\lambda(f,S) = \lambda(\pi^* f, \pi^{-1}(S))$. A special case of this is the pullback of a function by the map $M \longrightarrow \{pt\}$. Even the zero function is Morse-Bott, and it is a perfect Morse-Bott function!*

Example 2.7. *just as for the projective space, we may relax the usual requirement $a_1 > a_2 > \cdots > a_n > 0$ in the usual Morse function on $\mathrm{Gr}_k\mathbb{C}^n$, if we allow Morse-Bott functions. If we allow $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then $f(X) = \langle X, i\mathrm{diag}(a_1,\dots,a_n)\rangle$ is a perfect Morse-Bott function. One can show[16] that $V \in \mathrm{Gr}_k\mathbb{C}^n$ is a critical point if and only if V is spanned by eigenvectors of $\mathrm{diag}(a_1,\dots,a_n)$. If we write $\mathbb{C}^n = \oplus_k E_k$ as the eigenspace decomposition, then the critical submanifold containing V is then given by a product of Grassmannians*

$$\Pi_k \mathrm{Gr}_{a_k} E_k,$$

where $a_k = \dim_{\mathbb{C}}(V \cap E_k)$. In the crudest case, we may choose $\mathrm{diag}(1,0,\dots,0)$. Write $\mathbb{C}^n = \mathbb{C}e_1 \oplus \mathbb{C}^{n-1}$. then there are two critical submanifolds, according as whether $V \cap \mathbb{C}e_1 \neq \{0\}$ (the max), in which case $S_{max} \cong \mathrm{Gr}_{k-1}\mathbb{C}^{n-1}$, or $V \subset \mathbb{C}^n$ (the min), in which case $S_{min} \cong \mathrm{Gr}_k\mathbb{C}^{n-1}$. The index of the max is just the total dimension minus the dimension of S_{max}, i.e. $2(k(n-k) - (k-1)(n-k)) = 2(n-k)$, and so we get

$$P_{k,n}(t) = P_{k,n-1}(t) + t^{2(n-k)} P_{k-1,n-1}(t),$$

a recurrence which is solved by

$$P_{k,n} = \frac{\Pi_{i=1}^n (1-t^{2i})}{\Pi_{i=1}^k (1-t^{2i}) \Pi_{i=1}^{n-k}(1-t^{2i})}$$

[16] See Martin Guest's notes, for example