2.1 Morse theory of moment maps

Symplectic geometry provides[17] a huge source of Morse and Morse-Bott functions, and it is not unusual to find papers where properties of constructed spaces (such as moduli spaces) are determined by showing the space is symplectic, has a Hamiltonian S^1-action, and has a nice Morse-Bott function adapted to the symplectic form. Often one is interested in the cohomology ring, but there is a refinement of this technique which may be used to find the equivariant cohomology, which we will not discuss here[18].

A symplectic form on a manifold M is a 2-form $\omega \in \Omega^2(M, \mathbb{R})$ which is closed (i.e. $d\omega = 0$) and nondegenerate, in the sense that the map it determines $\omega : TM \longrightarrow T^*M$ via

$$X \mapsto i_X \omega$$

is an isomorphism. The most typical example is the cotangent bundle T^*N of any manifold N, equipped with the natural symplectic form $\omega = \sum_i dp_i \wedge dx^i$, where x^i is a coordinate chart and p_i are the induced fiber coordinates on T^*N. Other typical examples are orientable surfaces equipped with volume forms, products, covering spaces, and smooth complex projective varieties, such as $\mathbb{C}P^n$, which has a symplectic form known as the Fubini-Study form, given in homogeneous coordinates $[z_0, \ldots, z_n]$ by

$$\omega = i\partial\bar{\partial} \log |z|^2, \quad |z|^2 = \sum_0^n |z_k|^2.$$

A symplectic form provides a convenient setting for dynamics, since we only need to specify a function $f \in C^\infty(M, \mathbb{R})$ (the "Hamiltonian") and this determines a vector field (the "Hamiltonian vector field")

$$X_f = \omega^{-1}(df),$$

and this vector field is automatically a symmetry of ω:

$$L_{X_f}\omega = di_X\omega + i_X d\omega = d(df) = 0,$$

and it automatically "conserves" the Hamiltonian:

$$X_f(f) = i_{X_f} df = i_{X_f} i_{X_f} \omega = 0.$$

This is important for physics: If we only know the "energy" function on T^*N, then this function defines a flow on T^*N which conserves energy, determining a trajectory on N for a particle $x \in N$ with given initial lift $(x, p) \in T^*N$, i.e. initial "position and momentum".

A particularly convenient set-up which yields many Morse functions is when there is a Lie group G of symmetries of a symplectic manifold (M, ω), which is such that all the symmetries are generated by functions in the above way. In other words, there is a correspondence between symmetries and "conserved quantities", i.e. Hamiltonian functions.

Definition 15. Let a Lie group G act on the symplectic manifold (M, ω) by symplectomorphisms, so that it defines an infinitesimal action $\mathfrak{g} \longrightarrow \Gamma(M, TM)$ (a Lie algebra homomorphism) written as $a \mapsto X_a$. A *moment map* for this action is an equivariant[19] map $\mu : M \longrightarrow \mathfrak{g}^*$ such that

$$i_{X_a}\omega = d(a \circ \mu) \quad \forall a \in \mathfrak{g}.$$

We call the real-valued function $H_a = a \circ \mu$ the *Hamiltonian function* generating the vector field X_a. A symplectic group action $G \times M \longrightarrow M$ is called *Hamiltonian* when there is a moment map as above.

[17] See Atiyah, Bott, Frankel.
[18] See Atiyah-Bott and M. Guest's notes.
[19] This just means that the map is compatible with the natural G-actions on each side, i.e. $\mu(g \cdot x) = \mathrm{Ad}_g^*(\mu(x))$.

Example 2.8. *The unitary group $U(n+1)$ acts on $\mathbb{C}^{n+1}$ linearly and hence on $\mathbb{C}P^n$ via projective automorphisms. This action preserves the Fubini-Study symplectic form and is described by a moment map*

$$\mu : \mathbb{C}P^n \longrightarrow \mathfrak{u}(n+1)^*.$$

Using the inner product on $\mathfrak{u}(n+1)$ we may identify $\mathfrak{u}(n+1) \cong \mathfrak{u}(n+1)^*$ (sending the adjoint to the coadjoint action) and then we see that μ is simply the embedding of $\mathbb{C}P^n = \mathrm{Gr}_1\mathbb{C}^{n+1}$ as a coadjoint orbit.

Inside $U(n+1)$ we have the maximal torus T^{n+1} of diagonal unitary matrices $\mathrm{diag}(e^{i\theta_0}, \cdots, e^{i\theta_n})$, and inside this we have a $\iota : T^n \hookrightarrow U(n+1)$ of diagonal matrices with first entry 1. These act on $\mathbb{C}P^n$ with moment map $\mu_{\mathfrak{t}} = \iota^* \circ \mu$.

Given any Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω), we obtain the following large quantity of Morse-Bott functions:

Theorem 2.9. *Let the compact Lie group act on (M, ω) with moment map $\mu : M \longrightarrow \mathfrak{g}^*$. For any $a \in \mathfrak{g}$, the function $a \circ \mu$ is a Morse-Bott function. The critical manifolds are symplectic submanifolds and the Morse indices are even, and furthermore, $a \circ \mu$ is a perfect Morse-Bott function.*

Proof. a generates a vector field X_a and this generates a 1-parameter subgroup of G. The closure of this is a torus $H \subset G$ of some dimension. The function $f^a = a \circ \mu$ is **critical exactly when X_a vanishes, i.e. we are at a fixed point** for the action of H. So $\mathrm{Crit}(f^a) = M^H$. Such fixed point sets of symplectic group actions are easily shown to be symplectic submanifolds, and hence even dimensional. In a neighbourhood of a fixed point m, we can show that the action is equivalent to a T^k action on $\mathbb{C}^n$ of the form $(e^{i\theta_j}) \cdot (z_1, \ldots, z_n) = (e^{i\beta_1} z_1, \ldots, e^{i\beta_n} z_n)$, where $\beta_j : \mathfrak{t}^k \longrightarrow \mathbb{R}$ are "weights" i.e. linear maps with integral coefficients $\mathbb{R}^k \longrightarrow \mathbb{R}$. Then the moment map can be written

$$\mu(z_1, \ldots, z_n) = \tfrac{1}{2} \sum_j ||z_j||^2 \beta_j,$$

and $a \circ \mu = \frac{1}{2} \sum_j (x_j^2 + y_j^2) \beta_j(a)$, clearly having even index. ☐

In addition to providing a rich source of Morse-Bott functions, the torus action actually gives very precise information about the gradient flow trajectories, and we can pick out the stable and unstable manifolds of each critical point.

Theorem 2.10 (Atiyah-Guillemin-Sternberg). *Let $a \in \mathfrak{g}$ generate the Hamiltonian action of a torus T^k with moment map $\mu : M \longrightarrow \mathfrak{t}^*$. Then the image $\mu(M)$ is the convex hull of the finite set $\{\mu(m) \ : \ m \in \mathrm{Crit}(f^a)\}$. Furthermore, the closure V of an ascending or descending manifold[20] has image $\mu(V)$ given by the convex hull of the images of critical points of f^a contained in V.*

Example 2.11. *Consider $\mathbb{C}P^2$, which has a $T^3 \subset U(3)$ action with moment map*

$$\mu([z_0, z_1, z_2]) = \tfrac{1}{\sum |z_i|^2}(|z_1|^2, |z_2|^2, |z_3|^2),$$

and $\mu(\mathbb{C}P^2)$ is the convex hull of the basis vectors e_1, e_2, e_3. We could work with a T^2 action instead. The closures of the stable/unstable manifolds for the standard Morse function are of the form $[*, \cdots, *, 0, \cdots, 0]$ and $[0, \cdots, 0, *, \cdots, *]$ and we can see the images under μ. Can make comments about intersection theory.

[20] For a compatible Riemannian metric, e.g. for a Kähler manifold.