1.3 Genericity theorem

We shall explain the terms in, and then prove, the following statement to the effect that Morse functions are plentiful and stable under small perturbations.

Theorem 1.6. *Let M be a compact smooth manifold. Then the set of Morse functions is C^∞ dense in $C^\infty(M,\mathbb{R})$ and it is open with respect to the $C^2(M,\mathbb{R})$ topology.*

The vector space of continuous sections of a vector bundle E may be equipped with a norm, called the "sup norm", also called its "uniform norm", "C^0 norm", or "L^∞ norm":

$$||s||_{C^0} = \sup_{x\in M} |s(x)|,$$

where $|s(x)| = h(s(x), s(x))^{1/2}$ is the pointwise norm induced from a non-degenerate positive-definite bilinear form h chosen on E.

Convergence in this norm is precisely what is called "uniform convergence" and since the uniform limit of continuous functions is continuous (why?) we know that the space of continuous sections with finite C^0 norm actually forms a Banach[2] (but not a Hilbert space).

The vector space of k-times differentiable sections may similarly be equipped with a norm which takes the sum of the sup norms of the first k derivatives. For this to make sense, we need to be able to differentiate sections of vector bundles:

Definition 5. A connection ∇ on a vector bundle E is a linear map $\nabla : C^\infty(TM) \longrightarrow \mathrm{End}(C^\infty(E))$ such that $\nabla_X(fs) = f\nabla_X s + X(f)s$. This makes it a first-order differential operator on sections of E which depends linearly on a vector field; in essence it is a general way of differentiating sections of E.

Once we have a way of differentiating sections, we can define the uniform C^k norm:

$$||s||^2_{C^k} = ||s||^2_\infty + ||\nabla s||^2_\infty + \cdots + ||\nabla^k s||^2_\infty.$$

The C^k sections with finite C^k norm then also form a Banach space, called $C^k(M,E)$. It is not difficult to prove[3] that the resulting norm changes to an equivalent norm if we choose a different connection ∇.[4]

The C^k norms can be used to define a topology on the vector space $C^\infty(M,E) = \cap_0^\infty C^k(M,E)$. On a compact manifold, we say that (s_i) converges to $s \in C^\infty(M,E)$ if and only if $||s_i - s||_{C^k(K)} \to 0$ for all k. [5] This topology does not come from a norm, but does endow $C^\infty(M,E)$ with the structure of a *Fréchet space*.

To prove the theorem, we need some tranversality results, the most important of which are as follows:

Theorem 1.7 (Sard's theorem)**.** *Let $f : M \longrightarrow N$ be a smooth map of manifolds of dimension m, n, respectively. Let C be the set of critical points, i.e. points $x \in U$ with*

$$\mathit{rank}\ Df(x) < n.$$

Then $f(C)$ has measure zero.

Theorem 1.8 (Parametric transversality)**.** *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds. Suppose that F is transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ is transverse to g. (Note that F should be thought of as a family of maps $X \longrightarrow Y$ parametrized by S.*

[2] Recall that a Banach space is a normed vector space which is complete (i.e. every Cauchy sequence converges)

[3] Optional exercise

[4] We can even take the equivalent norm $||s||_{C^k} = \sum_0^k ||\nabla^i s||_\infty$.

[5] For a non-compact manifold we require C^k convergence on all compact sets, i.e. "compact convergence" or more pompously "uniform convergence on compacta" for the first k derivatives.

Proof of Theorem 1.6. Let $\{(U_\alpha \supset V_\alpha, \varphi_\alpha)\}$ be a finite regular[6] covering, which exists by compactness of M. Consider the coordinates $x^1, \ldots, x^n$ as functions h^i_α on V_α extended to smooth functions on all of M. Suppose the number of charts in the covering is N.

Then we can define a large family of sections of T^*M via the map $\Phi : \mathbb{R}^{Nn} \times M \longrightarrow T^*M$ given by

$$\Phi : (\lambda_{\alpha,i}, x) \mapsto df(x) + \sum_{\alpha, i=0,\ldots,n} \lambda_{\alpha,i} dh^i_\alpha(x).$$

First we claim that Φ is transverse to the zero section of T^*M. This is because any point $p \in M$ is contained in some V_α, and if we set $\lambda_{\beta,i} = 0$ for all $\beta \neq \alpha$ and all i, then we obtain

$$df + \sum_{i=1,\ldots,n} \lambda_{\alpha,i} dx^i.$$

By varying the constants $\lambda_{\alpha,i}$ above, we can span all of $T^*_x M$, showing Φ is a submersion.

By Sard's theorem, the set of parameters $\lambda_{\alpha,i}$ such that Φ_λ is transversal to the zero-section is dense. Hence we can find λ arbitrarily small such that

$$f + \sum \lambda_{\alpha,i} h^i_\alpha$$

is a Morse function. Since λ can be arbitrarily small, this means that we can approach f arbitrarily in each of the C^k norms, which is what the C^∞ topology requires.

Now we show that Morse functions are open in C^2 norm. We use the following general fact: If $g : Y \longrightarrow Z$ is a smooth proper map and X is compact, then the set of smooth maps $f : X \longrightarrow Z$ transverse to g is C^1 open. This comes from the fact that transversality amounts to a local submersion condition, and that this depends on subdeterminants of the derivative being nonzero. So if we are close in the C^1 norm, these nonzero values will remain nonzero.

Since f is Morse only when df is transverse to the zero section, we see that the C^2 bound on f implies a C^1 bound on f, implying the result. $\square$

Exercise 5. Let $M \subset \mathbb{R}^N$ be an embedded submanifold of Euclidean space. Outline the proof that for almost all $p \in M$, the function $f(x) = |x - p|^2$ is a Morse function. If $M \subset \mathbb{R}^N$ is a closed subset, show that f is *exhaustive* in the sense that its sublevel sets $\{x \mid f(x) \leq c\}$ are compact. Show that such exhaustive Morse functions satisfy the *Palais-Smale* condition: any sequence $x_n \in M$ such that $f(x_n)$ is bounded from above and $|df|_{x_n}|_g \longrightarrow 0$ contains a subsequence convergent to a critical point of f (here g is a Riemannian metric chosen on M).

The Palais-Smale condition was invented as a replacement for the compactness of sublevel sets in the infinite-dimensional setting.

Exercise 6. A Morse function is called *resonant* when there are two critical points with the same critical value. Show that a resonant Morse function can be arbitrarily well approximated in the C^2 topology by non-resonant ones.

Remark 4. *Both Sard's theorem and the transversality theorem have extremely important generalizations to Banach manifolds.*

A smooth map $f : X \longrightarrow Y$ of Banach manifolds is called a Fredholm mapping if the derivative $T_x f : T_x X \longrightarrow T_{f(x)} Y$ is a Fredholm operator (A continuous linear operator with finite dimensional kernel and cokernel, with closed range (recall the index *of such an operator is* $\dim\ker - \dim\operatorname{coker}$*, which is locally constant on the open set of Fredholm operators.))*

[6] A regular covering is a covering by coordinate charts $(U_\alpha, \varphi_\alpha)$ where $\varphi_\alpha(U_\alpha)$ is an open ball in $\mathbb{R}^n$, $\varphi_\alpha(V_\alpha)$ is a smaller open ball, and $(V_\alpha, \varphi_\alpha)$ is still a covering.

The Sard-Smale theorem states that if $f : X \longrightarrow Y$ is a smooth Fredholm map of Banach manifolds, then the set of regular values of f is residual *in Y. Residual means that it is a countable intersection of open dense sets, and the Baire category theorem says this is dense.*

The parametric transversality result also extends to Banach manifolds in the following form. If S, M, N are Banach manifolds and $F : S \times M \longrightarrow N$ is a submersion such that each $F_s : M \longrightarrow N$ is Fredholm, then for any finite dimensional submanifold $Z \subset N$, there is a residual set of parameters $s \in S$ for which F_s is transversal to Z.

Nevertheless, I don't know a proof of genericity for Morse functions on Banach manifolds.

1.4 Theorem A: Gradient vector fields

Theorem A of Morse theory is the simple statement that if $f^{-1}([a, b])$ is compact and contains no critical points, then the sublevel sets at a and b are diffeomorphic. It is a simple and intuitive result, however the proof involves an idea whose importance is impossible to exaggerate: this idea is to investigate the *dynamics* of f.

In mathematics, dynamics usually means flowing along a vector field – in our case this is the *gradient vector field* of f.

Definition 6. Let f be a smooth function on a Riemannian manifold (M, g). The gradient vector field of f, $\mathrm{grad}(f)$, is the unique vector field such that $df_p(v) = g_p(v, \mathrm{grad}(f))$ for all $v \in T_pM$.

Equivalently, viewing $g \in \Gamma(M, \mathrm{Sym}^2 T^*M)$ as a bundle isomorphism $g : TM \longrightarrow T^*M$, we can write $\mathrm{grad}(f) = g^{-1}(df)$. Also, in local coordinates x^i for which $g = g_{ij}dx^i dx^j$ and $g^{-1} = g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ with $g^{ik}g_{kj} = \delta_{ij}$, we have $\mathrm{grad}(f) = g^{ij}\partial_j f \frac{\partial}{\partial x^i}$.

Exercise 7. Draw the gradient vector field for the Morse function on $\mathbb{R}P^2$ defined earlier.

By the positivity of g, we see that $\mathrm{grad}(f)f = g(df, df) \geq 0$, so that $\mathrm{grad}(f)$ always points in the increasing direction of f, but vanishes on critical points. We normalize the gradient vector field in the following way: Define the smooth function with compact support

$$\rho = \begin{cases} 1/g(df, df) & \text{in } f^{-1}([a, b]) \\ 0 & \text{away from a compact neighbourhood of } f^{-1}([a, b]) \end{cases}$$

Then $X := \rho\,\mathrm{grad}(f)$ is a smooth vector field on M with compact support such that $X(f) = 1$ in the set $f^{-1}([a, b])$. This means that the flow ϕ^X_t, defined for all time, satisfies $f(\phi^X_t(p)) = f(p) + t$, for all $p \in f^{-1}([a, b])$ and $t \leq b - f(p)$. In particular, we obtain the following result:

Theorem 1.9. *Let f be a smooth function such that $f^{-1}([a, b])$ is compact and contains no critical points, and let X be as above. The flow ϕ^X_{b-a} is a diffeomorphism from the manifold with boundary $M^a = f^{-1}((-\infty, a])$ to the manifold with boundary $M^b = f^{-1}((-\infty, b])$. Furthermore, M^a is a deformation retract of M^b, so that the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof. We have already argued that the flow gives a diffeomorphism. To see the deformation[7] retract, define

$$r_t(q) = \begin{cases} q & \text{if } f(q) \leq a \\ \phi^X_{t(a-f(q))}(q) & \text{if } a \leq f(q) \leq b \end{cases}$$

Then $r_0 = \mathrm{Id}$ and r_1 is a retraction from M^b to M^a. □

Remark 5. *The condition that $f^{-1}([a, b])$ be compact is certainly required, as can be seen by removing a point $p \in f^{-1}([a, b])$ from a compact preimage.*

[7] A deformation retract is a map $F : X \times [0, 1] \longrightarrow X$ such that $F_0 = \mathrm{Id}$, $F_1(X) \subset A$, and $F_t|_A = \mathrm{Id}_A$.

1.5 Theorem B: surgery theory

To analyze the change in topology of a (sub)-level set going through a critical value of a Morse function, we need to introduce two types of modifications of manifolds: Surgery and handle attachment. This will give us enough vocabulary to state Theorem B and its corollaries.

1.5.1 Surgery

Let $\mathbb{D}^k$ be the closed unit k-dimensional disk. It is also known as the standard k-cell. A basic operation you can do with such a cell is to attach it to a topological space.

Definition 7. Given a map $\varphi : \partial\mathbb{D}^k \longrightarrow X$ for some topological space X, we say that the space

$$X \cup_\varphi \mathbb{D}^k := \frac{X \sqcup \mathbb{D}^k}{x \sim \varphi(x)},$$

equipped with the quotient topology, is obtained from X by *attaching a k-cell*.

Recall that a *CW* complex is a space obtained by starting with a discrete set of points and attaching 1-cells, then attaching 2-cells to the result, and so on increasing the dimension of the cells. For example, the k-sphere for $k > 0$ is obtained by attaching a k-cell to a 0-cell. The 2-torus can be constructed by attaching a pair of 1-cells to a 0-cell and then attaching a 2-cell.

Cell attachment is an operation in the topological category; we will extend it to a smooth operation, called surgery.

Let $S \hookrightarrow M$ be en embedding of the k-sphere in the n-manifold M, $k < n$, with trivial normal bundle $NS = TM|_S/TS$. Suppose we also choose a *framing* of the normal bundle, i.e. an isomorphism

$$\varphi : NS \longrightarrow S^k \times \mathbb{R}^{n-k}.$$

Using the tubular neighbourhood theorem to identify a neighbourhood of the zero section in NS with a neighbourhood $U \subset M$ containing S, we may view φ as a diffeomorphism[8]

$$\varphi : U \longrightarrow S^k \times \mathring{\mathbb{D}}^{n-k},$$

where $\mathring{\mathbb{D}}^k$ is the open k-disk.

Consider the two spaces $S^p \times \mathbb{D}^{q+1}$ and $\mathbb{D}^{p+1} \times S^q$, for $p + q + 1 = n$. These are both n-manifolds with boundary, and they have the same boundary $S^p \times S^q$. **The basic idea of surgery is that since these spaces have the same boundary, if either of these spaces are found in an n-manifold, we can carve it out and replace it by the other one.**

The proper way to do this is to observe that if we remove the central sphere we have a natural diffeomorphism

$$S^p \times (\mathring{\mathbb{D}}^{q+1} - \{0\}) \xrightarrow{s} (\mathring{\mathbb{D}}^{p+1} - \{0\}) \times S^q \tag{3}$$
$$(x, (y, t)) \mapsto ((x, t), y), \tag{4}$$

where we use polar coordinates $\mathring{\mathbb{D}}^{k+1} - \{0\} = S^k \times (0, 1)$.

Definition 8. Given an embedding $S \hookrightarrow M$ of the k-sphere, with framing φ as above, we say that the manifold

$$M(S, \varphi) = \frac{(M - S) \sqcup (\mathring{\mathbb{D}}^{k+1} \times S^{n-k-1})}{x \sim s(\varphi(x))}$$

is obtained from M by a surgery[9] of type (S, φ). Here s is the universal map (3) defined above.

[8] The diffeomorphism is defined up to isotopy by the framing.

[9] The diffeomorphism type of $M(S, \varphi)$ may depend on the isotopy class (i.e. a homotopy through embeddings) of the embedding of S and on the regular homotopy class of φ, i.e. a homotopy through immersions.