Example 1.10. *(Zero dimensional surgery) A zero dimensional surgery on M (which may not be connected) requires finding two points $p_\pm \in M$ and choosing a framing, which may be obtained from disjoint coordinate charts around the points. The surgery then consists in removing the points $p_\pm$ and gluing a cylinder $\mathbb{D}^1 \times S^{n-1}$ via the diffeomorphism s.*

If M is the disjoint union of manifolds $M_\pm$ and $p_\pm \in M_\pm$, then the surgery is usually known as the connected sum *operation.*

To see that the result may depend on the choice of framing, consider the surgery applied to $p_\pm \in \mathbb{R}$.

Example 1.11. *(Codimension 2 surgery) Suppose that M is compact and oriented and $i : S^{m-2} \hookrightarrow M$ is an embedding of a codimension 2 sphere, with trivializable normal bundle.*

Since S^{m-2} has a natural orientation and M has a chosen orientation, this induces a naturaly orientation on the normal bundle NS of $S = i(S^{m-2})$. Therefore once a framing φ_0 is chosen, the sphere bundle $S(NS)$ is identified with $S^{m-2} \times S^1$, any other framing differs from φ_0 by a section of this. This implies that the homotopy classes of framings are classified by $\pi_{m-2}S^1$, and hence all framings are equivalent unless $m = 3$, i.e. we are talking about a circle embedded in a 3-manifold, i.e. a knot.

The surgery removes the S^{m-2} and glues in a $S^1 \times \mathbb{D}^{m-1}$. A framing φ defines a section of the sphere bundle $S(NS)$; the image of this section is called the attaching sphere *of the surgery. After the surgery, this sphere will bound a disk $\{1\} \times \mathbb{D}^{m-1}$.*

Consider the standard torus in $\mathbb{R}^3 \sqcup \infty$; it can be filled inside or outside by solid tori. A different generating circle bounds in each case.

Example 1.12. *(surgery on knots in S^3) A Knot K in S^3 is always the boundary of an orientable surface X with boundary which is smoothly embedded in S^3, known as a Seifert surface of the knot. To construct this surface, draw the knot in the plane, orient it, and trace along the knot in the direction of the orientation. At each crossing, continue along the other part of the knot involved in the crossing, following orientation. This process produces a set of disjoint unknotted circles called* Seifert surfaces. *Filling the Seifert circles with disks, we then glue the disks to each other according to the crossing data.*

The framing of K induced by the inward-pointing normal vector along $\partial X = K$ is independent of the choice of the Seifert surface and is called the canonical framing *of the knot.*

Using this framing we get a diffeomorphism between a tubular neighbourhood U of the knot and $\mathbb{D}^2 \times S^1$. This framing gives a circle $\ell = \{1\} \times S^1 \subset \partial\mathbb{D}^2 \times S^1$ homotopic to K, called the longitude. *The homotopically trivial circle $\partial\mathbb{D}^2 \times \{1\}$ is called the* meridian *μ.*

Any other framing, being a section over K of the meridian circle bundle, has an attaching curve such that

$$[\varphi] = p[\mu] + [\ell],$$

where p is the winding number of the map $S^1 \longrightarrow S^1$ defined by the section. The integer p, called the coefficient of the surgery, *uniquely defines the surgery, and the resulting 3-manifold is denoted $S^3(K, p)$.*

A simple example of this is the unknot in S^3: to compute its 0-surgery, view S^3 as a union of solid tori along their common boundary: for $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \ : \ |z_0|^2 + |z_1|^2 = 1\}$, take

$$U_i = \{(z_0, z_1) \in S^3 \ : \ |z_i| \leq 1/\sqrt{2}\}$$

Then clearly $S^3 = U_0 \cup U_1$ with $U_0 \cap U_1 = T^2$, the "Clifford torus". Let $K_i = S^3 \cap \{z_i = 0\}$ be a pair of embedded circles, forming the link $K_0 \cup K_1$ known as the "Hopf link"; each of K_i is unknotted, as can be seen from stereographic projection or the fact that they bound embedded discs.

Zero surgery along K_0 would remove U_0 and replace it with a solid torus $\hat{U}_0$ whose longitude is given by the latitude K_1; the meridianal circles of K_1 would contract not only in U_1 but also in $\hat{U}_0$. The resulting manifold is diffeomorphic to $S^1 \times S^2$.

Exercise 8 (On Assignment 1)**.** What are $S^3(K_0, \pm 1)$? Work out the case $S^3(K_0, 0)$ in detail, this will help.

Exercise 9. Show that for K the trefoil knot, $S^3(K, -1)$ is the Poincaré sphere, a $\mathbb{Z}$-homology sphere with fundamental group given by the binary icosahedral group. It is diffeomorphic to

$$\{(x, y, z) \in \mathbb{C}^3 \;:\; x^2 + y^3 + z^5 = 0,\; |x|^2 + |y|^2 + |z|^2 = \epsilon > 0\}.$$

(See Nicolaescu's book for more information).

The Poincaré sphere has an alternative description via the 2-framed link diagram corresponding to the E_8 Dynkin diagram. The two links are related by a sequence of Kirby moves.

1.5.2 Handle attachment

Handle attachment is different from surgery in that it adds a "handle" onto the boundary of a n-manifold with boundary to create another n-manifold with boundary.

For n-dimensional manifolds, a λ-handle (or handle of index λ) is the manifold with corners

$$\mathbb{H}_\lambda = \mathbb{D}^\lambda \times \mathbb{D}^{n-\lambda}.$$

The disk $\mathbb{D}^\lambda \times \{0\} \subset \mathbb{H}_{\lambda,\mu}$ is called the *core* and the disk $\{0\} \times \mathbb{D}^{n-\lambda}$ is the *cocore*.

Example 1.13. *Draw a 0-handle and 1-handle and 2-handle of dimension 2, with cores indicated. Also draw handles of dimension 3.*

Attaching a λ-handle is supposed to be the smooth analogue of attaching a λ-cell. The main difference is that it should produce a smooth manifold and it depends on the precise way that the attachment is thickened, i.e. it should depend on a framing, just as in the case of surgery.

Attachment of a λ-handle to the n-manifold with boundary M requires the following data:

- An embedding with trivializable normal bundle $\iota : S^{\lambda-1} \hookrightarrow \partial M$; let $S = \iota(S^{\lambda-1})$.
- A framing $\varphi : N_{\partial M} S \longrightarrow S^{\lambda-1} \times \mathbb{R}^{n-\lambda}$ of the normal bundle.

The λ-handle is of course not a manifold with boundary – it is a manifold with corners. its boundary decomposes as $\partial \mathbb{H}_\lambda = \partial_- \mathbb{H}_\lambda \cup \partial_+ \mathbb{H}_\lambda$, with $\partial_- \mathbb{H}_\lambda = S^{\lambda-1} \times \mathbb{D}^{n-\lambda}$ and $\partial_+ \mathbb{H}_\lambda = \mathbb{D}^\lambda \times S^{n-\lambda-1}$.

Remark 6. *What we would like to do is the following: Use the framing to define a diffeomorphism* $\varphi : S^{\lambda-1} \times \mathbb{D}^{n-\lambda} \longrightarrow U$ *to a tubular neighbourhood* U *of* S *in* ∂M, *and then glue* $\mathbb{H}_\lambda$ *to* ∂X *along the identification* $\partial_- \mathbb{H}_\lambda \longrightarrow U$. *The resulting space* $X^+ = X(S, \varphi)$ *is not, however, naturally a smooth manifold.*

Rather than use the manifold with corners $\mathbb{H}_\lambda$, we use the homeomorphic space $\mathbb{D}^n$, $n = \lambda + \mu$. View the sphere $S^{\lambda-1}$ as a submanifold of $\mathbb{D}^n$. A standard tubular neighbourhood of S^λ is given by

$$T_\epsilon = \{x \in \mathbb{D}^n \mid x_\lambda^2 > \epsilon\}, \quad 0 \le \epsilon < 1.$$

We aim to attach $\mathbb{D}^n$ to M along this tubular neighbourhood, and we will use the following universal "switching" map $\alpha : T_\epsilon - S^{\lambda-1} \longrightarrow T_\epsilon - S^{\lambda-1}$, defined by

$$\alpha = q^{-1} \circ \sigma \circ q,$$

where $q : \mathbb{D}^n - S^{\lambda-1} \longrightarrow (\mathring{\mathbb{D}}^\lambda - \{0\}) \times \mathbb{D}^\mu$ is a stretching map given by

$$q : (x_\lambda, x_\mu) \mapsto (x_\lambda, \tfrac{x_\mu}{\sqrt{1 - x_\lambda^2}}),$$

and $\sigma : (\mathring{\mathbb{D}}^\lambda - \{0\}) \times \mathbb{D}^\mu \longrightarrow (\mathring{\mathbb{D}}^\lambda - \{0\}) \times \mathbb{D}^\mu$ is the following involution:

$$\sigma : (x_\lambda, x_\mu) \mapsto (\tfrac{x_\lambda}{|x_\lambda|} \sqrt{1 - x_\lambda^2 + \epsilon}, x_\mu).$$

In this way, α is a self-diffeomorphism of the punctured tubular neighbourhood $T_\epsilon - S^{\lambda-1}$ which reverses the radial direction.

Definition 9. Let $\iota : S^{\lambda-1} \hookrightarrow \partial M$ be an imbedding, and use a framing φ to extend this over the tubular neighbourhood to a diffeomorphism $\bar{\iota} : T_1 \longrightarrow M$. Then the manifold

$$M \cup_{\iota,\varphi} \mathbb{H}_\lambda = \frac{(M - \iota(S^{\lambda-1})) \sqcup (\mathbb{D}^n - S^{\lambda-1})}{x \sim \bar{\iota}\alpha(x), \quad x \in T_1 - S^{\lambda-1}}$$

is called M with a λ-handle attached along the attaching sphere $\iota(S^{\lambda-1})$. The subset $\mathbb{D}^n - S^{\lambda-1}$ is called the handle, $\mathbb{D}^\mu = \mathbb{D}^n \cap \{x_- = 0\}$ is the *belt disc*, and its boundary is the *belt sphere*.

Remark 7.

- *If M is oriented and $\bar{\iota}$ reverses orientation, then $M \cup_{\iota,\varphi} \mathbb{H}_\lambda$ admits an orientation extending the orientation of M as well as the orientation of $\mathbb{D}^n$.*
- *Attaching a 0-handle is simply taking a disjoint union with $\mathcal{D}^n$.*
- *If M_+, M_- are connected manifolds with boundary and $\iota : S^0 \mapsto \pm p \in M_\pm$, Then attaching a 1-handle to $M_- \sqcup M_+$ along the* 0*-sphere is known as the* boundary connected sum $M_- \#_b M_+$.
- *Handle attachment of a λ-handle to M may be viewed as a surgery performed on the attaching sphere. One simply needs to be aware that $S^{\lambda-1} \times (\mathbb{D}^\mu \cap H^n)$ and $\mathbb{D}^\lambda \times (S^{\mu-1} \times H^n)$ are manifolds with corners.*

Using Morse theory, we will prove the **Handle presentation theorem** of Milnor-Wallace: that every compact manifold can be built by consecutively attaching a λ-handle for each critical point of index λ. It is then a natural question when two such "handlebody decompositions" give diffeomorphic manifolds.