Exercise 1. Consider S^n and its two stereographic coordinate charts φ_S, φ_N to $\mathbb{R}^n$. Using standard coordinates on $\mathbb{R}^n$, write down the coordinate expressions for a smooth, nowhere-vanishing n-form on S^n with integral equal to 1.

Exercise 2. On $M = \mathbb{R}^2 \setminus \{0\}$ with coordinates (x, y), consider the differential form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}.$$

Prove that ω is closed, but not exact.

Exercise 3. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

where (r, ϕ, θ) are standard Cartesian coordinates on $\mathbb{R}^3$.

- Compute $\varphi^* dx, \varphi^* dy, \varphi^* dz$ where (x, y, z) are Cartesian coordinates for $\mathbb{R}^3$.
- Compute $\varphi^*(dx \wedge dy \wedge dz)$.
- For any vector field X, define i_X to be the unique degree -1 (i.e. it reduces degree by 1) derivation (i.e. $i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge i_X(\beta)$) of the algebra of differential forms such that $i_X(f) = 0$ and $i_X df = X(f)$ for $f \in \Omega^0(M)$. Compute the integral

  $$\int_{S^2} \iota_r^*(i_X(dx \wedge dy \wedge dz)),$$

  for the vector field $X = \varphi_* \frac{\partial}{\partial r}$, where $\iota_r : S^2 \to \mathbb{R}^3$ for $r > 0$ fixed is defined to be the embedding of S^2 as the sphere of radius r, i.e. $\iota_r(x) = rx$.

Exercise 4. Use Stokes' theorem if necessary:

1. Let M be a compact orientable smooth n-manifold (without boundary) and let $\mu \in \Omega^{n-1}(M)$. Prove there exists a point $p \in M$ with $d\mu(p) = 0$.
2. For any sphere S^k, let $\iota : S^k \to \mathbb{R}^{k+1}$ be the usual inclusion, and let $v_k \in \Omega^k(S^k)$ be given by

   $$v_k = \iota^* \sum_{i=0}^{k} (-1)^i x^i dx^0 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^k.$$

   Show that v_k is closed and that $[v_k] \neq 0$ in the top de Rham cohomology group $H^k(S^k)$.

Exercise 5. Compute the de Rham cohomology groups (Using Mayer-Vietoris if necessary) of the following spaces, for all degrees.

- $\mathbb{R}^3 - \{p\}$, for $p \in \mathbb{R}^3$ a point.
- $\mathbb{R}^3 - \{p_1 \cup p_2\}$ where p_i are distinct points?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$ where ℓ_i are non-intersecting lines?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$, assuming that l_1 intersects l_2 in exactly one point?

This question is a slightly easier version of the one John Nash asked in class in the movie "A Beautiful Mind".

Exercise 6 (Bonus)**.** Let $A : S^n \to S^n$ be the antipodal map which sends $x \to -x$. Begin by showing that this defines an action of S_2 on S^n with quotient S^n/S_2 diffeomorphic to $\mathbb{R}P^n$.

1. Prove that $\Omega^k(S^n)$ decomposes as $\Omega^k_+ \oplus \Omega^k_-$ into ± 1 eigenspaces of the operator A^*.
2. Prove that the exterior derivative preserves each of the subspaces $\Omega^\bullet_+$ and $\Omega^\bullet_-$, making each a sub-complex of $(\Omega^\bullet(S^n), d)$.
3. Let $\pi : S^n \to \mathbb{R}P^n$ be the quotient map. Show that $\pi^* : \Omega^\bullet(\mathbb{R}P^n) \to \Omega^\bullet_+$ is an isomorphism of complexes.
4. Use the above results together with knowledge of $H^\bullet(S^n)$ to compute $H^\bullet(\mathbb{R}P^n)$.