3.1 Stability

Transversality is a stable condition. In other words, if transversality holds, it will continue to hold for any sufficiently small perturbation (of the submanifolds or maps involved). Not only is transversality *stable*, it is actually *generic*, meaning that even if it does not hold, it can be made to hold by a small perturbation. In a sense, stability says that transversal maps form an open set, and genericity says that this open set is dense in the space of maps. To make this precise, we would introduce a topology on the space of maps, something which we leave for another course.

Definition 3.9. We call a smooth map

$$F : M \times [0, 1] \to N \tag{57}$$

a smooth homotopy from f_0 to f_1, where $f_t = F \circ j_t$ and $j_t : M \to M \times [0, 1]$ is the embedding $x \mapsto (x, t)$.

Definition 3.10. A property of a smooth map $f : M \longrightarrow N$ is *stable* under perturbations when for any smooth homotopy f_t with $f_0 = f$, there exists an $\epsilon > 0$ such that the property holds for all f_t with $t < \epsilon$.

Proposition 3.11. *If M is compact, then the property of $f : M \to N$ being an immersion (or submersion) is stable under perturbations.*

Proof. If $f_t, t \in [0, 1]$ is a smooth homotopy of the immersion f_0, then in any chart around the point $p \in M$, the derivative $Df_0(p)$ has a $m \times m$ submatrix with nonvanishing determinant, for $m = \dim M$. By continuity, this $m \times m$ submatrix must have nonvanishing determinant in a neighbourhood around $(p, 0) \in M \times [0, 1]$. We can cover $M \times \{0\}$ by a finite number of such neighbourhoods, since M is compact. Choose ϵ such that $M \times [0, \epsilon)$ is contained in the union of these intervals, giving the result. The proof for submersions is identical. □

Corollary 3.12. *If K is compact and $f : K \to M$ is transverse to the closed submanifold $L \subset M$ (this just means that f is transverse to the embedding $\iota : L \to M$), then the transversality is stable under perturbations of f.*

Proof. Let $F : K \times [0, 1] \to M$ be a homotopy with $f_0 = f$. We show that K has an open cover by neighbourhoods in which f_t is transverse for t in a small interval; we then use compactness to obtain a uniform interval.

First the points which do not intersect L: $F^{-1}(M \setminus L)$ is open in $K \times [0, 1]$ and contains $(K \setminus f^{-1}(L)) \times \{0\}$. So, for each $p \in K \setminus f^{-1}(L)$, there is a neighbourhood $U_p \subset K$ of p and an interval $I_p = [0, \epsilon_p)$ such that $F(U_p \times I_p) \cap L = \varnothing$.

Now, the points which do intersect L: L is a submanifold, so for each $p \in f^{-1}(L)$, we can find a neighbourhood $V \subset M$ containing $f(p)$ and a submersion $\psi : V \to \mathbb{R}^l$ cutting out $L \cap V$. Transversality of f and L is then the statement that ψf is a submersion at p. This implies there is a neighbourhood $\tilde{U}_p$ of $(p, 0)$ in $K \times [0, 1]$ where ψf_t is a submersion. Choose an open subset (containing $(p, 0)$) of the form $U_p \times I_p$, for $I_p = [0, \epsilon_p)$.

By compactness of K, choose a finite subcover of $\{U_p\}_{p \in K}$; the smallest ϵ_p in the resulting subcover gives the required interval in which f_t remains transverse to L. □

Remark 3.13. Transversality of two maps $f : M \to N$, $g : M' \to N$ can be expressed in terms of the transversality of $f \times g : M \times M' \to N \times N$ to the diagonal $\Delta_N \subset N \times N$. So, if M and M' are compact, we get stability for transversality of f, g under perturbations of both f and g.

Remark 3.14. Local diffeomorphism and embedding are also stable properties.