Exercise 1. Prove that $\mathbb{C}P^1$ is homeomorphic to S^2. Try proving this using the given coordinate charts.

Exercise 2. Consider the 3-sphere $S^3 \subset \mathbb{R}^4$. Using the isomorphism $\mathbb{R}^4 \cong \mathbb{C}^2$, we obtain the inclusion $\iota : S^3 \to \mathbb{C}^2 \setminus \{0\}$. Composing with the projection map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$, we obtain

$$p = \pi \circ \iota : S^3 \to \mathbb{C}P^1,$$

known as the Hopf fibration. Using the coordinate charts given in the notes for S^3 and $\mathbb{C}P^1$, compute p in coordinates (one chart on each of the domain and codomain will suffice).

Exercise 3. Let Γ be a group, and give it the discrete topology. Suppose Γ acts continuously on the topological n-manifold M, meaning that the action map

$$\begin{aligned} \Gamma \times M &\xrightarrow{\rho} M \\ (h, x) &\longmapsto h \cdot x \end{aligned}$$

is continuous. Suppose also that the action is *free*, i.e. the stabilizer of each point is trivial. Finally, suppose the action is *properly discontinuous*, meaning that each $x \in M$ has a neighbourhood U such that $h \cdot U$ is disjoint from U for all nontrivial $h \in \Gamma$, that is, for all $h \neq 1$.

i) Show that the quotient map $\pi : M \to M/\Gamma$ is a local homeomorphism, where M/Γ is given the quotient topology. Conclude that M/Γ is locally homeomorphic to $\mathbb{R}^n$.

ii) Show that π is an open map.

iii) Let $f : M \to N$ be a continuous map such that

$$f(h \cdot x) = f(x) \text{ for all } (h, x) \in \Gamma \times M.$$

Show there is a unique continuous map $\bar{f} : M/\Gamma \to N$ such that $\bar{f}(\pi(x)) = f(x)$ for all $x \in M$.

iv) Give an example where M/Γ is not Hausdorff.

Exercise 4. Let (Γ, M, ρ) be as in Exercise 3. Prove that M/Γ is Hausdorff if and only if the image of the map

$$\begin{aligned} \Gamma \times M &\longrightarrow M \times M \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

is closed in $M \times M$.

Exercise 5. Let the group of order two, $C_2 = \{1, -1\}$, act on S^n via $x \mapsto -x$. Show that S^n/C_2 is homeomorphic to the projective space $\mathbb{R}P^n$, as it was defined in class.

Exercise 6. Recall that in the description of $\mathbb{R}P^3$, the space of 1-dimensional subspaces of $\mathbb{R}^4$, we represented each point of $\mathbb{R}P^3$ as the equivalence class

$$[x_0 : x_1 : x_2 : x_3] = [(x_0, x_1, x_2, x_3)]$$

for the relation on 4-vectors defined by the action of the group $\mathbb{R}^*$: that is, $x \sim y \Leftrightarrow y = \lambda x$ for $\lambda \in \mathbb{R}^*$. Each coordinate defines a hyperplane $H_i = \{x \in \mathbb{R}^4 \ : \ x_i = 0\}$ and therefore an open set $U_i = \mathbb{R}^4 \backslash H_i$. We made these into coordinate charts by sending $x \in U_i$ to the 3-vector obtained by rescaling x by x_i^{-1} and deleting the i^{th} coordinate (which has value 1 due to the rescaling).

We now apply the same strategy to study $Gr(2,4)$, the Grassmannian of 2-dimensional linear subspaces of $\mathbb{R}^4$. Every point P in the Grassmannian is a 2-dimensional subspace of $\mathbb{R}^4$ and so we can choose a basis for it: write this basis as a 2×4 matrix where the rows are the basis vectors:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

Notice that we are going back to the traditional way of numbering coordinates (x_1, x_2, x_3, x_4) starting from 1 rather than 0.

1. Describe precisely what condition on the above 2×4 matrix guarantees that its rows span a 2-dimensional subspace. Prove that such matrices form an open subset of all 2×4 matrices.

2. What is the appropriate equivalence relation for such 2×4 matrices? That is, when do two matrices represent the same point $P \in Gr(2,4)$? Express this equivalence relation as the action of a group.

3. Suppose we focus on the first coordinate x_1: it defines a hyperplane $H_1 = \{x \in \mathbb{R}^4 \ : x_1 = 0\}$. Note that the intersection of P with H_1 must have dimension either 1 or 2. Suppose that $\dim P \cap H_1 = 1$. Show that this condition defines an open set in $Gr(2,4)$, and prove that any element of this open set can be described by a matrix of the form

$$\begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & b_2 & b_3 & b_4 \end{bmatrix}$$

4. Suppose that $\dim P \cap H_1 = 1$. Now consider the other coordinate x_2 and think about the hyperplane H_{12} it defines *inside* H_1 – this has dimension 2. Notice that $P \cap H_1$ has dimension 1 and H_{12} has dimension 2 in the 3-dimensional space H_1. As a result their intersection must have dimension 0 or 1. Show that the simultaneous requirements

$$\dim(P \cap H_1) = 1 \text{ and } \dim(P \cap H_{12}) = 0$$

define an open set $U_{12} \subset Gr(2,4)$, and show that any element of this open set may be described uniquely by a matrix of the form

$$\begin{bmatrix} 1 & 0 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \end{bmatrix}$$

5. Generalize the above by considering other pairs of coordinates besides (x_1, x_2), i.e. consider also $(13), (14), (23), (24)$, and (34). In this way construct an atlas of six coordinate charts for $Gr(2,4)$, and prove that it is a smooth 4-dimensional manifold.