1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We want to make this precise.

1.1 Topological manifolds

Definition 1.1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

"Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \subset \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold is called an *atlas*.

We now give examples of topological manifolds. The simplest is, technically, the empty set. Then we have a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.2 (open subsets)**.** Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M. For instance, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{1}$$

known as the general linear group, which is a topological manifold.

Example 1.3 (Circle)**.** The circle is defined as the subspace of unit vectors in $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

Let $N = (0, 1)$ be the north pole and let $S = (0, -1)$ be the south pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x, y) \mapsto (1 + y)^{-1} x, \tag{2}$$

$$\varphi_S : (x, y) \mapsto (1 - y)^{-1} x. \tag{3}$$

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product of manifolds is a manifold.

Example 1.4 (n-torus)**.** $S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with an atlas consisting of the 2^n charts given by all possible n-fold products of the charts φ_N, φ_S defined above.

The circle is a 1-dimensional sphere; we now describe general spheres.

Example 1.5 (Spheres)**.** The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \; : \; \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the north pole and let $S = (-1, 0, \ldots, 0)$ be the south pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}, \tag{4}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}. \tag{5}$$

Remark 1.6. We have endowed the sphere S^n with a certain topology, but is it possible for another topological n-manifold X to be homotopy equivalent to S^n *without* being homeomorphic to it? Recall that homotopy equivalence between the topological spaces M, N means the existence of continuous maps $F : M \to N$ and $G : N \to M$ such that both $F \circ G$ and $G \circ F$ are homotopic (i.e. continuously deformable) to identity maps.

The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the classification of topological 1- and 2-manifolds.

Remark 1.7 (The Hausdorff and second countability axioms)**.** Without the Hausdorff assumption, we would have examples such as the following: take the disjoint union $R_1 \sqcup R_2$ of two copies of the real line, i.e. $R_1 = R_2 = \mathbb{R}$, and form the quotient by the equivalence relation

$$R_1 \setminus \{0\} \ni x \sim \varphi(x) \in R_2 \setminus \{0\}, \tag{6}$$

where φ is the obvious identification $R_1 \to R_2$ (i.e. $\varphi(x) = x$). The resulting quotient topological space is locally homeomorphic to $\mathbb{R}$ but the points $[0 \in R_1], [0 \in R_2]$ cannot be separated by open neighbourhoods.

Second countability is not as crucial, but will be necessary for the proof of the Whitney embedding theorem, among other things.

Example 1.8 (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.

More precisely, let $X = \mathbb{K}^{n+1} \backslash \{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists \lambda \in \mathbb{K}^* = \mathbb{K} \backslash \{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an *open* map, since if $U \subset X$ is open, then tU is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t \in \mathbb{K}^*} tU = \pi^{-1}(\pi(U))$

is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but *not* quotients), we show that the graph of the equivalence relation is closed in $X \times X$. Since π, and hence $\pi \times \pi$ are open, this implies that the diagonal is closed in $\mathbb{K}P^n \times \mathbb{K}P^n$, which is equivalent to the Hausdorff property. The graph in question is by definition

$$\Gamma_\sim = \{(x,y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x,y) = (x_i y_j - x_j y_i) \quad i \neq j,$$

implying at once that it is a closed subset.

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{7}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.9 (Connected sum)**.** Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\begin{array}{ccc} B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} & \xrightarrow{\ \phi\ } & B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} \\ x & \longmapsto & \frac{2\epsilon^2}{|x|^2}x \end{array} \tag{8}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the *connected sum* $M \# N$, as the quotient $X/\sim$, where

$$X = (M \backslash \overline{\varphi^{-1}(B(0,\epsilon))}) \sqcup (N \backslash \overline{\psi^{-1}(B(0,\epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}.$$

Remark 1.10. The connected sum operation as described above may be viewed as an operation on the pair $(L, \{p,q\})$, where $L = M \sqcup N$ is the manifold formed by the disjoint union of M and N and $\{p,q\} \subset L$ is a set of two distinct points. The output of the connected sum is then the manifold $X/\sim$, where $\sim$ is as above and

$$X = L \backslash (\overline{\varphi^{-1}(B(0,\epsilon))} \sqcup \overline{\psi^{-1}(B(0,\epsilon))}).$$

The advantage of this formulation is that p, q need not be in the same connected component: indeed we may perform the connected sum of any manifold L with itself along a pair of points.

Remark 1.11. The homeomorphism type of the connected sum of connected manifolds M, N is independent of the choices of p, q and φ, ψ, except that it may depend on the two possible orientations of the gluing map $\psi^{-1}\phi\varphi$. To prove this, one must appeal to the so-called *annulus theorem.*

Remark 1.12. By iterated connect sum of S^2 with T^2 and $\mathbb{R}P^2$, we can obtain all compact 2-dimensional manifolds.

Example 1.13. Let F be a topological space. A fiber bundle with fiber F is a triple (E, p, B), where E, B are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood U of b and a homeomorphism

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over b".

When B, F are topological manifolds, then clearly E becomes one as well. We will often encounter such manifolds.

Example 1.14 (General gluing construction)**.** To construct a topological manifold "from scratch", we glue open subsets of $\mathbb{R}^n$ together using homeomorphisms, as follows.

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \ , \tag{9}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be *homeomorphisms.*

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij} \cap U_{jk}}$ for all i, j, k. This will ensure that the equivalence relation in (11) is well-defined.

Second countability of the glued manifold is guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that the graph of φ_{ij}, namely

$$\{(x, \varphi_{ij}(x)) \ : x \in U_{ij}\} \tag{10}$$

is a closed subset of $U_i \times U_j$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{11}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, for all i, j. This space has a distinguished atlas $\mathcal{A}$, whose charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

Example 1.15 (Quotient construction)**.** Let Γ be a group, and give it the discrete topology. Suppose Γ acts continuously on the topological n-manifold M, meaning that the action map

$$\begin{aligned} \Gamma \times M &\xrightarrow{\rho} M \\ (h, x) &\longmapsto h \cdot x \end{aligned}$$

is continuous. Suppose also that the action is *free*, i.e. the stabilizer of each point is trivial. Suppose the action is *properly discontinuous*, meaning that each $x \in M$ has a neighbourhood U such that $h \cdot U$ is disjoint from U for all nontrivial $h \in \Gamma$, that is, for all $h \neq 1$. Finally, assume that the following subset is closed:

$$\{(x, y) \in M \times M \ : \ y = h \cdot x \text{ for some } h \in \Gamma\}$$

Then M/Γ is a topological manifold and $\pi : M \to M/\Gamma$ is a local homeomorphism.

Example 1.16 (Mapping torus)**.** Let M be a topological manifold and $\phi : M \to M$ a homeomorphism. Then

$$M_\phi = (M \times \mathbb{R})\,/\mathbb{Z}$$

is a manifold, where $k \in \mathbb{Z}$ acts via $k \cdot (p, t) = (\phi^k(p), t + k)$. This is called the mapping torus of ϕ and is a fibre bundle over $\mathbb{R}/\mathbb{Z} \cong S^1$ with fibre M.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, we can compare them along the intersection $U_{ij} = U_i \cap U_j$, by forming the "gluing map"

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}). \tag{12}$$

This is a homeomorphism, since it is a composition of homeomorphisms. In this sense, topological manifolds are glued together by homeomorphisms.

This means that a given function on the manifold may happen to be differentiable in one chart but not in another, if the gluing map between the charts is not smooth – there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, where the gluing maps are *smooth*.

Remark 1.17 (Aside on smooth maps of vector spaces)**.** Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. We say f is differentiable at $p \in U$ if there is a linear map $Df(p) : V \longrightarrow W$ which approximates f near p, meaning that

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{|f(p + x) - f(p) - Df(p)(x)|}{|x|} = 0. \tag{13}$$

Notice that $Df(p)$ is uniquely characterized by the above property.

We have implicitly chosen inner products, and hence norms, on V and W in the above definition, though the differentiability of f is independent of this choice, since all norms are equivalent in finite dimensions. This is no longer true for infinite-dimensional vector spaces, where the norm or topology must be clearly specified and $Df(p)$ is required to be a continuous linear map. Most of what we do in this course can be developed in the setting of Banach spaces, i.e. complete normed vector spaces.

A basis for V has a corresponding dual basis $(x_1, \dots, x_n)$ of linear functions on V, and we call these "coordinates". Similarly, let $(y_1, \dots, y_m)$ be coordinates on W. Then the vector-valued function f has m scalar components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written, relative to the chosen bases for V, W, as an $m \times n$ matrix, called the *Jacobian matrix* of f at p.

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{14}$$

We say that f is differentiable in U when it is differentiable at all $p \in U$, and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W) \tag{15}$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

Notice that the first derivative Df is itself a map from U to a vector space $\mathrm{Hom}(V, W)$, so if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)), \tag{16}$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and so we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W). \tag{17}$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $\mathrm{Sym}^2(V^*) \otimes W$, since "mixed partials are equal".

Definition 1.18. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, as explained next.

Definition 1.19. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}) \tag{18}$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Remark 1.20. Note that the gluing maps $\varphi_j \circ \varphi_i^{-1}$ are not necessarily defined on all of $\mathbb{R}^n$. They only need be smooth on the open subset $\varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$.

Remark 1.21. Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$). This is how we define C^k, real-analytic, and complex manifolds, respectively.

We may now verify that all the examples from §1.1 are actually smooth manifolds:

Example 1.22 (Spheres)**.** The charts for the n-sphere given in Example 1.5 form a smooth atlas, since

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0}\vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\vec{z} \tag{19}$$

is a smooth map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, as required.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.4, is smooth. Example 1.2 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.23 (Projective spaces)**.** The charts for projective spaces given in Example 1.8 form a smooth atlas, since

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1}z_2, \ldots, z_1^{-1}z_n), \tag{20}$$

which is smooth on $\mathbb{R}^n \setminus \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.9 is clearly smooth since ϕ is a smooth map, and any topological manifold from Example 1.14 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.

1.3 Manifolds with boundary

Manifolds with boundary relate manifolds of different dimension. Since manifolds are not defined as subsets of another topological space, the notion of boundary is not the usual one from point set topology. To introduce boundaries, we change the local model for manifolds to

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}, \tag{21}$$

with the induced topology from $\mathbb{R}^n$.

Definition 1.24. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart. It follows that $M = \partial M \sqcup \operatorname{Int} M$.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, with smoothness as defined below.

Definition 1.25. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

Example 1.26. The function $f(x, y) = y$ is smooth on H^2 but $f(x, y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Remark 1.27. If M is an n-manifold with boundary, then $\operatorname{Int} M$ is a usual n-manifold (without boundary). Also, ∂M is an $n-1$-manifold without boundary. This is sometimes phrased as the equation

$$\partial^2 = 0. \tag{22}$$

Example 1.28 (Möbius strip)**.** Consider the quotient of $\mathbb{R} \times [0, 1]$ by the identification $(x, y) \sim (x + 1, 1 - y)$. The result E is a manifold with boundary. It is also a fiber bundle over S^1, via the map $\pi : [(x, y)] \mapsto e^{2\pi i x}$. The boundary, ∂E, is isomorphic to S^1, so this provides us with our first example of a non-trivial fiber bundle, since the trivial fiber bundle $S^1 \times [0, 1]$ has disconnected boundary.

1.4 Cobordism

Compact $(n+1)$-Manifolds with boundary provide us with a natural equivalence relation on compact n-manifolds, called *cobordism*.

Definition 1.29. Compact n-manifolds M_1, M_2 are *cobordant* when there exists N, a compact $n+1$-manifold with boundary, such that ∂N is isomorphic to the disjoint union $M_1 \sqcup M_2$. All manifolds cobordant to M form the *cobordism class* of M. We say that M is null-cobordant if $M = \partial N$ for N a compact $n + 1$–manifold with boundary.

Remark 1.30. It is important to assume compactness, otherwise all manifolds are null-cobordant, by taking Cartesian product with the noncompact manifold with boundary $[0, 1)$.

Let Ω^n be the set of cobordism classes of compact n-manifolds, including the empty set $\varnothing$ as a compact n-manifold. Using the disjoint union operation $[M_1] + [M_2] = [M_1 \sqcup M_2]$, we see that Ω^n is an abelian group with identity $[\varnothing]$. The additive inverse of $[M]$ is actually $[M]$ itself:

Proposition 1.31. *The cobordism ring is 2-torsion, i.e.* $x + x = 0 \ \ \forall x$.

Proof. For any manifold M, the manifold with boundary $M \times [0, 1]$ has boundary $M \sqcup M$. Hence $[M] + [M] = [\varnothing] = 0$, as required. □

The direct sum $\Omega^\bullet = \oplus_{n \geq 0} \Omega^n$ is then endowed with another operation,

$$[M_1] \cdot [M_2] = [M_1 \times M_2], \tag{23}$$

rendering $\Omega^\bullet$ into a commutative ring, called the *cobordism ring*. It has a multiplicative unit $[*]$, the class of the 0-manifold consisting of a single point. It is also graded by dimension.

Example 1.32. The n-sphere S^n is null-cobordant (i.e. cobordant to $\varnothing$), since $\partial \overline{B_{n+1}(0, 1)} \cong S^n$, where $B_{n+1}(0, 1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.

Example 1.33. Any oriented compact 2-manifold is null-cobordant: we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary.

We now state an amazing theorem of Thom, which is a complete description of the cobordism ring of smooth compact n-manifolds.

Theorem 1.34. *The cobordism ring is a (countably generated) polynomial ring over* $\mathbb{F}_2$ *with generators in every dimension* $n \neq 2^k - 1$*, i.e.*

$$\Omega^\bullet = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots]. \tag{24}$$

This theorem implies that there are 3 cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2 + x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives?

Remark 1.35. Thom showed that for k even we can take $x_k = [\mathbb{R}P^k]$. Dold showed that the family of manifolds

$$P(m, n) = (S^m \times \mathbb{C}P^n)/((x, y) \sim (-x, \bar{y})),$$

and showed that for $k = 2^r(2s + 1) - 1$, we can take $x_k = [P(2^r - 1, s2^r)]$.

Remark 1.36. Two manifolds are cobordant if and only if their Stiefel-Whitney characteristic numbers are the same. These numbers are built out of the Stiefel-Whitney classes, which are topological invariants associated to the tangent bundle of a manifold.

1.5 Smooth maps

For topological manifolds M, N of dimension m, n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous, we obtain a "category" of topological manifolds and continuous maps.

A category is a collection of objects $\mathcal{C}$ (in our case, topological manifolds) and a collection of arrows $\mathcal{A}$ (in our case, continuous maps). Each arrow goes from an object (the source) to another object (the target), meaning that there are "source" and "target" maps from $\mathcal{A}$ to $\mathcal{C}$:

$$\mathcal{A} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C} \tag{25}$$

Also, a category has an identity arrow for each object, given by a map $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{A}$ (in our case, the identity map of any manifold to itself). Furthermore, there is an associative composition operation on arrows.

Conventionally, we write the set of arrows from X to X as $\mathrm{Hom}(X, Y)$, i.e.

$$\mathrm{Hom}(X, Y) = \{a \in \mathcal{A} \ : \ s(a) = X \text{ and } t(a) = Y\}. \tag{26}$$

Then the associative composition of arrows mentioned above becomes a map

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z). \tag{27}$$

We have described the category of topological manifolds; we now describe the category of smooth manifolds by defining the notion of a smooth map.

Definition 1.37. A continuous map $f : M \to N$ is called *smooth* when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth map where it is defined, i.e. from the open set $\varphi(f^{-1}(V))$ to $\mathbb{R}^n$:

The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M, N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

Proposition 1.38. *If $g : L \to M$ and $f : M \to N$ are smooth maps, then so is the composition $f \circ g$.*

Proof. If charts φ, χ, ψ for L, M, N are chosen near $p \in L$, $g(p) \in M$, and $(fg)(p) \in N$, then $\psi \circ (f \circ g) \circ \varphi^{-1} = A \circ B$, for $A = \psi f \chi^{-1}$ and $B = \chi g \varphi^{-1}$ both smooth mappings $\mathbb{R}^n \to \mathbb{R}^n$. By the chain rule, $A \circ B$ is differentiable at p, with derivative $D_{\phi(p)}(A \circ B) = (D_{\chi(g(p))}A)(D_{\phi(p)}B)$ (matrix multiplication). $\square$

Now we have a new category, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic. In fact, the definitions above carry over, word for word, to the setting of manifolds with boundary. Hence we have defined another category, the category of smooth manifolds with boundary.

In defining the arrows for the category of manifolds with boundary, we may choose to consider all smooth maps, or only those smooth maps which send the boundary to the boundary, i.e. boundary-preserving maps.

The operation ∂ of "taking the boundary" sends a manifold with boundary to a usual manifold. Furthermore, if $\psi : M \to N$ is a boundary-preserving smooth map, then we can "take its boundary" by restricting it to the boundary, i.e. $\partial\psi = \psi|_{\partial M}$. Since ∂ takes objects to objects and arrows to arrows in a manner which respects compositions and identity maps, it is called a "functor" from the category of manifolds with boundary (and boundary-preserving smooth maps) to the category of smooth manifolds.

Example 1.39. The smooth inclusion $j : S^1 \to \mathbb{C}$ induces a smooth inclusion $j \times j$ of the 2-torus $T^2 = S^1 \times S^1$ into $\mathbb{C}^2$. The image of $j \times j$ does not include zero, so we may compose with the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ and the diffeomorphism $\mathbb{C}P^1 \to S^2$, to obtain a smooth map

$$\pi \circ (j \times j) : T^2 \to S^2. \tag{28}$$

Remark 1.40 (Exotic smooth structures)**.** The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called *exotic* spheres.

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a *monoid* structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are 28 smooth spheres, which we will encounter later on.

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called *fake* $\mathbb{R}^4$s.

Remark 1.41. The maps $\alpha : x \mapsto x$ and $\beta : x \mapsto x^3$ are both homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. Each one defines, by itself, a smooth atlas on $\mathbb{R}$. These two smooth atlases are not compatible (why?), so they do not define the same smooth structure on $\mathbb{R}$. Nevertheless, the smooth structures are equivalent, since there is a diffeomorphism taking one to the other. What is it?

Example 1.42 (Lie groups)**.** A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a *topological group*. If G is a given a smooth structure and m, ι are smooth maps, the result is a *Lie group*.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their Cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called *left multiplication* and *right multiplication*. Note that the group axioms imply that $R_g L_h = L_h R_g$.