2 The derivative

The derivative of a smooth map is an absolutely central topic in differential geometry. To make sense of the derivative, however, we must introduce the notion of tangent vector and, further, the space of all tangent vectors, known as the tangent bundle. In this section, we describe the tangent bundle intrinsically, without reference to any embedding of the manifold in a vector space. The definition of the tangent bundle is simplest for an open subset $U \subset \mathbb{R}^n$. In this case, a tangent vector to a point $p \in U$ is simply a vector in $\mathbb{R}^n$, and so the tangent bundle, which consists of all tangent vectors to all points in U, is simply given by

$$TU = U \times \mathbb{R}^n. \tag{29}$$

We now investigate the problem of generalizing the tangent bundle to other manifolds, where the convenience of being an open set in a vector space is not available.

2.1 The tangent bundle

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M. As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

We may define a tangent vector v is as an equivalence class of smooth curves. Let a smooth curve through p be a smooth map $\gamma : I \to M$ from an open interval around zero $I \subset \mathbb{R}$ to the manifold M, such that $\gamma(0) = p$. Then we say two such curves γ_1, γ_2 are equivalent when they have the same *velocity* at p, which we take to mean the following: in a chart (U, φ) containing p, we have

$$\tfrac{d}{dt}\Big|_{t=0}(\varphi \circ \gamma_1) = \tfrac{d}{dt}\Big|_{t=0}(\varphi \circ \gamma_2).$$

Note that the above differentiation makes sense since $\varphi \circ \gamma_i$ are maps between Euclidean spaces, which we know how to differentiate. Also note that if this condition holds in one chart, then it clearly holds in any other chart, by the chain rule.

Inspired by the above definition, which uses charts to make sense of the derivative of a curve, we now present an alternative definition which emphasizes the importance of the charts and makes it more clear how tangent spaces at different points may be unified to obtain a single tangent bundle. We use as main ingredient the definition (29) of the tangent bundle of an open set in Euclidean space.

Definition 2.1. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM. It is a real vector space of dimension $\dim M$, since both $T_{\varphi(p)}\varphi(U)$ and $T_{\psi(p)}\psi(V)$ are, and $D(\psi \circ \varphi^{-1})$ is a linear isomorphism.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_p M, \tag{30}$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_x M$.

We now give it a manifold structure in a natural way.

Proposition 2.2. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U) \tag{31}$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n, \tag{32}$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

Remark 2.3. This is a more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \;\Leftrightarrow\; y = \varphi_j \circ \varphi_i^{-1}(x) \;\text{ and }\; v = D(\varphi_j \circ \varphi_i^{-1})_x u, \tag{33}$$

and verify the conditions of the general gluing construction 1.14. The choice of a different atlas yields a canonically diffeomorphic manifold.

2.2 The derivative

A description of the tangent bundle is not complete without defining the derivative of a general smooth map of manifolds $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha}$ (defined where the composition makes sense) which satisfy (again, at all points where the composition is defined)

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1}). \tag{34}$$

Differentiating, we obtain

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1}). \tag{35}$$

Equation 35 shows that $Df_{i\alpha}$ and $Df_{j\beta}$ glue together to define a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Df : TM \longrightarrow TN$. Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \tag{36}$$

Each fiber $\pi^{-1}(x) = T_xM \subset TM$ is a vector space, and the map $Df : T_xM \longrightarrow T_{f(x)}N$ is a linear map. In fact, (f, Df) defines a homomorphism of vector bundles from TM to TN.

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Df \circ Dg = Dh$. As a result, we obtain the following category-theoretic statement.

Proposition 2.4. *The mapping T which assigns to a manifold M its tangent bundle TM, and which assigns to a map $f : M \longrightarrow N$ its derivative $Df : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself*[1].

For this reason, the derivative map Df is sometimes called the "tangent mapping" Tf.

---

[1] We can also say that it is a functor from manifolds to the category of smooth vector bundles.