2.5 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U,φ), (V,ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi\circ f\circ\varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem.*

Theorem 2.15 (Inverse function theorem). *Let $f : (M,p) \to (N,q)$ be a smooth map of n-dimensional manifolds and suppose that $Df(p) : T_pM \to T_qN$ is invertible. Then f has a local smooth inverse. That is, there are neighbourhoods U, V of p, q and a smooth map $g : V \to U$ such that $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$.*

This theorem provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.16 (Constant rank theorem). *Let $f : M^m \to N^n$ be a smooth map such that Df has constant rank k in a neighbourhood of $p \in M$. Then there are charts (U,φ) and (V,ψ) containing $p, f(p)$ such that*

$$\psi\circ f\circ\varphi^{-1} : (x_1,\ldots,x_m) \mapsto (x_1,\ldots,x_k,0,\ldots,0). \tag{42}$$

Proof. Begin by choosing charts so that without loss of generality M is an open set in $\mathbb{R}^m$ and N is $\mathbb{R}^n$.

Since $\mathrm{rk}\, Df = k$ at p, there is a $k\times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1,\ldots,x_k,y_1,\ldots y_{m-k})$ on the domain and $(u_1,\ldots u_k,v_1,\ldots,v_{n-k})$ on the codomain.

Then we may write $f(x,y) = (Q(x,y), R(x,y))$, where Q is the projection to $u = (u_1,\ldots,u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x,y) = (Q(x,y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix} \tag{43}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x,y) = (A(x,y), B(x,y))$. Since it's an inverse this means $(x,y) = \phi(\phi^{-1}(x,y)) = (Q(A,B),B)$, which implies that $B(x,y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, S = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k\times k} & 0 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \end{pmatrix} \tag{44}$$

we conclude that $\frac{\partial S}{\partial y} = 0$, meaning that we have eliminated the y-dependence:

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)). \tag{45}$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0), \tag{46}$$

as required. □

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining submanifolds.

There are several ways to define the notion of submanifold. We will use a definition which works for topological and smooth manifolds, based on the local model of inclusion of a vector subspace. These are sometimes called *regular* or *embedded* submanifolds.

Definition 2.17. A subspace $L \subset M$ of an m-manifold is called a submanifold of codimension k when each point $x \in L$ is contained in a chart (U, φ) for M such that

$$L \cap U = f^{-1}(0), \tag{47}$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \ldots, x_m)$. A submanifold of codimension 1 is usually called a *hypersurface*.

Proposition 2.18. *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k submanifold. □

Example 2.19. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.22.

The previous example leads to the following special case.

Proposition 2.20. *If $f : M \longrightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to* $\dim N$ *along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Definition 2.21. If $f : M \longrightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \varnothing$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 2.22. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*.

Proposition 2.23. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $f(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Restricting the coordinates to V', we see that $f(M)$ is cut out by $(x_{m+1}, \ldots, x_n)$, where $n = \dim N$. $\square$

Example 2.24. If $\iota : M \longrightarrow N$ is an embedding of M into N, then $D\iota : TM \longrightarrow TN$ is also an embedding (hence so are $D^k\iota : T^kM \longrightarrow T^kN$), showing that TM is a submanifold of TN.