3.4 Genericity

Theorem 3.26 (Transversality theorem)**.** *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds where only X has boundary. Suppose that F and ∂F are transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ and ∂f_s are transverse to g.*

Proof. Due to the transversality, the fiber product $W = (X \times S) \times_Y Z$ is a submanifold (with boundary) of $X \times S \times Z$ and projects to S via the usual projection map π. We show that any $s \in S$ which is a regular value for both the projection map $\pi : W \longrightarrow S$ and its boundary map $\partial\pi$ gives rise to a f_s which is transverse to g. Then by Sard's theorem the s which fail to be regular in this way form a set of measure zero.

Suppose that $s \in S$ is a regular value for π. Suppose that $f_s(x) = g(z) = y$ and we now show that f_s is transverse to g there. Since $F(x, s) = g(z)$ and F is transverse to g, we know that

$$\mathrm{im} DF_{(x,s)} + \mathrm{im} Dg_z = T_y Y.$$

Therefore, for any $a \in T_y Y$, there exists $b = (w, e) \in T(X \times S)$ with $DF_{(x,s)} b - a$ in the image of Dg_z. But since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)} W$. Hence we observe that

$$(Df_s)(w-w')-a = DF_{(x,s)}[(w,e)-(w',e)]-a = (DF_{(x,s)}b-a)-DF_{(x,s)}(w',e),$$

where both terms on the right hand side lie in $\mathrm{im} Dg_z$, since $(w', e, c') \in T_{(x,y,z)} W$ means $Dg_z(c') = DF_{(x,y)}(w', e)$.

Precisely the same argument (with X replaced with ∂X and F replaced with ∂F) shows that if s is regular for $\partial\pi$ then ∂f_s is transverse to g. This gives the result. □

The previous result immediately shows that transversal maps to $\mathbb{R}^n$ are generic, since for any smooth map $f : M \longrightarrow \mathbb{R}^n$ we may produce a family of maps

$$F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{73}$$

via $F(x, s) = f(x) + s$. This new map F is clearly a submersion and hence is transverse to any smooth map $g : Z \longrightarrow \mathbb{R}^n$. For arbitrary target manifolds, we will imitate this argument, but we will require a (weak) version of Whitney's embedding theorem for manifolds into $\mathbb{R}^n$.

In the next section we will show that any manifold Y can be embedded via $\iota : Y \to \mathbb{R}^N$ in some large Euclidean space, and in such a way that the image has a "tubular neighbourhood" $U \subset \mathbb{R}^N$ of radius $\epsilon(y)$ (for a positive real-valued function $\epsilon : Y \to \mathbb{R}$) equipped with a projection $\pi : U \to Y$ such that $\pi\iota = \mathrm{id}_Y$.

Corollary 3.27. *Let X be a manifold with boundary and $f : X \longrightarrow Y$ be a smooth map to a manifold Y. Then there is an open ball $S = B(0, 1) \subset \mathbb{R}^N$ and a smooth map $F : X \times S \longrightarrow Y$ such that $F(x, 0) = f(x)$ and for fixed x, the map $f_x : s \mapsto F(x, s)$ is a submersion $S \longrightarrow Y$.*

In particular, F and ∂F are submersions, so are transverse to any $g : Z \to Y$.

Proof. Use the embedding of $\iota : Y \to \mathbb{R}^N$ and the tubular neighbourhood $\pi : U \to Y$ to define

$$F(x,s) = \pi(\iota(f(x)) + \epsilon(y)s). \tag{74}$$

□

The transversality theorem then guarantees that given any smooth $g : Z \longrightarrow Y$, for almost all $s \in S$ the maps $f_s, \partial f_s$ are transverse to g. We improve this slightly to show that f_s may be chosen to be *homotopic* to f.

Corollary 3.28 (Transversality homotopy theorem)**.** *Given any smooth maps $f_0 : X \longrightarrow Y$, $g : Z \longrightarrow Y$, where only X has boundary, there exists a smooth map $f_1 : X \longrightarrow Y$ homotopic to f_0 with $f_1, \partial f_1$ both transverse to g.*

Proof. Let S, F be as in the previous corollary. Away from a set of measure zero in S, the functions $f_s, \partial f_s$ are transverse to g, by the transversality theorem. But these f_s are all homotopic to f via the homotopy $X \times [0,1] \longrightarrow Y$ given by

$$(x,t) \mapsto F(x,ts). \tag{75}$$

□

The last theorem we shall prove concerning transversality is a very useful extension result which is essential for intersection theory:

Theorem 3.29 (Homotopic transverse extension of boundary map)**.** *Let X be a manifold with boundary and $f_0 : X \longrightarrow Y$ a smooth map to a manifold Y. Suppose that ∂f_0 is transverse to the closed map $g : Z \longrightarrow Y$. Then there exists a map $f_1 : X \longrightarrow Y$, homotopic to f and with $\partial f_1 = \partial f_0$, such that f_1 is transverse to g.*

Proof. First observe that since ∂f_0 is transverse to g on ∂X, f_0 is also transverse to g there, and furthermore since g is closed, f_0 is transverse to g in a neighbourhood U of ∂X. (for example, if $x \in \partial X$ but x not in $f_0^{-1}(g(Z))$ then since the latter set is closed, we obtain a neighbourhood of x for which f_0 is transverse to g.)

Now choose a smooth function $\gamma : X \longrightarrow [0,1]$ which is 1 outside U but 0 on a neighbourhood of ∂X. (why does γ exist? exercise.) Then set $\tau = \gamma^2$, so that $d\tau(x) = 0$ wherever $\tau(x) = 0$. Recall the map $F : X \times S \longrightarrow Y$ we used in proving the transversality homotopy theorem and modify it via

$$G(x,s) = F(x, \tau(x)s). \tag{76}$$

The claim is that G and ∂G are transverse to g. This is clear for x such that $\tau(x) \neq 0$. But if $\tau(x) = 0$,

$$TG_{(x,s)}(v,w) = TF_{(x,0)}(v,0) = T(f_0)_x(v), \tag{77}$$

but $\tau(x) = 0$ means that $x \in U$, in which f is transverse to g.

Since transversality holds, there exists s such that $f_1 : x \mapsto G(x,s)$ and ∂f_1 are transverse to g (and homotopic to f_0, as before). Finally, if x is in the neighbourhood of ∂X for which $\tau = 0$, then $f_1(x) = F(x,0) = f_0(x)$. □

Corollary 3.30. *If $f_0 : X \longrightarrow Y$ and $f_1 : X \longrightarrow Y$ are homotopic smooth maps of manifolds, each transverse to the closed map $g : Z \longrightarrow Y$, then the fiber products $W_0 = X_{f_0}\times_g Z$ and $W_1 = X_{f_1}\times_g Z$ are cobordant.*

Proof. if $F : X \times [0,1] \longrightarrow Y$ is the homotopy between f_0, f_1, then by the previous theorem, we may find a (homotopic) homotopy $G : X \times [0,1] \longrightarrow Y$ which is transverse to g, without changing F on the boundary. Hence the fiber product $U = (X \times [0,1])_G\times_g Z$ is a cobordism with boundary $W \sqcup W'$. □

3.5 Intersection theory

The previous corollary allows us to make the following definition:

Definition 3.31. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X and Z compact, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \#(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g.

Example 3.32. If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.

Example 3.33. If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle\cos(\theta/2)e_1 + \sin(\theta/2)e_2\rangle$ and $\iota_2 : \theta \mapsto \langle\cos(\theta/2)e_2 + \sin(\theta/2)e_3\rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2\rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points.

Example 3.34. Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2\dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.

Example 3.35. Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.

Example 3.36. By the previous argument, any compact manifold is not contractible.

Example 3.37. Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 3.38. Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-homotopic, we cannot deform the planes to remove all intersection.

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 3.39. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

Example 3.40. Let $f : S^1 \longrightarrow S^1$ be given by $z \mapsto z^k$. Then $\deg_2(f) = k \pmod 2$.

Example 3.41. If $p : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a polynomial of degree k, then as a map $S^2 \longrightarrow S^2$ we have $\deg_2(p) = k \pmod 2$, and hence any odd polynomial has at least one root. To get the fundamental theorem of algebra, we must consider *oriented cobordism*

Even if submanifolds C, C' do not intersect, it may be that there are more sophisticated geometrical invariants which cause them to be "intertwined" in some way. One example of this is linking number.

Definition 3.42. Suppose that $M, N \subset \mathbb{R}^{k+1}$ are compact embedded submanifolds with $\dim M + \dim N = k$, and let us assume they are transverse, meaning they do not intersect at all.

Then define $\lambda : M \times N \longrightarrow S^k$ via

$$(x, y) \mapsto \frac{x - y}{|x - y|}.$$

Then we define the $\pmod 2$ linking number of M, N to be $\deg_2(\lambda)$.

Example 3.43. Consider the standard Hopf link in $\mathbb{R}^3$. Then it is easy to calculate that $\deg_2(\lambda) = 1$. On the other hand, the standard embedding of disjoint circles (differing by a translation, say) has $\deg_2(\lambda) = 0$. Hence it is impossible to deform the circles through embeddings of $S^1 \sqcup S^1 \longrightarrow \mathbb{R}^3$, so that they are unlinked. Why must we stay within the space of embeddings, and not allow the circles to intersect?