4.2 Vector fields vs. derivations

The space $C^\infty(M,\mathbb{R})$ of smooth functions on M is not only a vector space but also a ring, with multiplication $(fg)(p) := f(p)g(p)$. Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$, given by $f \mapsto f \circ \varphi$. This is called the pullback of functions, and defines a homomorphism of rings.

The association $M \mapsto C^\infty(M,\mathbb{R})$ and $\varphi \mapsto \varphi^*$ is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M,\mathbb{R})$, but actually all automorphisms are of this form (Exercise!).

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 4.12. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$.

If automorphisms of $C^\infty(M,\mathbb{R})$ correspond to diffeomorphisms, then it is natural to ask what derivations correspond to. We now show that they correspond to vector fields.

The vector fields $\mathfrak{X}(M)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless M is a finite set). They also form a module over the ring of smooth functions $C^\infty(M,\mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M,\mathbb{R})$ and $X \in \mathfrak{X}(M)$, $fX : x \mapsto f(x)X(x)$ is a smooth vector field.

The important property of vector fields which we are interested in is that they act as derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i (\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M,\mathbb{R})$.

Definition 4.13. The differential of a function $f \in C^\infty(M,\mathbb{R})$ is the function on TM given by composing $Tf : TM \to T\mathbb{R}$ with the second projection $p_2 : T\mathbb{R} = \mathbb{R}\times\mathbb{R} \to \mathbb{R}$:

$$df = p_2 \circ Tf \tag{78}$$

To remove any confusion, df evaluates at the point $(x, v) \in TM$ to give the derivative of f at x in the direction v:

$$df(x, v) = Df|_x(v).$$

Definition 4.14. Let X be a vector field. Then we define

$$X(f) = df \circ X.$$

This is called the directional (or Lie) derivative of f along X.

In coordinates, if $X = \sum a_i \partial/\partial x_i$, then $X(f) = \sum a_i \partial f/\partial x_i$, coinciding with the usual directional derivative mentioned above. This shows that $f \mapsto X(f)$ has the derivation property (since it satisfies it locally), but we can alternatively see that it is a derivation by using the property

$$d(fg) = fdg + gdf$$

of the differential of a product (here fdg is really $(\pi^* f)dg$).

Theorem 4.15. *The map $X \mapsto (f \mapsto X(f))$ is an isomorphism*

$$\mathfrak{X}(M) \to Der(C^\infty(M, \mathbb{R})).$$

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y))g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i)g_i(z) - \sum_i (x^i(z) - x^i(y))Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i an associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p)D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set W, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside W; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero in W).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. $\square$