1

Introduction

1.1 Some history

In the words of S.S. Chern, "*the fundamental objects of study in differential geometry are manifolds.*" [1] Roughly, an n-dimensional manifold is a mathematical object that "locally" looks like $\mathbb{R}^n$. The theory of manifolds has a long and complicated history. For centuries, manifolds have been studied as subsets of Euclidean space, given for example as level sets of equations. The term 'manifold' goes back to the 1851 thesis of Bernhard Riemann, *"Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"* ("foundations for a general theory of functions of a complex variable") and his 1854 habilitation address *"Über die Hypothesen, welche der Geometrie zugrunde liegen"* ("on the hypotheses underlying geometry").

However, in neither reference does Riemann make an attempt to give a precise definition of the concept. This was done subsequently by many authors, including Riemann himself. [2] Henri Poincaré in his 1895 work *analysis situs*, introduces the idea of a *manifold atlas*.

The first rigorous axiomatic definition of manifolds was given by Veblen and Whitehead only in 1931.

We will see below that the concept of a manifold is really not all that complicated; and in hindsight it may come as a bit of a surprise that it took so long to evolve. Quite possibly, one reason is that for quite a while, the concept as such was mainly regarded as just a change of perspective (away from level sets in Euclidean spaces, towards the 'intrinsic' notion of manifolds). Albert Einstein's theory of General Relativity from 1916 gave a major boost to this new point of view; In his theory, space-time is regarded as a 4-dimensional 'curved' manifold with no distinguished coordinates (not even a distinguished separation into 'space' and 'time'); a local observer may want to introduce local *xyzt* coordinates to perform measurements, but all physically meaningful quantities must admit formulations that are coordinate-free. At the same time, it would seem unnatural to try to embed the 4-dimensional curved space-time continuum into some higher-dimensional flat space, in the absence of any physical significance for the additional dimensions. Some years later, *gauge theory* once again emphasized coordinate-free formulations, and provided physics-based motivations for more elaborate constructions such as fiber bundles and connections.

Since the late 1940s and early 1950s, differential geometry and the theory of manifolds has developed with breathtaking speed. It has become part of the basic education of any mathematician or theoretical physicist, and with applications in other areas of science such as engineering and economics. There are many sub-branches, for example complex geometry, Riemannian geometry, and symplectic geometry, which further subdivide into sub-sub-branches.

1.2 The concept of manifolds: Informal discussion

To repeat, an n-dimensional manifold is something that "locally" looks like $\mathbb{R}^n$. The prototype of a manifold is the surface of planet earth:

It is (roughly) a 2-dimensional sphere, but we use local charts to depict it as subsets of 2-dimensional Euclidean spaces. [3]

To describe the entire planet, one uses an atlas with a collection of such charts, such that every point on the planet is depicted in at least one such chart.

This idea will be used to give an 'intrinsic' definition of manifolds, as essentially a collection of charts glued together in a consistent way. One can then try to develop analysis on such manifolds – for example, develop a theory of integration and differentiation, consider ordinary and partial differential equations on manifolds, by working in charts; the task is then to understand the 'change of coordinates' as one leaves the domain of one chart and enters the domain of another.

1.3 Manifolds in Euclidean space

In multivariable calculus, you will have encountered manifolds as solution sets of equations. For example, the solution set of an equation of the form $f(x,y,z) = a$ in $\mathbb{R}^3$ defines a 'smooth' hypersurface $S \subseteq \mathbb{R}^3$ provided the gradient of f is non-vanishing at all points of S. We call such a value of f a *regular value*, and hence $S = f^{-1}(a)$ a *regular level set*. Similarly, the joint solution set C of two equations

$$f(x,y,z) = a, \quad g(x,y,z) = b$$

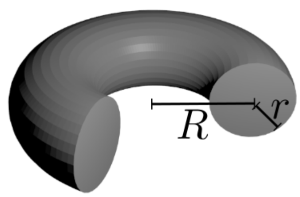
defines a smooth curve in $\mathbb{R}^3$, provided (a,b) is a regular value of (f,g) in the sense that the gradients of f and g are linearly independent at all points of C. A familiar example of a manifold is the 2-dimensional sphere S^2, conveniently described as a level surface inside $\mathbb{R}^3$:

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}.$$

There are many ways of introducing local coordinates on the 2-sphere: For example, one can use spherical polar coordinates, cylindrical coordinates, stereographic projections, or orthogonal projections onto the coordinate planes. We will discuss some of these coordinates below. More generally*, one has the n-dimensional sphere S^n inside $\mathbb{R}^{n+1}$,

$$S^n = \{(x^0,\dots,x^n) \in \mathbb{R}^{n+1} |\ (x^0)^2 + \cdots + (x^n)^2 = 1\}.$$

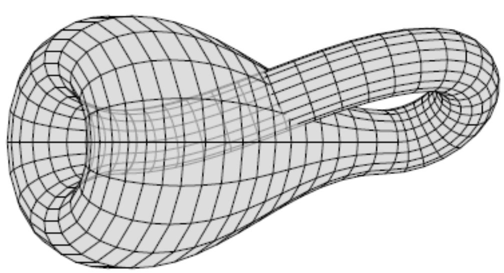
The 0-sphere S^0 consists of two points, the 1-sphere S^1 is the *unit circle*. Another example is the *2-torus*, T^2. It is often depicted as a surface of revolution: Given real numbers r,R with $0 < r < R$, take a circle of radius r in the $x-z$ plane, with center at $(R,0)$, and rotate about the z-axis.



The resulting surface is given by an equation,

$$T^2 = \{(x,y,z) |\ \big(\sqrt{x^2+y^2} - R\big)^2 + z^2 = r^2\}. \tag{1.1}$$

Not all surfaces can be realized as 'embedded' in $\mathbb{R}^3$; for non-orientable surfaces one needs to allow for self-intersections. This type of realization is referred to as an *immersion*: We don't allow edges or corners, but we do allow that different parts of the surface pass through each other. An example is the *Klein bottle*



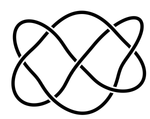
* Note that we adopt the superscript notation for indices, so that a point in say $\mathbb{R}^4$ is writen as (x^1,x^2,x^3,x^4). The justification for this convention has to do with tensor calculus which is beyond the scope of the current book.

The Klein bottle is an example of a *non-orientable surface*: It has only one side. (In fact, the Klein bottle contains a Möbius band – see exercises.) It is not possible to represent it as a regular level set $f^{-1}(0)$ of a function f: For any such surface one has one side where f is positive, and another side where f is negative.

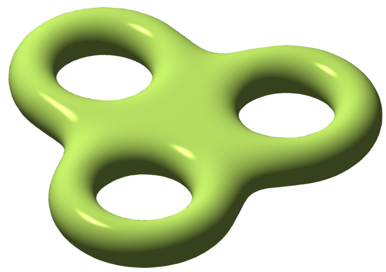
1.4 Intrinsic descriptions of manifolds

In this course, we will mostly avoid concrete embeddings of manifolds into any $\mathbb{R}^N$. Here, the term 'embedding' is used in an intuitive sense, for example as the realization as the level set of some equations. (Later, we will give a precise definition.) There are a number of reasons for why we prefer developing an 'intrinsic' theory of manifolds.

1. Embeddings of simple manifolds in Euclidean space can look quite complicated. The following one-dimensional manifold



is intrinsically, 'as a manifold', just a closed curve, that is, a circle. The problem of distinguishing embeddings of a circle into $\mathbb{R}^3$ is one of the goals of *knot theory*, a deep and difficult area of mathematics.
2. Such complications disappear if one goes to higher dimensions. For example, the above knot (and indeed any knot in $\mathbb{R}^3$) can be disentangled inside $\mathbb{R}^4$ (with $\mathbb{R}^3$ viewed as a subspace). Thus, in $\mathbb{R}^4$ they become *unknots*.
3. The intrinsic description is sometimes much simpler to deal with than the extrinsic one. For instance, the equation describing the torus $T^2 \subseteq \mathbb{R}^3$ is not especially simple or beautiful. But once we introduce the following parametrization of the torus
$$x = (R + r\cos\varphi)\cos\theta, \ \ y = (R + r\cos\varphi)\sin\theta, \ \ z = r\sin\varphi,$$
where θ, φ are determined up to multiples of 2π, we recognize that T^2 is simply a product:
$$T^2 = S^1 \times S^1. \tag{1.2}$$
That is, T^2 consists of ordered pairs of points on the circle, with the two factors corresponding to θ, φ. In contrast to (1.1), there is no distinction between 'small' circle (of radius r) and 'large circle' (of radius R). The new description suggests an embedding of T^2 into $\mathbb{R}^4$ which is 'nicer' then the one in $\mathbb{R}^3$. (But does it help?)
4. Often, there is no natural choice of an embedding of a given manifold inside $\mathbb{R}^N$, at least not in terms of concrete equations. For instance, while the triple torus

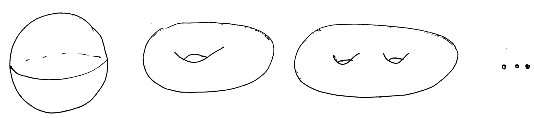


is easily pictured in 3-space $\mathbb{R}^3$, it is hard to describe it concretely as the level set of an equation.

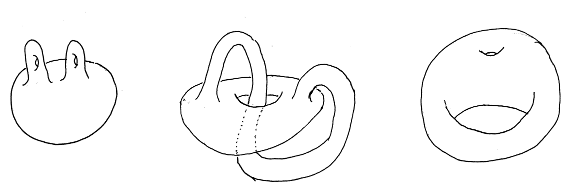
5. While many examples of manifolds arise naturally as level sets of equations in some Euclidean space, there are also many examples for which the initial construction is different. For example, the set M whose elements are all affine lines in $\mathbb{R}^2$ (that is, straight lines that need not go through the origin) is naturally a 2-dimensional manifold. But some thought is required to realize it as a surface in $\mathbb{R}^3$.

1.5 Surfaces

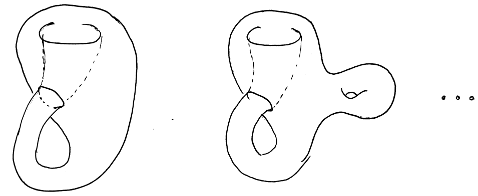
Let us briefly give a very informal discussion of *surfaces*. A surface is the same thing as a 2-dimensional manifold. We have already encountered some examples: The sphere, the torus, the double torus, triple torus, and so on:



All of these are 'orientable' surfaces, which essentially means that they have two sides which you might paint in two different colors. It turns out that these are *all* the orientable surfaces, if we consider the surfaces 'intrinsically' and only consider surfaces that are compact in the sense that they don't go off to infinity and do not have a boundary (thus excluding a cylinder, for example). For instance, each of the following drawings depicts a double torus:



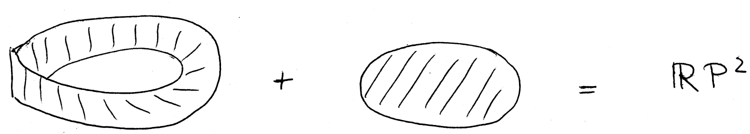
We also have one example of a non-orientable surface: The Klein bottle. More examples are obtained by attaching handles (just like we can think of the torus, double torus and so on as a sphere with handles attached).



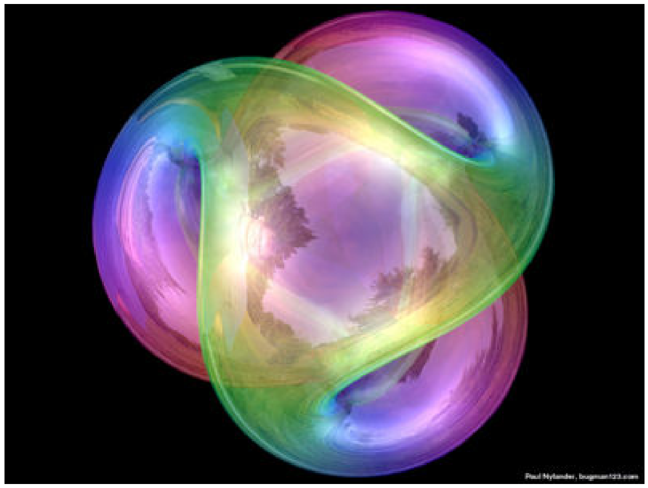
Are these *all* the non-orientable surfaces? In fact, the answer is *no*. We have missed what is in some sense the simplest non-orientable surface. Ironically, it is the surface which is hardest to visualize in 3-space. This surface is called the *projective plane* or *projective space*, and is denoted $\mathbb{R}\mathrm{P}^2$. One can define $\mathbb{R}\mathrm{P}^2$ as the set of all lines (i.e., 1-dimensional subspaces) in $\mathbb{R}^3$. It should be clear that this is a 2-dimensional manifold, since it takes 2 parameters to specify such a line. We can label such lines by their points of intersection with S^2, hence we can also think of $\mathbb{R}\mathrm{P}^2$ as the set of antipodal (i.e., opposite) points on S^2. In other words, it is obtained from S^2 by identifying antipodal points. To get a better idea of how $\mathbb{R}\mathrm{P}^2$ looks like, let us subdivide the sphere S^2 into two parts:

(i) points having distance $\leq \varepsilon$ from the equator,
(ii) points having distance $\geq \varepsilon$ from the equator.

If we perform the antipodal identification for (i), we obtain a Möbius strip. If we perform antipodal identification for (ii), we obtain a 2-dimensional disk (think of it as the points of (ii) lying in the upper hemisphere). Hence, $\mathbb{R}\mathrm{P}^2$ can also be regarded as gluing the boundary of a Möbius strip to the boundary of a disk:



Now, the question arises: *Is it possible to realize* $\mathbb{R}\mathrm{P}^2$ *smoothly as a surface inside* $\mathbb{R}^3$*, possibly with self-intersections* (similar to the Klein bottle)? Simple attempts of joining the boundary circle of the Möbius strip with the boundary of the disk will always create sharp edges or corners – try it. Around 1900, David Hilbert posed this problem to his student Werner Boy, who discovered that the answer is *yes*. The following picture of *Boy's surface* was created by Paul Nylander.



There are some nice videos illustrating the construction of the surface: See in particular

`https://www.youtube.com/watch?v=9gRx66xKXek`

and

`www.indiana.edu/~minimal/archive/NonOrientable/NonOrientable/Bryant-anim/web/`

While these pictures are very beautiful, it certainly makes the projective space appear more complicated than it actually is. If one is only interested in $\mathbb{R}P^2$ itself, rather than its realization as a surface in $\mathbb{R}^3$, it is much simpler to work with the definition (as a sphere with antipodal identification).

Going back to the classification of surfaces: It turns out that all closed, connected surfaces are obtained from either the 2-sphere S^2, the Klein bottle, or $\mathbb{R}P^2$, by attaching handles.

Remark 1.1. Another operation for surfaces, generalizing the procedure of 'attaching handles', is the *connected sum*. Given two surfaces Σ_1 and Σ_2, remove small disks around given points $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$, to create two surfaces with boundary circles. Then glue-in a cylinder connecting the two boundary circles, without creating edges. The resulting surface is denoted

$$\Sigma_1 \sharp \Sigma_2.$$

For example, the connected sum $\Sigma \sharp T^2$ is Σ with a handle attached. You may want to think about the following questions: What is the connected sum of two $\mathbb{R}P^2$'s? And what is the connected sum of $\mathbb{R}P^2$ with a Klein bottle? Both must be in the list of 2-dimensional surfaces given above.

Notes

[1]Page 332 of Chern, Chen, Lam: Lectures on Differential Geometry, World Scientific

[2]See e.g. the article by Scholz http://www.maths.ed.ac.uk/ aar/papers/scholz.pdf for the long list of names involved.

[3]Note that such a chart will always give a somewhat ‘distorted’ picture of the planet; the distances on the sphere are never quite correct, and either the areas or the angles (or both) are wrong. For example, in the standard maps of the world, Canada always appears somewhat bigger than it really is. (Even more so Greenland, of course.)