2

Manifolds

It is one of the goals of these lectures to develop the theory of manifolds in intrinsic terms, although we may occasionally use immersions or embeddings into Euclidean space in order to illustrate concepts. In physics terminology, we will formulate the theory of manifolds in terms that are 'manifestly coordinate-free'.

2.1 Atlases and charts

As we mentioned above, the basic feature of manifolds is the existence of 'local coordinates'. The transition from one set of coordinates to another should be *smooth*. We recall the following notions from multivariable calculus.

Definition 2.1. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. A map $F: U \to V$ is called* smooth *if it is infinitely differentiable. The set of smooth functions from U to V is denoted $C^\infty(U,V)$. The map F is called a* diffeomorphism *from U to V if it is invertible and the inverse map $F^{-1}: V \to U$ is again smooth.*

Example 2.1. The exponential map $\exp: \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x) = e^x$ is smooth. It may be regarded as a map onto $\mathbb{R}_{>0} = \{y | y > 0\}$, and is a diffeomorphism

$$\exp: \mathbb{R} \to \mathbb{R}_{>0}$$

with inverse $\exp^{-1} = \log$ (the natural logarithm). Similarly,

$$\tan: \{x \in \mathbb{R} | -\pi/2 < x < \pi/2\} \to \mathbb{R}$$

is a diffeomorphism, with inverse arctan.

Definition 2.2. *For a smooth map $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and any $x \in U$, one defines the* Jacobian matrix *$DF(x)$ to be the $n \times m$-matrix of partial derivatives*

$$DF(x) = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(x) & \cdots & \frac{\partial F^1}{\partial x^m}(x) \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x^1}(x) & \cdots & \frac{\partial F^n}{\partial x^m}(x) \end{pmatrix}$$

Its determinant is called the Jacobian determinant *of F at x. In the expression above, we are using the standard coordinates $(x^1,\ldots,x^m)$ on $\mathbb{R}^m$ and we view F as a vector-valued function with n components $F^1,\ldots,F^n$, each of which is a real-valued function $F^i : U \to \mathbb{R}$.*

Recall that the *inverse function theorem* states that if $F \in C^\infty(U,V)$ has invertible Jacobian at a point $x \in U$, then there is an open neighbourhood $U_x \subseteq U$ of x such that $F(U_x) \subseteq V$ is open and $F : U_x \to F(U_x)$ is a diffeomorphism. One also has a formula for the Jacobian of the inverse map F^{-1}: at the point $F(x)$ we have $(D(F^{-1}))(F(x)) = (DF(x))^{-1}$.

The following definition formalizes the concept of introducing local coordinates.

Definition 2.3 (Charts). *Let M be a set.*

1. *An m-dimensional* (coordinate) chart *(U,φ) on M is a subset $U \subseteq M$ together with a map $\varphi :\ U \to \mathbb{R}^m$, such that $\varphi(U) \subseteq \mathbb{R}^m$ is open and φ is a bijection from U to $\varphi(U)$.*
2. *Two charts (U,φ) and (V,ψ) are called* compatible *if the subsets $\varphi(U\cap V)$ and $\psi(U\cap V)$ are open, and the* transition map

$$\psi\circ\varphi^{-1} :\ \varphi(U\cap V) \to \psi(U\cap V)$$

is a diffeomorphism.

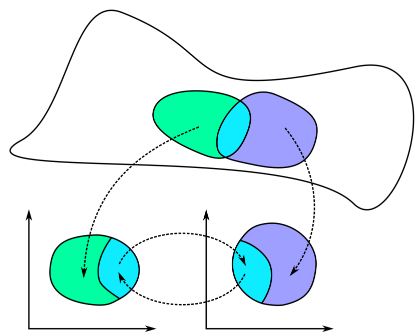
As a special case, charts with $U \cap V = \emptyset$ are always compatible.

Exercise 1. The bijection requirement on φ plays an important role. Prove the following (we shall use the properties below without further comment in future chapters):
Let $f : A \to B$, and suppose $C,D \subseteq A$, and $E,F \subseteq B$.

1. Show that $f(C\cup D) = f(C)\cup f(D)$.
2. Give an example where $f(C\cap D) \neq f(C)\cap f(D)$. Is there any relation between the two sides of the inequality? Likewise, for $f(C\setminus D) \neq f(C)\setminus f(D)$.
3. Show that if f is injective, then $f(C\cap D) = f(C)\cap f(D)$ and $f(C\setminus D) = f(C)\setminus f(D)$.

Exercise 2. Is compatibility of charts an equivalence relation?

Let (U,φ) be a coordinate chart. Given a point $p\in U$, and writing $\varphi(p) = (u^1,\dots,u^m)$, we say that the u^i are the *coordinates of* p in the given chart*. (Letting p vary, these become real-valued functions $p\mapsto u^i(p)$; they are simply the component functions of φ.) The transition maps $\psi\circ\varphi^{-1}$ amount to a *change of coordinates*. Here is a picture of a 'coordinate change':



Definition 2.4 (Atlas). *Let M be a set. An m-dimensional* atlas *on M is a collection of coordinate charts $\mathscr{A} = \{(U_\alpha,\varphi_\alpha)\}$ such that*

1. *The U_α cover all of M, i.e., $\bigcup_\alpha U_\alpha = M$.*
2. *For all indices α,β, the charts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β) are compatible.*

Example 2.2 (An atlas on the 2-sphere). Let $S^2\subseteq\mathbb{R}^3$ be the unit sphere, consisting of all $(x,y,z)\in\mathbb{R}^3$ satisfying the equation $x^2+y^2+z^2=1$. We shall define an atlas with two charts (U_+,φ_+) and (U_-,φ_-). Let $n=(0,0,1)$ be the north pole, let $s=(0,0,-1)$ be the south pole, and put

$$U_+ = S^2\setminus\{s\},\quad U_- = S^2\setminus\{n\}.$$

Regard $\mathbb{R}^2$ as the coordinate subspace of $\mathbb{R}^3$ on which $z=0$. Let

$$\varphi_+:\ U_+\to\mathbb{R}^2,\quad p\mapsto\varphi_+(p)$$

be *stereographic projection from the south pole*. That is, $\varphi_+(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and s. Similarly,

$$\varphi_-:\ U_-\to\mathbb{R}^2,\quad p\mapsto\varphi_-(p)$$

is *stereographic projection from the north pole*, where $\varphi_-(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and n. A picture of φ_-, with $p' = \varphi_-(p)$ (the picture uses capital letters):

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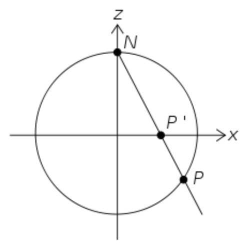
* Note the convention of indexing by superscripts; be careful not to confuse indices with powers.



Exercise 3. Let $p=(x,y,z)$. Find explicit formulas for $\varphi_+(x,y,z)$ and $\varphi_-(x,y,z)$.

Both $\varphi_\pm : U_\pm \to \mathbb{R}^2$ are bijections onto $\mathbb{R}^2$. Let us verify this in detail for the map φ_+. Given (u,v) we may solve the equation $(u,v)=\varphi_+(x,y,z)$, using the condition that $x^2+y^2+z^2=1$ and $z>-1$. One has

$$u^2+v^2=\frac{x^2+y^2}{(1+z)^2}=\frac{1-z^2}{(1+z)^2}=\frac{(1-z)(1+z)}{(1+z)^2}=\frac{1-z}{1+z},$$

from which one obtains

$$z=\frac{1-(u^2+v^2)}{1+(u^2+v^2)},$$

and since $x=u(1+z),\ y=v(1+z)$ one obtains

$$\varphi_+^{-1}(u,v)=\Big(\frac{2u}{1+(u^2+v^2)},\frac{2v}{1+(u^2+v^2)},\ \frac{1-(u^2+v^2)}{1+(u^2+v^2)}\Big).$$

For the map φ_-, we obtain by a similar calculation

$$\varphi_-^{-1}(u,v)=\Big(\frac{2u}{1+(u^2+v^2)},\frac{2v}{1+(u^2+v^2)},\ \frac{(u^2+v^2)-1}{1+(u^2+v^2)}\Big).$$

(Actually, it is also clear from the geometry that $\varphi_+^{-1},\varphi_-^{-1}$ only differ by the sign of the z-coordinate.) Note that $\varphi_+(U_+\cap U_-)=\mathbb{R}^2\backslash\{(0,0)\}$. The transition map on the overlap of the two charts is

$$(\varphi_-\circ\varphi_+^{-1})(u,v)=\Big(\frac{u}{u^2+v^2},\ \frac{v}{u^2+v^2}\Big)$$

which is smooth on $\mathbb{R}^2\backslash\{(0,0)\}$ as required. □

Here is another simple, but less familiar example where one has an atlas with two charts.

Example 2.3 (Affine lines in $\mathbb{R}^2$*).* A *line* in a vector space E is the same as a 1-dimensional subspace. By an *affine line*, we mean a subset $\ell\subseteq E$, such that the set of differences $\{v-w\,|\,v,w\in\ell\}$ is a 1-dimensional subspace. Put differently, ℓ is obtained by adding a fixed vector v_0 to all elements of a 1-dimensional subspace. In plain terms, an affine line is simply a straight line that does not necessarily pass through the origin.

Let M be a set of affine lines in $\mathbb{R}^2$. Let $U \subseteq M$ be the subset of lines that are not vertical, and $V \subseteq M$ the lines that are not horizontal. Any $\ell \in U$ is given by an equation of the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept. The map $\varphi:\ U \to \mathbb{R}^2$ taking ℓ to (m,b) is a bijection. On the other hand, lines in V are given by equations of the form

$$x = ny + c,$$

and we also have the map $\psi:\ V \to \mathbb{R}^2$ taking such ℓ to (n,c). The intersection $U \cap V$ are lines ℓ that are neither vertical nor horizontal. Hence, $\varphi(U \cap V)$ is the set of all (m,b) such that $m \neq 0$, and similarly $\psi(U \cap V)$ is the set of all (n,c) such that $n \neq 0$.

Exercise 4. Describe the transition maps $\psi \circ \varphi^{-1}$, $\varphi^{-1} \circ \psi$ and show they are smooth.

We conclude that U, V define an 2-dimensional atlas on M.

Question: What is the resulting surface?

As a first approximation, we may take an m-dimensional manifold to be a set with an m-dimensional atlas. This is almost the right definition, but we will make a few adjustments. A first criticism is that we may not want any *particular* atlas as part of the definition: For example, the 2-sphere with the atlas given by stereographic projections onto the xy-plane, and the 2-sphere with the atlas given by stereographic projections onto the yz-plane, should be one and the same manifold: S^2. To resolve this problem, we will use the following notion.

Definition 2.5. *Suppose* $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ *is an m-dimensional atlas on M, and let* (U, φ) *be another chart. Then* (U, φ) *is said to be* compatible *with* $\mathscr{A}$ *if it is compatible with all charts* $(U_\alpha, \varphi_\alpha)$ *of* $\mathscr{A}$.

Example 2.4. On the 2-sphere S^2, we had constructed the atlas

$$\mathscr{A} = \{(U_+, \varphi_+),\ (U_-, \varphi_-)\}$$

given by stereographic projection. Consider the chart (V, ψ), with domain V the set of all $(x,y,z) \in S^2$ such that $y < 0$, with $\psi(x,y,z) = (x,z)$. To check that it is compatible (U_+, φ_+), note that $U_+ \cap V = V$, and

$$\varphi_+(U_+ \cap V) = \{(u,v) |\ v < 0\}, \quad \psi(U_+ \cap V) = \{(x,z) |\ x^2 + z^2 < 1\}.$$

Exercise 5. Find explicit formulas for $\psi \circ \varphi_+^{-1}$ and $\varphi_+ \circ \psi^{-1}$. Conclude that (V, ψ) is compatible with (U_+, φ_+).

Note that (U, φ) is compatible with the atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ if and only if the union $\mathscr{A} \cup \{(U, \varphi)\}$ is again an atlas on M. This suggests defining a bigger atlas, by using *all* charts that are compatible with the given atlas. In order for this to work, we need the new charts to be compatible not only with the charts of $\mathscr{A}$, but also with each other.

Lemma 2.1. *Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a given atlas on the set M. If two charts (U, φ), (V, ψ) are compatible with $\mathscr{A}$, then they are also compatible with each other.*

Proof. For every chart U_α, the sets $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi_\alpha(V \cap U_\alpha)$ are open, hence their intersection is open. This intersection is

$$\varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(V \cap U_\alpha) = \varphi_\alpha(U \cap V \cap U_\alpha).$$

Since $\varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \to \varphi(U \cap U_\alpha)$ is a diffeomorphism, it follows that

$$\varphi(U \cap V \cap U_\alpha) = (\varphi \circ \varphi_\alpha^{-1})\big(\varphi_\alpha(U \cap V \cap U_\alpha)\big)$$

is open. Taking the union over all α, we see that

$$\varphi(U \cap V) = \bigcup_\alpha \varphi(U \cap V \cap U_\alpha)$$

is open. A similar argument applies to $\psi(U \cap V)$. The transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth since for all α, its restriction to $\varphi(U \cap V \cap U_\alpha)$ is a composition of two smooth maps $\varphi_\alpha \circ \varphi^{-1} : \varphi(U \cap V \cap U_\alpha) \longrightarrow \varphi_\alpha(U \cap V \cap U_\alpha)$ and $\psi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap V \cap U_\alpha) \longrightarrow \psi(U \cap V \cap U_\alpha)$. Likewise, the composition $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. $\square$

Exercise 6. One of the steps in the proof above is missing a justification. Find it and fix it.
(Hint: Recall the first exercise of the chapter.)

Theorem 2.1. *Given an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M, let $\widetilde{\mathscr{A}}$ be the collection of all charts (U, φ) that are compatible with $\mathscr{A}$. Then $\widetilde{\mathscr{A}}$ is itself an atlas on M, containing $\mathscr{A}$. In fact, $\widetilde{\mathscr{A}}$ is the largest atlas containing $\mathscr{A}$.*

Proof. Note first that $\widetilde{\mathscr{A}}$ contains $\mathscr{A}$, since the set of charts compatible with $\mathscr{A}$ contains the charts from the atlas $\mathscr{A}$ itself. In particular, the charts in $\widetilde{\mathscr{A}}$ cover M. By the lemma above, any two charts in $\widetilde{\mathscr{A}}$ are compatible. Hence $\widetilde{\mathscr{A}}$ is an atlas. If (U, φ) is a chart compatible with all charts in $\widetilde{\mathscr{A}}$, then in particular it is compatible with all charts in $\mathscr{A}$; hence $(U, \varphi) \in \widetilde{\mathscr{A}}$ by the definition of $\widetilde{\mathscr{A}}$. This shows that $\widetilde{\mathscr{A}}$ cannot be extended to a larger atlas.

Definition 2.6. *An atlas $\mathscr{A}$ is called* maximal *if it is not properly contained in any larger atlas. Given an arbitrary atlas $\mathscr{A}$, one calls $\widetilde{\mathscr{A}}$ (as in Theorem 2.1) the* maximal atlas determined by $\mathscr{A}$.

Remark 2.1. Although we will not need it, let us briefly discuss the notion of equivalence of atlases. (For background on equivalence relations, see the appendix to this chapter, Section 2.9.2.) Two atlases $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ and $\mathscr{A}' = \{(U'_\alpha, \varphi'_\alpha)\}$ are called *equivalent* if every chart of $\mathscr{A}$ is compatible with every chart in $\mathscr{A}'$. For example, the atlas on the 2-sphere given by the two stereographic projections to the *xy*-plane is equivalent to the atlas $\mathscr{A}'$ given by the two stereographic projections to the *yz*-plane. Using Lemma 2.1, one sees that equivalence of atlases is indeed an equivalence relation. (In fact, two atlases are equivalent if and only if their union is an atlas.) Furthermore, two atlases are equivalent if and only if they are contained in the same maximal atlas. That is, *any maximal atlas determines an equivalence class of atlases, and vice versa.*

2.2 Definition of manifold

As our next approximation towards the definition of manifolds, we can take an m-dimensional manifold to be a set M together with an m-dimensional *maximal* atlas. This is already quite close to what we want, but for technical reasons we would like to impose two further conditions.

First of all, we insist that M can be covered by *countably many* coordinate charts. In most of our examples, M is in fact covered by finitely many coordinate charts. This countability condition is used for various arguments involving a proof by induction.

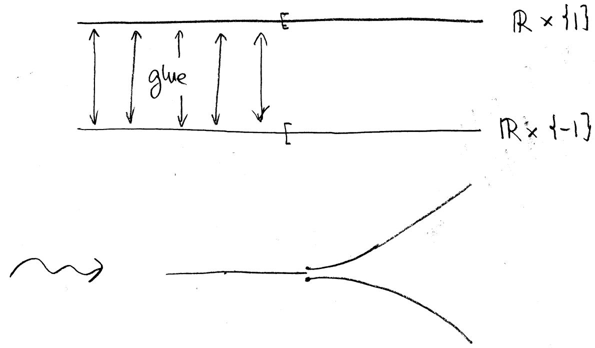
Example 2.5. A simple non-example that is not countable: Let $M = \mathbb{R}$, with $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ the 0-dimensional maximal (!) atlas, where each U_α consists of a single point, and $\varphi_\alpha : U_\alpha \to \{0\}$ is the unique map to $\mathbb{R}^0 = \{0\}$. Compatibility of charts is obvious. But M cannot be covered by countably many of these charts. Thus, we will not consider $\mathbb{R}$ to be a zero-dimensional manifold.

Secondly, we would like to avoid the following type of example.

Example 2.6. Let X be a disjoint union of two copies of the real line $\mathbb{R}$. We denote the two copies by $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{-1\}$, just so that we can tell them apart. Define an equivalence relation on X generated by

$$(x,1) \sim (x',-1) \;\Leftrightarrow x' = x < 0,$$

and let $M = X/\sim$ the set of equivalence classes. That is, we 'glue' the two real lines along their negative real axes (taking care that no glue gets on the origins of the axes). Here is a (not very successful) attempt to sketch the resulting space:



As a set, M is a disjoint union of $\mathbb{R}_{<0}$ with two copies of $\mathbb{R}_{\geq 0}$. Let $\pi:\ X \to M$ be the quotient map, and let

$$U = \pi(\mathbb{R}\times\{1\}), \quad V = \pi(\mathbb{R}\times\{-1\})$$

the images of the two real lines. The projection map $X \to \mathbb{R},\ (x,\pm 1)\mapsto x$ is constant on equivalence classes, hence it descends to a map $f:\ M \to \mathbb{R}$; let $\varphi:\ U \to \mathbb{R}$ be the restriction of f to U and $\psi:\ V \to \mathbb{R}$ the restriction to V. Then

$$\varphi(U) = \psi(V) = \mathbb{R}, \quad \varphi(U\cap V) = \psi(U\cap V) = \mathbb{R}_{<0},$$

and the transition map is the identity map. Hence, $\mathscr{A} = \{(U,\varphi),\ (V,\psi)\}$ is an atlas for M. A strange feature of M with this atlas is that the points

$$p = \varphi^{-1}(\{0\}), \quad q = \psi^{-1}(\{0\})$$

are 'arbitrarily close', in the sense that if $I,J \subseteq \mathbb{R}$ are any open subsets containing 0, the intersection of their pre-images is non-empty:

$$\varphi^{-1}(I)\cap\psi^{-1}(J) \neq \emptyset.$$

Yet, $p \neq q$! There is no really satisfactory way of drawing M (our picture above is inadequate), since it cannot be realized as a submanifold of any $\mathbb{R}^n$.

Since such a behaviour is inconsistent with the idea of a manifold that 'locally looks like $\mathbb{R}^n$' (where, e.g. every converging sequence has a unique limit), we shall insist that for any two distinct points $p,q \in M$, there are always disjoint coordinate charts separating the two points. This is called the *Hausdorff condition*, after *Felix Hausdorff* (1868-1942).

Definition 2.7. *An m-dimensional manifold is a set M, together with a maximal atlas* $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ *with the following properties:*

1. **(Countability condition)** *M is covered by countably many coordinate charts in* $\mathscr{A}$*. That is, there are indices* $\alpha_1, \alpha_2, \ldots$ *with*
$$M = \bigcup_i U_{\alpha_i}.$$
2. **(Hausdorff condition)** *For any two distinct points* $p, q \in M$ *there are coordinate charts* $(U_\alpha, \varphi_\alpha)$ *and* (U_β, φ_β) *in* $\mathscr{A}$ *such that* $p \in U_\alpha$*,* $q \in U_\beta$*, and*
$$U_\alpha \cap U_\beta = \emptyset.$$

The charts $(U, \varphi) \in \mathscr{A}$ *are called (coordinate)* charts on the manifold M.

Before giving examples, let us note the following useful fact concerning the Hausdorff condition. We shall use the following result:

Exercise 7. Suppose (U, φ) is a chart, with image $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Let $V \subseteq U$ be a subset such that $\widetilde{V} = \varphi(V) \subseteq \widetilde{U}$ is open, and let $\psi = \varphi|_V$ be the restriction of φ. Prove that (V, ψ) is again a chart, and is compatible with (U, φ). Furthermore, if (U, φ) is a chart from an atlas $\mathscr{A}$, then (V, ψ) is compatible with that atlas.

Lemma 2.2. *Let M be a set with a maximal atlas* $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$*, and suppose* $p, q \in M$ *are distinct points contained in a single coordinate chart* $(U, \varphi) \in \mathscr{A}$*. Then we can find indices* α, β *such that* $p \in U_\alpha$*,* $q \in U_\beta$*, with* $U_\alpha \cap U_\beta = \emptyset$*.*

Proof. Let (U, φ) be as in the lemma. Since
$$\widetilde{p} = \varphi(p), \quad \widetilde{q} = \varphi(q)$$
are distinct points in $\widetilde{U} \subseteq \mathbb{R}^m$, we can choose disjoint open subsets $\widetilde{U}_\alpha$ and $\widetilde{U}_\beta \subseteq \widetilde{U}$ containing $\widetilde{p} = \varphi(p)$ and $\widetilde{q} = \varphi(q)$, respectively.† Let U_α, $U_\beta \subseteq U$ be their preimages

† For instance, take these subsets to be the elements in $\widetilde{U}$ of distance less than $||\widetilde{p} - \widetilde{q}||/2$ from $\widetilde{p}$ and $\widetilde{q}$, respectively.

(why are they contained in U?), and take $\varphi_\alpha = \varphi|_{U_\alpha},\ \varphi_\beta = \varphi|_{U_\beta}$. Then $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in $\mathscr{A}$, with disjoint chart domains, and by construction we have that $p \in U_\alpha$ and $q \in U_\beta$. $\square$

Example 2.7. Consider the 2-sphere S^2 with the atlas given by the two coordinate charts (U_+, φ_+) and (U_-, φ_-). This atlas extends uniquely to a maximal atlas. The countability condition is satisfied, since S^2 is already covered by two charts. The Hausdorff condition is satisfied as well: Given distinct points $p, q \in S^2$, if both are contained in U_+ or both in U_-, we can apply the lemma. The only remaining case is if one point (say p) is the north pole and the other (say q) the south pole. But here we can construct U_α, U_β by replacing U_+ and U_- with the open upper hemisphere and open lower hemisphere, respectively. Alternatively, we can use the chart given by stereographic projection to the xz plane, noting that this is also in the maximal atlas.

Remark 2.2. As we explained above, the Hausdorff condition rules out some strange examples that don't quite fit our idea of a space that is locally like $\mathbb{R}^n$. Nevertheless, so-called *non-Hausdorff manifolds* (with non-Hausdorff more properly called *not necessarily Hausdorff*) do arise in some important applications. Much of the theory can be developed without the Hausdorff property, but there are some complications. For instance, initial value problems for vector fields need not have unique solutions for non-Hausdorff manifolds.

Remark 2.3 (Charts taking values in 'abstract' vector spaces). In the definition of an m-dimensional manifold M, rather than letting the charts $(U_\alpha, \varphi_\alpha)$ take values in $\mathbb{R}^m$ we could just as well let them take values in m-dimensional real vector spaces E_α:

$$\varphi_\alpha :\ U_\alpha \to E_\alpha.$$

Transition functions are defined as before, except they now take an open subset of E_β to an open subset of E_α. A choice of basis identifies $E_\alpha = \mathbb{R}^m$, and takes us back to the original definition.

As far as the definition of manifolds is concerned, nothing has been gained by adding this level of abstraction. However, it often happens that the E_α's are given to us 'naturally'. For example, if M is a surface inside $\mathbb{R}^3$, one would typically use xy-coordinates, or xz-coordinates, or yz-coordinates on appropriate chart domains. It can then be useful to regard the xy-plane, xz-plane, and yz-plane as the target spaces of the coordinate maps, and for notational reasons it may be convenient not to associate them with a single $\mathbb{R}^2$.