6

Differential forms

6.1 Review: Differential forms on $\mathbb{R}^m$

A differential k-form on an open subset $U \subseteq \mathbb{R}^m$ is an expression of the form

$$\omega = \sum_{i_1 \cdots i_k} \omega_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

where $\omega_{i_1 \dots i_k} \in C^\infty(U)$ are functions, and the indices are numbers

$$1 \le i_1 < \cdots < i_k \le m.$$

Let $\Omega^k(U)$ be the vector space consisting of such expressions, with pointwise addition. It is convenient to introduce a short hand notation $I = \{i_1, \dots, i_k\}$ for the index set, and write $\omega = \sum_I \omega_I \mathrm{d}x^I$ with

$$\omega_I = \omega_{i_1 \dots i_k}, \quad \mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Since a k-form is determined by these functions ω_I, and since there are $\frac{m!}{k!(m-k)!}$ ways of picking k-element subsets from $\{1, \dots, m\}$, the space $\Omega^k(U)$ can be identified with vector-valued smooth functions,

$$\Omega^k(U) = C^\infty(U,\ \mathbb{R}^{\frac{m!}{k!(m-k)!}}).$$

The $\mathrm{d}x^I$ are just formal expressions; at this stage they do not have any particular meaning. They are used, however, to define an associative product operation

$$\Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$

by the 'rule of computation'

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$$

for all i, j; in particular $\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$. In turn, using the product structure we may define the *exterior differential*

$$\mathrm{d}:\ \Omega^k(U)\to\Omega^{k+1}(U),\ \ \mathrm{d}\Big(\sum_I \omega_I \mathrm{d}x^I\Big)=\sum_{i=1}^m\sum_I \frac{\partial\omega_I}{\partial x^i}\mathrm{d}x^i\wedge\mathrm{d}x^I. \tag{6.1}$$

The key property of the exterior differential is the following fact:

Proposition 6.1. *The exterior differential satisfies*

$$\mathrm{d}\circ\mathrm{d}=0,$$

i.e. $\mathrm{dd}\omega=0$ *for all* ω.

Proof. By definition,

$$\mathrm{dd}\omega=\sum_{j=1}^m\sum_{i=1}^m\sum_I\frac{\partial^2\omega_I}{\partial x^j\partial x^i}\mathrm{d}x^j\wedge\mathrm{d}x^i\wedge\mathrm{d}x^I,$$

which vanishes by equality of mixed partials $\frac{\partial\omega_I}{\partial x^i\partial x^j}=\frac{\partial\omega_I}{\partial x^j\partial x^i}$. (We have $\mathrm{d}x^i\wedge\mathrm{d}x^j=-\mathrm{d}x^j\wedge\mathrm{d}x^i$, but the coefficients in front of $\mathrm{d}x^i\wedge\mathrm{d}x^j$ and $\mathrm{d}x^j\wedge\mathrm{d}x^i$ are the same.) □

Exercise 75.

(a) A 0-form on $\mathbb{R}^3$ is simply a smooth function $f\in\Omega^0(\mathbb{R}^3)$. Use the definition of the exterior differential above to compute the resulting 1-form df.

(b) A general 1-form $\omega\in\Omega^1(\mathbb{R}^3)$ is an expression

$$\omega=f\mathrm{d}x+g\mathrm{d}y+h\mathrm{d}z$$

with smooth functions $f,g,h\in C^\infty(\mathbb{R}^3)$. Use the definition of the exterior differential above to compute the resulting 2-form $d\omega$.

(c) A general 2-form $\omega\in\Omega^2(\mathbb{R}^3)$ may be written

$$\omega=a\ \mathrm{d}y\wedge\mathrm{d}z+b\ \mathrm{d}z\wedge\mathrm{d}x+c\ \mathrm{d}x\wedge\mathrm{d}y,$$

with $A=(a,b,c):\ U\to\mathbb{R}^3$. Use the definition of the exterior differential above to compute the resulting 3-form $d\omega$.

(d) Relate your results from the previous parts to familiar vector-calculus operators, and conclude that the usual properties

$$\mathrm{curl}(\mathrm{grad}(f))=0,\quad \mathrm{div}(\mathrm{curl}(F))=0$$

are both special cases of $\mathrm{d}\circ\mathrm{d}=0$.

(e) Write an expression for a general 3-form $\rho\in\Omega^3(\mathbb{R}^3)$. What is $d\rho$?

(f) Write an expression for a general element of $\Omega^4(\mathbb{R}^3)$. Generalize to $\Omega^m(\mathbb{R}^n)$ for $m>n$ (prove your assertion by using the 'rule of computation').

The *support* $\mathrm{supp}(\omega)\subseteq U$ of a differential form is the smallest closed subset such that ω vanishes on $U\backslash\mathrm{supp}(\omega)$. Suppose $\omega\in\Omega^m(U)$ is a compactly supported form of top degree $k=m$. Such a differential form is an expression

$$\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f \in C^\infty(U)$ is a compactly supported function. One defines the integral of ω to be the usual Riemann integral:

$$\int_U \omega = \int_{\mathbb{R}^m} f(x^1, \ldots, x^m)\mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{6.2}$$

Note that we can regard ω as a form on all of $\mathbb{R}^m$, due to the compact support condition.

Our aim is now to define differential forms on manifolds, beginning with 1-forms. Even though 1-forms on $U \subseteq \mathbb{R}^m$ are identified with functions $U \to \mathbb{R}^m$, they should not be regarded as vector fields, since their transformation properties under coordinate changes are different. In fact, while vector fields are sections of the tangent bundle, the 1-forms are sections of its dual, the cotangent bundle. We will therefore begin with a review of dual spaces in general.

6.2 Dual spaces

For any real vector space E, we denote by $E^* = L(E, \mathbb{R})$ its dual space, consisting of all linear maps $\alpha: E \to \mathbb{R}$. We will assume that E is finite-dimensional. Then the dual space is also finite-dimensional, and $\dim E^* = \dim E$. * It is common to write the value of $\alpha \in E^*$ on $v \in E$ as a *pairing*, using the bracket notation:†

$$\langle \alpha, v \rangle := \alpha(v).$$

This pairing notation emphasizes the duality between α and v. In the notation $\alpha(v)$ we think of α as a function acting on elements of E, and in particular on v. However, one may just as well think of v as acting on elements of E^* by evaluation: $v(\alpha) = \alpha(v)$ for all $\alpha \in E^*$. This symmetry manifests notationally in the pairing notation.

Let $e_1, \ldots, e_r$ be a basis of E. Any element of E^* is determined by its values on these basis vectors. For $i = 1, \ldots, r$, let $e^i \in E^*$ (with *upper* indices) be the linear functional such that

$$\langle e^i,\ e_j \rangle = \delta^i{}_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

The elements $e^1, \ldots, e^r$ are a basis of E^*; this is called the *dual basis*. The element $\alpha \in E^*$ is described in terms of the dual basis as

$$\alpha = \sum_{j=1}^{r} \alpha_j e^j, \quad \alpha_j = \langle \alpha, e_j \rangle.$$

* For possibly infinite-dimensional vector spaces, the dual space E^* is not isomorphic to E, in general.

† In physics, one also uses the *Dirac bra-ket* notation $\langle \alpha |\ v \rangle := \alpha(v)$; here $\alpha = \langle \alpha |$ is the 'bra' and $v = | v \rangle$ is the 'ket'.

Similarly, for vectors $v \in E$ we have

$$v = \sum_{i=1}^{r} v^i e_i, \quad v^i = \langle e^i, v \rangle.$$

Notice the placement of indices: In a given summation over $i, j, \ldots$, upper indices are always paired with lower indices.

Remark 6.1. As a special case, for $\mathbb{R}^r$ with its standard basis, we have a canonical identification $(\mathbb{R}^r)^* = \mathbb{R}^r$. For more general E with $\dim E < \infty$, there is no *canonical* isomorphism between E and E^* unless more structure is given.

Exercise 76. Let V be a finite dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Every vector $v \in V$ determines a linear transformation A_v by

$$v \mapsto \langle v, \cdot \rangle.$$

(a) Show that if $V = \mathbb{R}^n$, then $A_v = v^*$.
(b) For a general V, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis. Show that $\{A_{e_1}, \ldots, A_{e_n}\}$ is the corresponding dual basis.
(c) Conversely, for some element $A \in V^*$, show how to use the inner product in order to recover a $v \in V$ such that $A = A_v$. Show that this v is unique.

We see that equipping a finite-dimensional real vector space with an inner product gives it an additional structure that allows for a canonical isomorphism between V and V^*.

Given a linear map $R : E \to F$ between vector spaces, one defines the *dual map*

$$R^* : F^* \to E^*$$

(note the direction), by setting

$$\langle R^* \beta, v \rangle = \langle \beta, R(v) \rangle$$

for $\beta \in F^*$ and $v \in E$. This satisfies $(R^*)^* = R$, and under the composition of linear maps,

$$(R_1 \circ R_2)^* = R_2^* \circ R_1^*.$$

In terms of basis $e_1, \ldots, e_r$ of E and $f_1, \ldots, f_s$ of F, and the corresponding dual bases (with upper indices), a linear map $R : E \to F$ is given by the matrix with entries

$$R_i{}^j = \langle f^j, R(e_i) \rangle,$$

while R^* is described by the *transpose* of this matrix (the roles of i and j are reversed). Namely,‡

‡ In bra-ket notation, we have $R_i{}^j = \langle f^j | R | e_i \rangle$, and

$$|Re_i\rangle = R|e_i\rangle = \sum_j |f_j\rangle\langle f^j|R|e_i\rangle, \qquad \langle R^*(f^j)| = \langle (f^j)|R = \langle f^j|R|e_i\rangle\langle e^i|$$

$$R(e_i) = \sum_{j=1}^{s} R_i{}^j f_j, \qquad R^*(f^j) = \sum_{i=1}^{r} R_i{}^j\, f^i.$$

Thus,

$$(R^*)^j{}_i = R_i{}^j.$$

6.3 Cotangent spaces

Definition 6.1. *The dual of the tangent space T_pM of a manifold M is called the* cotangent space *at p, denoted*

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called cotangent vectors, or simply covectors*. Given a smooth map $F \in C^\infty(M,N)$, and any $p \in M$ we have the* cotangent map

$$T_p^*F = (T_pF)^* : \; T^*_{F(p)}N \to T_p^*M$$

defined as the dual to the tangent map.

Thus, a co(tangent) vector at p is a linear functional on the tangent space, assigning to each tangent vector at p a number. The very definition of the tangent space suggests one such functional: Every function $f \in C^\infty(M)$ defines a linear map, $T_pM \to \mathbb{R}, \;\; v \mapsto v(f)$. This linear functional is denoted $(\mathrm{d}f)_p \in T_p^*M$.[§]

Definition 6.2. *Let $f \in C^\infty(M)$ and $p \in M$. The covector*

$$(\mathrm{d}f)_p \in T_p^*M, \quad \langle (\mathrm{d}f)_p, v\rangle = v(f).$$

is called the differential of f at p.

Lemma 6.1. *For $F \in C^\infty(M,N)$ and $g \in C^\infty(N)$,*

$$\mathrm{d}(F^*g)_p = T_p^*F((\mathrm{d}g)_{F(p)}).$$

Exercise 77. Prove the lemma.
(Hint: every element of the dual space is completely determined by its action on vectors; so it suffices to show that the pairings are the same.)

Consider an open subset $U \subseteq \mathbb{R}^m$, with coordinates $x^1, \ldots, x^m$. Here $T_pU \cong \mathbb{R}^m$, with basis

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p \in T_pU \tag{6.3}$$

[§] Note that this is actually the same as the tangent map $T_pf : \; T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$.

The basis of the dual space T_p^*U, dual to the basis (6.3), is given by the differentials of the coordinate functions:

$$(\mathrm{d}x^1)_p, \ \dots, \ (\mathrm{d}x^m)_p \in T_p^*U.$$

Indeed,

$$\Big\langle (\mathrm{d}x^i)_p, \ \frac{\partial}{\partial x^j}\Big|_p \Big\rangle = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \delta^i{}_j$$

as required. For $f \in C^\infty(M)$, the coefficients of $(\mathrm{d}f)_p = \sum_i \langle (\mathrm{d}f)_p, \ e_i \rangle e^i$ are determined as

$$\Big\langle (\mathrm{d}f)_p, \ \frac{\partial}{\partial x^j}\Big|_p \Big\rangle = \frac{\partial}{\partial x^j}\Big|_p (f) = \frac{\partial f}{\partial x^j}\Big|_p.$$

Thus,

$$(\mathrm{d}f)_p = \sum_{i=1}^m \frac{\partial f}{\partial x^i}\Big|_p \ (\mathrm{d}x^i)_p.$$

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, with coordinates $x^1, \dots, x^m$ and $y^1, \dots, y^n$. For $F \in C^\infty(U, V)$, the tangent map is described by the Jacobian matrix, with entries

$$(D_pF)_i{}^j = \frac{\partial F^j}{\partial x^i}(p)$$

for $i = 1, \dots, m, \ j = 1, \dots, n$. We have:

$$(T_pF)\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_{j=1}^n (D_pF)_i{}^j \ \frac{\partial}{\partial y^j}\Big|_{F(p)},$$

hence dually

$$(T_pF)^*(\mathrm{d}y^j)_{F(p)} = \sum_{i=1}^m (D_pF)_i{}^j \ (\mathrm{d}x^i)_p. \tag{6.4}$$

Thought of as matrices, *the coefficients of the cotangent map are the transpose of the coefficients of the tangent map*.

Exercise 78. Consider $\mathbb{R}^3$ with coordinates x, y, z, and $\mathbb{R}^2$ with coordinates u, v. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$(x, y, z) \mapsto (x^2 y + e^z, yz - x).$$

Let $p = (1, 1, 1)$. What is $T_pF\left(\frac{\partial}{\partial x}\Big|_p\right)$? What is $T_p^*F(du)_{F(p)}$?

6.4 1-forms

Similar to the definition of vector fields, one can define *co-vector fields*, more commonly known as *1-forms*: Collections of covectors $\alpha_p \in T_p^*M$ depending smoothly on the base point. One approach of making precise the smooth dependence on the base point is to endow the *cotangent bundle*

$$T^*M = \bigsqcup_p T_p^*M.$$

(disjoint union of all cotangent spaces) with a smooth structure, and require that the map $p \mapsto \alpha_p$ be smooth. The construction of charts on T^*M is similar to that for the tangent bundle: Charts (U, φ) of M give cotangent charts $(T^*U, T^*\varphi^{-1})$ of T^*M, using the fact that $T^*(\varphi(U)) = \varphi(U) \times \mathbb{R}^m$ canonically (since $\varphi(U)$ is an open subset of $\mathbb{R}^m$). Here $T^*\varphi^{-1}: T^*U \to T^*\varphi(U)$ is the union of inverses of all cotangent maps $T_p^*\varphi: T^*_{\varphi(p)}\varphi(U) \to T_p^*U$.

Exercise 79. Carry out this construction to prove that T^*M is naturally a $2m$-dimensional manifold.

A second approach is observe that in local coordinates, 1-forms are given by expressions $\sum_i f_i \mathrm{d}x^i$, and smoothness should mean that the coefficient functions are smooth.

We will use the following (equivalent) approach.

Definition 6.3. *A 1-form on M is a linear map*

$$\alpha: \mathfrak{X}(M) \to C^\infty(M), \quad X \mapsto \alpha(X) = \langle \alpha, X \rangle,$$

which is $C^\infty(M)$-linear *in the sense that*

$$\alpha(fX) = f\alpha(X)$$

for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$. *The space of 1-forms is denoted* $\Omega^1(M)$.

Let us verify that a 1-form can be regarded as a collection of covectors:

Lemma 6.2. *Let* $\alpha \in \Omega^1(M)$ *be a 1-form, and* $p \in M$. *Then there is a unique covector* $\alpha_p \in T_p^*M$ *such that*

$$\alpha(X)_p = \alpha_p(X_p)$$

for all $X \in \mathfrak{X}(M)$.

(We indicate the value of the function $\alpha(X)$ at p by a subscript, just like we did for vector fields.)

Proof. We have to show that $\alpha(X)_p$ depends only on the value of X at p. By considering the difference of vector fields having the same value at p, it is enough to show that if $X_p = 0$, then $\alpha(X)_p = 0$. But any vector field vanishing at p can be written as a finite sum $X = \sum_i f_i Y_i$ where $f_i \in C^\infty(M)$ vanish at p. ¶ By C^∞-linearity, this implies that

$$\alpha(X) = \alpha(\sum_i f_i Y_i) = \sum_i f_i \alpha(Y_i)$$

vanishes at p. □

¶ For example, using local coordinates, we can take the Y_i to correspond to $\frac{\partial}{\partial x^i}$ near p, and the f_i to the coefficient functions.

The first example of a 1-form is described in the following definition.

Definition 6.4. *The* exterior differential *of a function $f \in C^\infty(M)$ is the 1-form*

$$\mathrm{d}f \in \Omega^1(M),$$

defined in terms of its pairings with vector fields $X \in \mathfrak{X}(M)$ as $\langle \mathrm{d}f,\, X\rangle = X(f)$.

Clearly, $\mathrm{d}f$ is the 1-form defined by the family of covectors $(\mathrm{d}f)_p$. Note that critical points of f may be described in terms of this 1-form: $p \in M$ is a critical point of f if and only if $(\mathrm{d}f)_p = 0$.

Similar to vector fields, 1-forms can be multiplied by functions; hence one has more general examples of 1-forms as finite sums,

$$\alpha = \sum_i f_i \,\mathrm{d}g_i$$

where $f_i, g_i \in C^\infty(M)$.

Let us examine what the 1-forms are for open subsets $U \subseteq \mathbb{R}^m$. Given $\alpha \in \Omega^1(U)$, we have

$$\alpha = \sum_{i=1}^m \alpha_i \,\mathrm{d}x^i$$

with coefficient functions $\alpha_i = \left\langle \alpha,\, \frac{\partial}{\partial x^i}\right\rangle \in C^\infty(U)$. (Indeed, the right hand side takes on the correct values at any $p \in U$, and is uniquely determined by those values.) General vector fields on U may be written

$$X = \sum_{j=1}^m X^j \frac{\partial}{\partial x^j}$$

(to match the notation for 1-forms, we write the coefficients as X^i rather than a^i, as we did in the past), where the coefficient functions are recovered as $X^j = \langle \mathrm{d}x^j,\, X\rangle$. The pairing of the 1-form α with the vector field X is then

$$\langle \alpha,\, X\rangle = \sum_{i=1}^m \alpha_i X^i.$$

Lemma 6.3. *Let $\alpha:\ p \mapsto \alpha_p \in T_p^*M$ be a collection of covectors. Then α defines a 1-form, with*

$$\alpha(X)_p = \alpha_p(X_p)$$

for $p \in M$, if and only if for all charts (U, φ), the coefficient functions for α in the chart are smooth.

Exercise 80. Prove Lemma 6.3.

6.5 Pull-backs of function and 1-forms

Recall again that for any manifold M, the vector space $C^\infty(M)$ of smooth functions is an algebra, with product the pointwise multiplication. Any smooth map $F:\ M\to M'$ between manifolds defines an algebra homomorphism, called the *pull-back*

$$F^*:\ C^\infty(M')\to C^\infty(M),\ \ f\mapsto F^*(f):=f\circ F.$$

Exercise 81. Show that the pull-back is indeed an algebra homomorphism by showing that it preserves sums and products:

$$F^*(f)+F^*(g)=F^*(f+g)\quad ;\quad F^*(f)F^*(g)=F^*(fg).$$

Next, show that if $F,F':\ M\to M'$ are two smooth maps between manifolds, then

$$(F'\circ F)^*=F^*\circ(F')^*.$$

(With order reversed.)

Let $F\in C^\infty(M,N)$ be a smooth map. Recall that for vector fields, there is no general 'push-forward' or 'pull-back' operations, unless F is a diffeomorphism. For 1-forms the situation is better. Indeed, for any $p\in M$ one has the dual to the tangent map

$$T_p^*F=(T_pF)^*:\ T^*_{F(p)}N\to T_p^*M.$$

For a 1-form $\beta\in\Omega^1(N)$, we can therefore define

$$(F^*\beta)_p:=(T_p^*F)(\beta_{F(p)}).$$

This gives us a collection of covectors in T_p^*M at each point $p\in M$. The following Lemma shows that these form a 1-form.

Lemma 6.4. *The collection of co-vectors $(F^*\beta)_p\in T_p^*M$ depends smoothly on p, defining a 1-form $F^*\beta\in\Omega^1(M)$.*

Proof. We shall use Lemma 6.3. By working on local coordinates, we may assume that M is an open subset $U\subseteq\mathbb{R}^m$, and N is an open subset $V\subseteq\mathbb{R}^n$. Write

$$\beta=\sum_{j=1}^n\beta_j(y)\mathrm{d}y^j.$$

By (6.4), the pull-back of β is given by

$$F^*\beta=\sum_{i=1}^m\Big(\sum_{j=1}^n\beta_j(F(x))\frac{\partial F^j}{\partial x^i}\Big)\mathrm{d}x^i.$$

In particular, the coefficients are smooth. □

Lemma 6.4 shows that we have a well-defined pull-back map

$$F^* : \ \Omega^1(N) \to \Omega^1(M), \ \ \beta \mapsto F^*\beta.$$

Note that with respect to composition of two maps

$$(F_1 \circ F_2)^* = F_2^* \circ F_1^*$$

with order reversed.

A nice property of the pull-back of forms is its relation to the pull-back of functions. Lemma 6.1 shows that for $g \in C^\infty(N)$,

$$F^*(\mathrm{d}g) = \mathrm{d}(F^*g)$$

(Note that on the left we are pulling-back a form, and on the right a function.)

Exercise 82. In this exercise we shall use coordinates x, y, z on the domain and u, v, w on the target space. Consider the maps $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x,y,z) = (x^3 e^{yz}, \sin x)$$
$$g(u,v) = (u+v)^2 e^{uv}.$$

(a) Compute $F^*(\mathrm{d}u)$ and $F^*(v \cos u \mathrm{d}v)$. Compute $F^*(v \cos u \mathrm{d}u + \sin u \mathrm{d}v)$ by using Lemma 6.1.
(b) Verify Lemma 6.1 by computing $\mathrm{d}g$, $F^*(\mathrm{d}g)$, as well as F^*g and $\mathrm{d}(F^*g)$.

Recall once again that while $F \in C^\infty(M,N)$ induces a tangent map $TF \in C^\infty(TM,TN)$, there is no natural push-forward operation for vector fields. By contrast, for cotangent bundles there is no naturally induced map from T^*N to T^*M (or the other way), yet there is a natural pull-back operation for 1-forms!

In the case of vector fields, rather than working with '$F_*(X)$' one has the notion of related vector fields, $X \sim_F Y$. We know that 1-forms act on vector fields, how do they act on related vector fields?

Exercise 83. Show that for any related vector fields $X \sim_F Y$, and $\beta \in \Omega^1(N)$,

$$(F^*\beta)(X) = F^*(\beta(Y)).$$

(Notice once again the different notions of pull-back that we are using.)

Exercise 84. Recall that for a given vector field on a manifold $X \in \mathfrak{X}(M)$, the smooth curve $\gamma \in C^\infty(J,M)$ (where $J \subseteq \mathbb{R}$) is a solution curve iff

$$\frac{\partial}{\partial t} \ \gamma X.$$

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be given by

$$t \mapsto (\cos t, \sin t).$$

Let $X \in \mathfrak{X}(M)$ be given by

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Finally, let $\beta \in \Omega^1(M)$ be given by $\mathrm{d}x - \mathrm{d}y$.

(a) Show by computing directly that $\frac{\partial}{\partial t}$ is F-related to X.
(b) Verify the conclusion of Exercise 83 by computing each of $\beta(X)$, $F^*\beta$, $F^*\beta(\frac{\partial}{\partial t})$, and $F^*(\beta(X))$.

6.6 Integration of 1-forms

Given a curve $\gamma\colon J \to M$ in a manifold, and any 1-form $\alpha \in \Omega^1(M)$, we can consider the pull-back $\gamma^*\alpha \in \Omega^1(J)$. By the description of 1-forms on $\mathbb{R}$, this is of the form

$$\gamma^*\alpha = f(t)\mathrm{d}t$$

for a smooth function $f \in C^\infty(J)$.

To discuss integration, it is convenient to work with closed intervals rather than open intervals. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A map $\gamma\colon\ [a,b] \to M$ into a manifold will be called *smooth* if it extends to a smooth map from an open interval containing $[a,b]$. We will call such a map a smooth *path*.

Definition 6.5. *Given a smooth path $\gamma\colon\ [a,b] \to M$, we define the integral of a 1-form $\alpha \in \Omega^1(M)$ along γ as*

$$\int_\gamma \alpha = \int_a^b \gamma^*\alpha.$$

The fundamental theorem of calculus has the following consequence for manifolds. It is a special case of *Stokes' theorem*.

Proposition 6.2. *Let $\gamma\colon\ [a,b] \to M$ be a smooth path, with $\gamma(a) = p,\ \gamma(b) = q$. For any $f \in C^\infty(M)$, we have*

$$\int_\gamma \mathrm{d}f = f(q) - f(p).$$

In particular, the integral of $\mathrm{d}f$ *depends only on the end points of the path, rather than the path itself.*

Proof. We have

$$\gamma^*\mathrm{d}f = \mathrm{d}\gamma^* f = \mathrm{d}(f\circ\gamma) = \frac{\partial(f\circ\gamma)}{\partial t}\mathrm{d}t.$$

Integrating from a to b, we obtain, by the fundamental theorem of calculus, $f(\gamma(b)) - f(\gamma(a))$. $\square$

Exercise 85. Given a diffeomorphism $\kappa:\ [c,d] \to [a,b]$ one defines the corresponding *reparametrization*

$$\gamma \circ \kappa:\ [c,d] \to M.$$

The diffeomorphism (or the reparametrization) is called *orientation preserving* if $\kappa(c) = a,\ \kappa(d) = b$, orientation reversing if $\kappa(c) = b,\ \kappa(d) = a$.
Prove that the integral is invariant under orientation preserving reparametrization

$$\int_\gamma \alpha = \int_{\gamma\circ\kappa} \alpha,$$

while an orientation reversing reparametrization gives $\int_\gamma \alpha = -\int_{\gamma\circ\kappa} \alpha$.

Exercise 86. Consider the 1-form

$$\alpha = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y \in \Omega(\mathbb{R}^2).$$

Find the integral of α along the path

$$\gamma:\ [0,1] \to M,\ \ t \mapsto (\sin(\pi t/2), t^3).$$

A 1-form $\alpha \in \Omega^1(M)$ such that $\alpha = \mathrm{d}f$ for some function $f \in C^\infty(M)$ is called *exact*. Proposition 6.2 gives a necessary condition for exactness: The integral of α along paths should depend only on the end points. This condition is also sufficient, since we can define f on the connected components of M, by fixing a base point p_0 on each such component, and putting $f(p) = \int_\gamma \alpha$ for any path from p_0 to p.

If M is an open subset $U \subseteq \mathbb{R}^m$, so that $\alpha = \sum_i \alpha_i \mathrm{d}x^i$, then $\alpha = \mathrm{d}f$ means that $\alpha_i = \frac{\partial f}{\partial x^i}$. A necessary condition is the equality of partial derivatives,

$$\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i},$$

In multivariable calculus one learns that this condition is also sufficient, provided U is simply connected (e.g., convex). Using the exterior differential of forms in $\Omega^1(U)$, this condition becomes $\mathrm{d}\alpha = 0$. Since α is a 1-form, $\mathrm{d}\alpha$ is a 2-form. Thus, to obtain a coordinate-free version of the condition, we need higher order forms.

6.7 k-forms

To get a feeling for higher degree forms, and constructions with higher forms, we first discuss 2-forms.

6.7.1 2-forms.

Definition 6.6. *A 2-form on M is a $C^\infty(M)$-bilinear skew-symmetric map*

$$\alpha: \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M), \quad (X,Y) \mapsto \alpha(X,Y).$$

Here skew-symmetry means that $\alpha(X,Y) = -\alpha(Y,X)$ for all vector fields X,Y, while $C^\infty(M)$-bilinearity means

$$\alpha(fX,Y) = f\alpha(X,Y) = \alpha(X,fY)$$

for $f \in C^\infty(M)$, as well as $\alpha(X'+X'',Y) = \alpha(X',Y)+\alpha(X'',Y)$, and similarly in the second argument. (Actually, by skew-symmetry it suffices to require $C^\infty(M)$-linearity in the first argument.) By the same argument as for 1-forms, the value $\alpha(X,Y)_p$ depends only on the values X_p, Y_p. Also, if α is a 2-form then so is $f\alpha$ for any smooth function f.

Our first examples of 2-forms are obtained from 1-forms: Let $\alpha, \beta \in \Omega^1(M)$. Then we define a *wedge product* $\alpha \wedge \beta \in \Omega^2(M)$, as follows:

$$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{6.5}$$

Exercise 87. Show that Equation (6.5) indeed defines a 2-form.

For an open subset $U \subseteq \mathbb{R}^m$, a 2-form $\omega \in \Omega^2(U)$ is uniquely determined by its values on coordinate vector fields. By skew-symmetry the functions

$$\omega_{ij} = \omega\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)$$

satisfy $\omega_{ij} = -\omega_{ji}$; hence it suffices to know these functions for $i < j$. As a consequence, we see that the most general 2-form on U is

$$\omega = \frac{1}{2}\sum_{i,j=1}^{m} \omega_{ij} \mathrm{d}x^i \wedge dx^j = \sum_{i<j} \omega_{ij} \mathrm{d}x^i \wedge dx^j.$$

Exercise 88. Compute the following 2-forms on $\mathbb{R}^3$ with coordinates (x,y,z) (that is, write them in normal form as above):

(a) $(3\mathrm{d}x) \wedge (-7\mathrm{d}y) + (xdy) \wedge (dx)$.
(b) $(\mathrm{d}x - \mathrm{d}y) \wedge (\mathrm{d}x + \mathrm{d}z)$.
(c) $\mathrm{d}(x^2 + xyz) \wedge \mathrm{d}(ze^{x^3y^2})$.

We now generalize to forms of arbitrary degree.

6.7.2 k-forms

Definition 6.7. *Let k be a non-negative integer. A k-*form *on M is a $C^\infty(M)$-multilinear, skew-symmetric map*

$$\alpha: \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ times}} \to C^\infty(M).$$

The space of k-forms is denoted $\Omega^k(M)$; in particular $\Omega^0(M) = C^\infty(M)$.

Here, *skew-symmetry* means that $\alpha(X_1, \ldots, X_k)$ changes sign under exchange of any two of its elements. For example, $\alpha(X_1, X_2, X_3, \ldots) = -\alpha(X_2, X_1, X_3, \ldots)$. More generally, if S_k is the group of permutations of $\{1, \ldots, k\}$, and $\mathrm{sgn}(s)$ is the sign of a permutation $s \in S_k$ ($+1$ for an even permutation, -1 for an odd permutation) then

$$\alpha(X_{s(1)}, \ldots, X_{s(k)}) = \mathrm{sgn}(s)\alpha(X_1, \ldots, X_k).$$

The $C^\infty(M)$*-multilinearity* means $C^\infty(M)$-linearity in each argument, similarly to the condition for 2-forms. It implies, in particular, α is *local* in the sense that the value of $\alpha(X_1, \ldots, X_k)$ at any given $p \in M$ depends only on the values $X_1|_p, \ldots, X_k|_p \in T_pM$. One thus obtains a skew-symmetric multilinear form

$$\alpha_p: \ T_pM \times \cdots \times T_pM \to \mathbb{R},$$

for all $p \in M$.

If $\alpha_1, \ldots, \alpha_k$ are 1-forms, then one obtains a k-form $\alpha =: \alpha_1 \wedge \ldots \wedge \alpha_k$ by 'wedge product'.

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, \ldots, X_k) = \sum_{s \in S_k} \mathrm{sign}(s)\alpha_1(X_{s(1)}) \cdots \alpha_k(X_{s(k)}).$$

(More general wedge products will be discussed below.)

Exercise 89. Show that the wedge-product above indeed defines a k-form.

Using C^∞-multilinearity, a k-form on $U \subseteq \mathbb{R}^m$ is uniquely determined by its values on coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$, i.e. by the functions

$$\alpha_{i_1 \ldots i_k} = \alpha\Big(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\Big).$$

Moreover, by skew-symmetry we only need to consider *ordered* index sets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$, that is, $i_1 < \cdots < i_k$. Using the wedge product notation, we obtain

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \mathrm{d}x^{i_k}.$$

Exercise 90. Compute the following 3-forms on $\mathbb{R}^5$ with coordinates $(x_1,\dots,x_5)$:

(a) $(\mathrm{d}x_1+\mathrm{d}x_2)\wedge(\mathrm{d}x_2+\mathrm{d}x_3)\wedge(\mathrm{d}x_4+\mathrm{d}x_5)$.
(b) $\mathrm{d}(x_1x_2^3e^{x_5}+\sin x_3\cos x_4)\wedge\mathrm{d}(\cos(x_1^2x_5)-e^{x_3})\wedge\mathrm{d}(x_4)$.

6.7.3 Wedge product

We next turn to the definition of a *wedge product* of forms of arbitrary degree $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^l(M)$. A permutation $s\in S_{k+l}$ is called a k,l *shuffle* if it satisfies

$$s(1)<\cdots<s(k),\quad s(k+1)<\cdots<s(k+l).$$

Definition 6.8. *The wedge product of* $\alpha\in\Omega^k(M),\ \beta\in\Omega^l(M)$ *is the element*

$$\alpha\wedge\beta\in\Omega^{k+l}(M)$$

given as

$$(\alpha\wedge\beta)(X_1,\dots,X_{k+l})=\sum\operatorname{sgn}(s)\alpha(X_{s(1)},\dots,X_{s(k)})\,\beta(X_{s(k+1)},\dots,X_{s(k+l)})$$

where the sum is over all k,l*-shuffles.*

Exercise 91.

(a) Show that Definition 6.8 is consistent with our previous definition of the wedge-product of two 1-forms (Equation (6.5)).
(b) Show that Definition 6.8 indeed defines a $(k+l)$-form.

Exercise 92.

(a) For $\alpha,\beta,\rho\in\Omega^2(M)$, and $T,U,V,W,X,Y,Z\in\mathfrak{X}(M)$, compute

$$(\alpha\wedge\beta)(W,X,Y,Z)$$

and

$$(\alpha\wedge\beta)\wedge\rho(T,U,V,W,X,Y,Z).$$

(b) For $\alpha,\beta\in\Omega^3(M)$, and $T,U,V,W,X,Y,Z\in\mathfrak{X}(M)$, compute

$$(\alpha\wedge\beta)(T,U,V,W,X,Y,Z).$$

Exercise 93.

(a) Prove that the wedge product is *graded commutative*: If $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^l(M)$ then

$$\alpha\wedge\beta=(-1)^{kl}\beta\wedge\alpha.$$

(b) Prove that the wedge product is associative: Given $\alpha_i \in \Omega_{k_i}(M)$ we have

$$(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3).$$

So, we may in fact drop the parentheses when writing wedge products.

6.7.4 Exterior differential

Recall that we have defined the exterior differential on functions by the formula

$$(df)(X) = X(f). \tag{6.6}$$

we will now extend this definition to all forms.

Theorem 6.1. *There is a unique collection of linear maps* $\mathrm{d}: \ \Omega^k(M) \to \Omega^{k+1}(M)$*, extending the map (6.6) for* $k = 0$*, such that* $\mathrm{d}(\mathrm{d}f) = 0$ *and satisfying the graded product rule,*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta \tag{6.7}$$

for $\alpha \in \Omega^k(M)$ *and* $\beta \in \Omega^l(M)$*. This* exterior differential *satisfies* $\mathrm{d} \circ \mathrm{d} = 0$.

Proof. Suppose first that such an exterior differential is given. Then d is local, in the sense that for any open subset $U \subseteq M$ the restriction $(\mathrm{d}\alpha)|_U$ depends only on $\alpha|_U$, or equivalently $(\mathrm{d}\alpha)|_U = 0$ when $\alpha|_U = 0$. Indeed, if this is the case and $p \in U$, we may choose $f \in C^\infty(M) = \Omega^0(M)$ such that f vanishes on $M \backslash U$ and $f|_p = 1$. Then $f\alpha = 0$, hence the product rule (6.7) gives

$$0 = \mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha.$$

Evaluating at p we obtain $(\mathrm{d}\alpha)_p = 0$ as claimed. Using locality, we may thus work in local coordinates. If $\alpha \in \Omega^1(M)$ is locally given by

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

then the product rule together with $\mathrm{d}\mathrm{d}x^i = 0$ forces us to define

$$\mathrm{d}\alpha = \sum_{i_1 < \cdots < i_k} \mathrm{d}\alpha_{i_1 \cdots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{l=1}^{m} \sum_{i_1 < \cdots < i_k} \frac{\partial \alpha_{i_1 \cdots i_k}}{\partial x^l} \mathrm{d}x^l \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Conversely, we may use this explicit formula (cf. (6.1)) to *define* $\mathrm{d}\alpha|_U$ for a coordinate chart domain U; by uniqueness the local definitions on overlas of coordinate chart domains agree. Proposition 6.1 shows that $(\mathrm{d}\mathrm{d}\alpha)|_U = 0$, hence it also holds globally. $\square$

Exercise 94. Find the exterior differential of each of the following forms on $\mathbb{R}^3$ (with coordinates (x,y,z)).

(a) $\alpha = y^2 e^x \mathrm{d}y + 2ye^x \mathrm{d}x$.
(b) $\beta = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y$.
(c) $\rho = e^{x^2 y} \sin z \mathrm{d}x \wedge \mathrm{d}y + 2\cos(z^3 y)\mathrm{d}x$.
(d) $\omega = \frac{\sin e^{xy} - \cos \sin z^3 x}{1+(x+y+z)^4+(7xy)^6} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Definition 6.9. *A k-form $\omega \in \Omega^k(M)$ is called* exact *if $\omega = \mathrm{d}\alpha$ for some $\alpha \in \Omega^{k-1}(M)$. It is called* closed *if* $\mathrm{d}\omega = 0$.

Since $\mathrm{d} \circ \mathrm{d} = 0$, the exact k-forms are a subspace of the space of closed k-forms. For the case of 1-forms, we had seen that the integral $\int_\gamma \alpha$ of an exact 1-form $\alpha = \mathrm{d}f$ along a smooth path $\gamma\colon\ [a,b] \to M$ is given by the difference of the values at the end points; a necessary condition for α to be exact is that it is closed. An example of a 1-form that is closed but not exact is

$$\alpha = \frac{y\mathrm{d}x - x\mathrm{d}y}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \backslash \{0\}).$$

Remark 6.2. The quotient space (closed k-forms modulo exact k-forms) is a vector space called the k-th (de Rham) *cohomology*

$$H^k(M) = \frac{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is closed }\}}{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is exact }\}}.$$

It turns out that whenever M is compact (and often also if M is non-compact), $H^k(M)$ is a finite-dimensional vector space. The dimension of this vector space

$$b_k(M) = \dim H^k(M)$$

is called the *k-th Betti number* of M; these numbers are important invariants of M which one can use to distinguish non-diffeomorphic manifolds. For example, if $M = \mathbb{C}\mathrm{P}^n$ one can show that

$$b_k(\mathbb{C}\mathrm{P}^n) = 1 \quad \text{for} \quad k = 0, 2, \ldots, 2n$$

and $b_k(\mathbb{C}\mathrm{P}^n) = 0$ otherwise. For $M = S^N$ the Betti numbers are

$$b_k(S^n) = 1 \quad \text{for} \quad k = 0, n$$

while $b_k(S^n) = 0$ for all other k. Hence $\mathbb{C}\mathrm{P}^n$ cannot be diffeomorphic to S^{2n} unless $n = 1$.

6.8 Lie derivatives and contractions

Given a vector field X, and a k-form $\alpha \in \Omega^k(M)$, we can define a $(k-1)$-form

$$\iota_X \alpha \in \Omega^{k-1}(M)$$

by *contraction*: Thinking of α as a multi-linear form, one simply puts X into the first slot:

$$(\iota_X \alpha)(X_1,\ldots,X_{k-1}) = \alpha(X,X_1,\ldots,X_{k-1}).$$

Contractions have the following compatibility with the wedge product, similar to that for the exterior differential:

$$\iota_X(\alpha\wedge\beta) = \iota_X\alpha\wedge\beta + (-1)^k \alpha\wedge\iota_X\beta, \tag{6.8}$$

for $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$,

Exercise 95. Prove Equation (6.8).

Another important operator on forms is the *Lie derivative*:

Theorem 6.2. *Given a vector field X, there is a unique collection of linear maps $L_X : \Omega^k(M) \to \Omega^k(M)$, such that*

$$L_X(f) = X(f), \;\; L_X(\mathrm{d}f) = \mathrm{d}X(f),$$

and satisfying the product rule,

$$L_X(\alpha\wedge\beta) = L_X\alpha\wedge\beta + \alpha\wedge L_X\beta \tag{6.9}$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.

Proof. As in the case of the exterior differential, we can use the product rule to show that L_X is local: $(L_X\alpha)|_U$ depends only on $\alpha|_U$ and $X|_U$. Since any differential form is a sum of wedge products of 1-forms, L_X is uniquely determined by its action on functions and differential of functions. This proves uniqueness. For existence, we give the following formula:

$$L_X = \mathrm{d}\circ\iota_X + \iota_X\circ\mathrm{d}.$$

On functions, this gives the correct result since

$$L_X f = \iota_X \mathrm{d}f = X(f),$$

and also on differentials of functions since

$$L_X\mathrm{d}f = \mathrm{d}\iota_X\mathrm{d}f = \mathrm{d}L_X f = 0.$$

$\square$

Exercise 96. For each of the following vector-fields X and differential forms α on $\mathbb{R}^3$ (with coordinates (x,y,z)) compute $L_X\alpha$

(a) $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and $\alpha = -y\mathrm{d}x - x\mathrm{d}y - z\mathrm{d}z$.
(b) The Wikipedia example: $X = \sin x\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial x}$ and $\alpha = x^2 - \sin(y)$.
(c) The Wikipedia example: $X = \sin x\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial x}$ and $\alpha = (x^2+y^2)\mathrm{d}x\wedge\mathrm{d}z$.

To summarize, we have introduced three operators

$$\mathrm{d}:\ \Omega^k(M)\to\Omega^{k+1}(M),\ \ L_X:\ \Omega^k(M)\to\Omega^k(M),\ \ \iota_X:\ \Omega^k(M)\to\Omega^{k-1}(M).$$

These have the following compatibilities with the wedge product: For $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^l(M)$ one has

$$\begin{aligned}
\mathrm{d}(\alpha\wedge\beta) &= (\mathrm{d}\alpha)\wedge\beta + (-1)^k\alpha\wedge\mathrm{d}\beta,\\
L_X(\alpha\wedge\beta) &= (L_X\alpha)\wedge\beta + \alpha\wedge L_X\beta,\\
\iota_X(\alpha\wedge\beta) &= (\iota_X\alpha)\wedge\beta + (-1)^k\alpha\wedge\iota_X\beta.
\end{aligned}$$

One says that L_X is an *even derivation* relative to the wedge product, whereas d,ι_X are *odd derivations*. They also satisfy important relations among each other:

Exercise 97. Prove the following relations:

$$\begin{aligned}
\mathrm{d}\circ\mathrm{d} &= 0\\
L_X\circ L_Y - L_Y\circ L_X &= L_{[X,Y]}\\
\iota_X\circ\iota_Y + \iota_Y\circ\iota_X &= 0\\
\mathrm{d}\circ L_X - L_X\circ\mathrm{d} &= 0\\
L_X\circ\iota_Y - \iota_Y\circ L_X &= \iota_{[X,Y]}\\
\iota_X\circ\mathrm{d} + \mathrm{d}\circ\iota_X &= L_X.
\end{aligned}$$

Again, the signs are determined by the even/odd parity of these operators; one should think of the left hand side as 'graded' commutators, where a plus sign appears whenever two entries are odd. Writing $[\cdot,\cdot]$ for the graded commutators (with the agreement that the commutator of two odd operators has a sign built in) the identities become

$$\begin{aligned}
[\mathrm{d},\mathrm{d}] &= 0,\\
[L_X,L_Y] &= L_{[X,Y]},\\
[\iota_X,\iota_Y] &= 0,\\
[\mathrm{d},L_X] &= 0,\\
[L_X,\iota_Y] &= \iota_{[X,Y]},\\
[\mathrm{d},\iota_X] &= L_X.
\end{aligned}$$

This collection of identities is referred to as the *Cartan calculus*, after Élie Cartan (1861-1951), and in particular the last identity (which certainly is the most intriguing) is called the *Cartan formula*. Basic contributions to the theory of differential forms were made by his son Henri Cartan (1906-1980), who also wrote a textbook on the subject.

Exercise 98.

(a) As an illustration of the Cartan identities, let us prove the following formula for the exterior differential of a 1-form $\alpha \in \Omega^1(M)$:

$$(\mathrm{d}\alpha)(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]).$$

(In the Cartan Calculus, we prefer to write $L_X f$ instead of $X(f)$ since expressions such as $X(\alpha(Y))$ would look too confusing.)

(b) Prove a similar formula for the exterior differential of a 2-form, and try to generalize to arbitrary k-form.

Exercise 99. Show that $\iota_X \circ \iota_X = 0$.

Exercise 100. Prove the Jacobi-identity for the Lie derivative: for any $X,Y,Z \in \mathfrak{M}$ we have

$$L_{[X,[Y,Z]]} + L_{[Y,[Z,X]]} + L_{[Z,[X,Y]]} = 0.$$

6.8.1 Pull-backs

Similar to the pull-back of functions (0-forms) and 1-forms, we have a pull-back operation for k-forms,

$$F^* : \ \Omega^k(N) \to \Omega^k(M)$$

for any smooth map between manifolds, $F \in C^\infty(M,N)$. Its evaluation at any $p \in M$ is given by

$$(F^*\beta)_p(v_1,\ldots,v_k) = \beta_{F(p)}(T_pF(v_1),\ldots,T_pF(v_k)).$$

The pull-back map satisfies $\mathrm{d}(F^*\beta) = F^*\mathrm{d}\beta$, and for a wedge product of forms,

$$F^*(\beta_1 \wedge \beta_2) = F^*\beta_1 \wedge F^*\beta_2.$$

In local coordinates, if $F : \ U \to V$ is a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, with coordinates x,y, the pull-back just amounts to 'putting $y = F(x)$'.

Exercise 101. If $F : \ \mathbb{R}^3 \to \mathbb{R}^2$ is given by $(u,v) = F(x,y,z) = (y^2z,\ x)$, compute

$$F^*(\mathrm{d}u \wedge \mathrm{d}v).$$

The next example is very important, hence we state it as a proposition. It is the 'key fact' toward the definition of an integral.

Proposition 6.3. *Let $U \subseteq \mathbb{R}^m$ with coordinates x^i, and $V \subseteq \mathbb{R}^n$ with coordinates y^j. Suppose $m = j$, and $F \in C^\infty(U,V)$. Then*

$$F^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n) = J\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

where $J(x)$ is the determinant of the Jacobian matrix,

$$J(x) = \det\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j=1}^n.$$

Proof.

$$\begin{aligned}
F^*\beta &= \mathrm{d}F^1 \wedge \cdots \wedge \mathrm{d}F^n \\
&= \sum_{i_1 \dots i_n} \frac{\partial F^1}{\partial x^{i_1}} \cdots \frac{\partial F^n}{\partial x^{i_n}} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\
&= \sum_{s \in S_n} \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^{s(1)} \wedge \cdots \wedge \mathrm{d}x^{s(n)} \\
&= \sum_{s \in S_n} \mathrm{sign}(s) \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= J\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,
\end{aligned}$$

Here we noted that the wedge product $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$ is zero unless $i_1, \dots, i_n$ are a permutation of $1, \dots, n$. $\square$

One may regard this result as giving a new, 'better' definition of the Jacobian determinant.

Remark 6.3. The Lie derivative $L_X\alpha$ of a differential form with respect to a vector field X has an important interpretation in terms of the flow Φ_t of X. Assuming for simplicity that X is complete (so that Φ_t is a globally defined diffeomorphism), one has the formula

$$L_X\alpha = \frac{d}{dt}\Big|_{t=0} \Phi_t^*\alpha.$$

(If X is incomplete, the flow Φ_t is defined only locally, but the definition still works.) To prove this identity, it suffices to check that the right hand side satisfies a product rule with respect to the wedge product of forms, and that it takes on the correct values on functions and on differentials of functions. The formula shows that L_X measures to what extent α is invariant under the flow of X.

Exercise 102. Prove the identity by following the suggestion above.

6.9 Integration of differential forms

Differential forms of top degree can be integrated over *oriented* manifolds. Let M be an oriented manifold of dimension m, and $\omega \in \Omega^m(M)$. Let $\mathrm{supp}(\omega)$ be the support of ω. [‖]

If $\mathrm{supp}(\omega)$ is contained in an oriented coordinate chart (U, φ), then one defines

$$\int_M \omega = \int_{\mathbb{R}^m} f(x)\mathrm{d}x^1 \cdots \mathrm{d}x^m$$

where $f \in C^\infty(\mathbb{R}^m)$ is the function, with $\mathrm{supp}(f) \subseteq \varphi(U)$, determined from

$$(\varphi^{-1})^*\omega = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

This definition does not depend on the choice of oriented chart. Indeed, suppose (V, ψ) is another oriented chart with $\mathrm{supp}(\omega) \subseteq V$, and write

$$(\psi^{-1})^*\omega = g\ \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m.$$

where we write $y^1, \ldots, y^m$ for the coordinates on V. Letting $F = \psi \circ \varphi^{-1}$ be the change of coordinates $y = F(x)$, Proposition 6.3 says that

$$F^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m) = J(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

where $J(x) = \det(DF(x))$ is the determinant of the Jacobian matrix of F at x. Hence, $f(x) = g(F(x))J(x)$, and we obtain

$$\int_{\psi(U)} g(y)\mathrm{d}y^1 \cdots y^m = \int_{\varphi(U)} g(F(x))J(x)\mathrm{d}x^1 \cdots \mathrm{d}x^m = \int_{\varphi(U)} f(x)\mathrm{d}x^1 \cdots \mathrm{d}x^m,$$

as required.

Remark 6.4. Here we used the *change-of-variables formula* from multivariable calculus. It was very important that the charts are oriented, so that $J > 0$ everywhere. Indeed, for general changes of variables, the change-of-variables formula involves $|J|$ rather than J itself.

If ω is not necessarily supported in a single oriented chart, we proceed as follows. Let $(U_i, \varphi_i),\ \ i = 1, \ldots, r$ be a finite collection of oriented charts covering $\mathrm{supp}(\omega)$. Together with $U_0 = M \backslash \mathrm{supp}(\omega)$ this is an open cover of M.

Lemma 6.5. *Given a finite open cover of a manifold there exists a* partition of unity subordinate to the cover, *i.e. functions* $\chi_i \in C^\infty(M)$ *with* $\mathrm{supp}(\chi_i) \subseteq U_i$ *and* $\sum_{i=0}^r \chi_i = 1$.

[‖] The support of a form is defined similar to the support of a function, or support of a vector field. For any differential form $\alpha \in \Omega^k(M)$, we define the *support* $\mathrm{supp}(\alpha)$ to be the smallest closed subset of M outside of which α is zero. (Equivalently, it is the *closure* of the subset over which α is non-zero.)

Indeed, partitions of unity exists for *any* open cover, not only finite ones. A proof is given in the appendix on 'topology of manifolds'.

Let $\chi_0,\ldots,\chi_r$ be a partition of unity subordinate to this cover. We define

$$\int_M \omega = \sum_{i=1}^r \int_M \chi_i \omega$$

where the summands are defined as above, since $\chi_i\omega$ is supported in U_i for $i \geq 1$. (We didn't include the term for $i = 0$, since $\chi_0\omega = 0$.) We have to check that this is well-defined, independent of the choices. Thus, let (V_j, ψ_j) for $j = 1,\ldots,s$ be another collection of oriented coordinate charts covering $\mathrm{supp}(\omega)$, put $V_0 = M - \mathrm{supp}(\omega)$, and let $\sigma_0,\ldots,\sigma_s$ a corresponding partition of unity subordinate to the cover by the V_i's.

Then the $U_i \cap V_j$ form an open cover, with the collection of $\chi_i\sigma_j$ as a partition of unity. We obtain

$$\sum_{j=1}^s \int_M \sigma_j \omega = \sum_{j=1}^s \int_M (\sum_{i=1}^r \chi_i)\, \sigma_j \omega = \sum_{j=1}^s \sum_{i=1}^r \int_M \sigma_j \chi_i \omega.$$

This is the same as the corresponding expression for $\sum_{i=1}^r \int_M \chi_i \omega$.

6.10 Integration over oriented submanifolds

Let M be a manifold, not necessarily oriented, and S is a k-dimensional oriented submanifold, with inclusion $i\colon\ S \to M$. We define the integral over S, of any k-form $\omega \in \Omega^k(M)$ such that $S \cap \mathrm{supp}(\omega)$ is compact, as follows:

$$\int_S \omega = \int_S i^* \omega.$$

Of course, this definition works equally well for *any* smooth map from S into M. For example, the integral of compactly supported 1-forms along arbitrary paths $\gamma\colon\ \mathbb{R} \to M$ is defined. Note also that M itself does not have to be oriented, it suffices that S is oriented.

6.11 Stokes' theorem

Let M be an m-dimensional oriented manifold.

Definition 6.10. *A* region with (smooth) boundary *in M is a closed subset $D \subseteq M$ with the following property: There exists a smooth function $f \in C^\infty(M,\mathbb{R})$ such that* 0 *is a regular value of f, and*

$$D = \{p \in M \mid f(p) \leq 0\}.$$

We do not consider f itself as part of the definition of D, only the existence of f is required. The interior of a region with boundary, given as the largest open subset contained in D, is $\mathrm{int}(D) = \{p \in M | \, f(p) < 0$, and the boundary itself is

$$\partial D = \{p \in M | \, f(p) = 0\},$$

a codimension 1 submanifold (i.e., hypersurface) in M.

Example 6.1. The region with bounday defined by the function $f \in C^\infty(\mathbb{R}^2)$, given by $f(x,y) = x^2 + y^2 - 1$, is the unit disk $D \subseteq \mathbb{R}^2$; its boundary is the unit circle.

Example 6.2. Recall that for $0 < r < R$, zero is a regular value of the function on $\mathbb{R}^3$,

$$f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - R)^2 - r^2.$$

The corresponding region with boundary $D \subseteq \mathbb{R}^3$ is the solid torus, its boundary is the torus.

Recall that we are considering D inside an *oriented* manifold M. The boundary ∂D may be covered by oriented submanifold charts (U, φ), in such a way that ∂D is given in the chart by the condition $x^1 = 0$, and D by the condition $x^1 \leq 0$: **

$$\varphi(U \cap D) = \varphi(U) \cap \{x \in \mathbb{R}^m | \, x^1 \leq 0\}.$$

(Indeed, given an oriented submanifold chart for which D lies on the side where $x_1 \geq 0$, one obtains a region chart by composing with the orientation-preserving coordinate change $(x^1, \ldots, x^m) \mapsto (-x^1, -x^2, x^3 \ldots, x^m)$.) We call oriented submanifold charts of this kind 'region charts'.††

Lemma 6.6. *The restriction of the region charts to ∂D form an oriented atlas for ∂D.*

Proof. Let (U, φ) and (V, ψ) be two region charts, defining coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^m$, and let $F = \psi \circ \varphi^{-1} : \ \varphi(U \cap V) \to \psi(U \cap V), \ x \mapsto y = F(x)$. It restricts to a map

$$F_1 : \ \{x \in \varphi(U \cap V) | \, x_1 = 0\} \to \{y \in \psi(U \cap V) | \, y_1 = 0\}.$$

Since $y^1 > 0$ if and only if $x^1 > 0$, the change of coordinates satisfies

$$\frac{\partial y^1}{\partial x^1}\Big|_{x^1=0} > 0, \quad \frac{\partial y^1}{\partial x^j}\Big|_{x^1=0} = 0, \quad \text{for } j > 0.$$

** Note that while we originally defined submanifold charts in such a way that the last $m - k$ coordinates are zero on S, here we require that the first coordinate be zero. It doesn't matter, since one can simply reorder coordinates, but works better for our description of the 'induced orientation'.

†† This is not a standard name.

Hence, the Jacobian matrix $DF(x)|_{x^1=0}$ has a positive $(1,1)$ entry, and all other entries in the first row equal to zero. Using expansion of the determinant across the first row, it follows that

$$\det(DF(0,x^2,\ldots,x^m)) = \frac{\partial y^1}{\partial x^1}\Big|_{x^1=0} \det(DF'(x^2,\ldots,x^m)).$$

which shows that $\det(DF') > 0$.

In particular, ∂D is again an oriented manifold. To repeat: If $x^1,\ldots,x^m$ are local coordinates near $p \in \partial D$, compatible with the orientation and such that D lies on the side $x^1 \le 0$, then $x^2,\ldots,x^m$ are local coordinates on ∂D. This convention of 'induced orientation' is arranged in such a way that the Stokes' theorem holds without extra signs.

For an m-form ω such that $\mathrm{supp}(\omega) \cap D$ is compact, the integral

$$\int_D \omega$$

is defined similar to the case of $D = M$: One covers $D \cap \mathrm{supp}(\omega)$ by finitely many submanifold charts (U_i, φ_i) with respect to ∂D (this includes charts that are entirely in the interior of D), and puts

$$\int_D \omega = \sum \int_{D\cap U_i} \chi_i \omega$$

where the χ_i are supported in U_i and satisfy $\sum_i \chi_i$ over $D \cap \mathrm{supp}(\omega)$. By the same argument as for $D = M$, this definition of the integral is independent of the choice made.

Theorem 6.3 (Stokes' theorem). *Let M be an oriented manifold of dimension m, and $D \subseteq M$ a region with smooth boundary ∂D. Let $\alpha \in \Omega^{m-1}(M)$ be a form of degree $m-1$, such that* $\mathrm{supp}(\alpha) \cap D$ *is compact. Then*

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha.$$

As explained above, the right hand side means $\int_{\partial D} i^*\alpha$, where $i\colon\ \partial D \hookrightarrow M$ is the inclusion map.

Proof. We will see that Stokes' theorem is just a coordinate-free version of the fundamental theorem of calculus. Let (U_i, φ_i) for $i = 1,\ldots,r$ be a finite collection of region charts covering $\mathrm{supp}(\alpha) \cap D$. Let $\chi_1,\ldots,\chi_r \in C^\infty(M)$ be functions with $\chi_i \ge 0$, $\mathrm{supp}(\chi_i) \subseteq U_i$, and such that $\chi_1 + \ldots + \chi_r$ is equal to 1 on $\mathrm{supp}(\alpha) \cap D$. (E.g., we may take $U_1,\ldots,U_r$ together with $U_0 = M\backslash\mathrm{supp}(\omega)$ as an open covering, and take the $\chi_0,\ldots,\chi_r \in C^\infty(M)$ to be a partition of unity subordinate to this cover.) Since

$$\int_D \mathrm{d}\alpha = \sum_{i=1}^r \int_D \mathrm{d}(\chi_i\alpha), \qquad \int_{\partial D} \alpha = \sum_{i=1}^r \int_{\partial D} \chi_i \alpha,$$

it suffices to consider the case that α is supported in a region chart.

Using the corresponding coordinates, it hence suffices to prove Stokes' theorem for the case that $\alpha \in \Omega^{m-1}(\mathbb{R}^m)$ is a compactly supported form in $\mathbb{R}^m$, and $D = \{x \in \mathbb{R}^m | x^1 \leq 0\}$. That is, α has the form

$$\alpha = \sum_{i=1}^{m} f_i \, \mathrm{d}x^1 \wedge \cdots \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m,$$

with compactly supported f_i where the hat means that the corresponding factor is to be omitted. Only the $i = 1$ term contributes to the integral over $\partial D = \mathbb{R}^{m-1}$, and

$$\int_{\mathbb{R}^{m-1}} \alpha = \int f_1(0, x^2, \ldots, x^m) \, \mathrm{d}x^2 \cdots \mathrm{d}x^m.$$

On the other hand,

$$\mathrm{d}\alpha = \Big(\sum_{i=1}^{m} (-1)^{i+1} \, \frac{\partial f_i}{\partial x^i} \Big) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

Let us integrate each summand over the region D given by $x^1 \leq 0$. For $i > 1$, we have

$$\int_{\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_i}{\partial x_i} (x^1, \ldots, x^m) \mathrm{d}x^1 \cdots \mathrm{d}x^m = 0$$

where we used Fubini's theorem to carry out the x^i-integration first, and applied the fundamental theorem of calculus to the x^i-integration (keeping the other variables fixed, the integrand is the derivative of a compactly supported function). It remains to consider the case $i = 1$. Here we have, again by applying the fundamental theorem of calculus,

$$\begin{aligned} \int_D \mathrm{d}\alpha &= \int_{\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1} (x^1, \ldots, x^m) \mathrm{d}x^1 \cdots \mathrm{d}x^m \\ &= \int_{\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(0, x^2, \ldots, x^m) \mathrm{d}x^2 \cdots \mathrm{d}x^m = \int_{\partial D} \alpha \end{aligned}$$

□

As a special case, we have

Corollary 6.1. *Let* $\alpha \in \Omega^{m-1}(M)$ *be a compactly supported form on the oriented manifold* M. *Then*

$$\int_M \mathrm{d}\alpha = 0.$$

Note that it does not suffice that $\mathrm{d}\alpha$ has compact support. For example, if $f(t)$ is a function with $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for $t > 0$, then $\mathrm{d}f$ has compact support, but $\int_{\mathbb{R}} \mathrm{d}f = 1$.

A typical application of Stokes' theorem shows that for a closed form $\omega \in \Omega^k(M)$, the integral of ω over an oriented compact submanifold does not change with smooth deformations of the submanifold.

Theorem 6.4. *Let $\omega \in \Omega^k(M)$ be a closed form on a manifold M, and S a compact, oriented manifold of dimension k. Let $F \in C^\infty(\mathbb{R} \times S, M)$ be a smooth map, thought of as a smooth family of maps*

$$F_t = F(t, \cdot) : \ S \to M.$$

Then the integrals

$$\int_S F_t^* \omega$$

do not depend on t.

If F_t is an embedding, then this is the integral of ω over the submanifold $F_t(S) \subseteq M$.

Proof. Let $a < b$, and consider the domain $D = [a,b] \times S \subseteq \mathbb{R} \times S$. The boundary ∂D has two components, both diffeomorphic to S. At $t = b$ the orientation is the given orientation on S, while at $t = a$ we get the opposite orientation. Hence,

$$0 = \int_D F^* \mathrm{d}\omega = \int_D \mathrm{d}F^* \omega = \int_{\partial D} F^* \omega = \int_S F_b^* \omega - \int_S F_a^* \omega.$$

Hence $\int_S F_b^* \omega = \int_S F_a^* \omega$. $\square$

Remark 6.5. Note that if one member of this family of maps, say the map F_1, takes values in a $k-1$-dimensional submanifold (for instance, if F_1 is a constant map), then $F_1^* \omega = 0$. (Indeed, the assumption means that $F_1 = j \circ F_1'$, where j is the inclusion of a $k-1$-submanifold and F_1' takes values in that submanifold. But $j^* \omega = 0$ for degree reasons.) It then follows that $\int_S F_t^* \omega = 0$ for all t.

Given a smooth map $\varphi : \ S \to M$, one refers to a smooth map $F : \ \mathbb{R} \times S \to M$ with $F_0 = \varphi$ as an 'smooth deformation' (or 'isotopy') of φ. We say that φ can be smoothly deformed into φ' if there exists a smooth isotopy F with $\varphi = F_0$ and $\varphi' = F_1$. The theorem shows that if S is oriented, and if there is a closed form $\omega \in \Omega^k(M)$ with

$$\int_S \varphi^* \omega \neq \int_S (\varphi')^* \omega$$

then φ cannot be smoothly deformed into φ'. This observation has many applications; here are some of them. ‡‡

Example 6.3. Suppose $\varphi : \ S \to M$ is a smooth map, where S is oriented of dimension k, and $\omega \in \Omega^k(M)$ is closed. If $\int_S \varphi^* \omega \neq 0$, then φ cannot smoothly be 'deformed' into a map taking values in a lower-dimensional submanifold. (In particular it cannot be deformed into a constant map.) Indeed, if φ' takes values in a lower-dimensional submanifold, then $\varphi' = j \circ \varphi_1'$ where j is the inclusion of that submanifold. But then $j^* \omega = 0$, hence $(\varphi')^* \omega = 0$. For instance, the inclusion $\varphi : \ S^2 \to M = \mathbb{R}^3 \backslash \{0\}$ cannot be smoothly deformed inside M so that φ' would take values in $\mathbb{R}^2 \backslash \{0\} \subseteq \mathbb{R}^3 \backslash \{0\}$.

‡‡ You may wonder if it is still possible to find a continuous deformation, rather than smooth. It turns out that it doesn't help: Results from differential topology show that two smooth maps can be smoothly deformed into each other if and only if they can be continuously deformed into each other.

Example 6.4 (Winding number). Let $\omega \in \Omega^2(\mathbb{R}^2\backslash\{0\})$ be the 1-form

$$\omega = \frac{1}{x^2+y^2}(x\mathrm{d}y - y\mathrm{d}x)$$

In polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, one has that $\omega = \mathrm{d}\theta$. Using this fact one sees that ω is closed (but not exact, since θ is not a globally defined function on $\mathbb{R}^2\backslash\{0\}$.) Hence, if

$$\gamma :\ S^1 \to \mathbb{R}^2\backslash\{0\}$$

is any smooth map (a 'loop'), then the integral

$$\int_{S^1} \gamma^*\omega$$

does not change under deformations (isotopies) of the loop. In particular, γ cannot be deformed into a constant map, unless the integral is zero. The number

$$w(\gamma) = \frac{1}{2\pi}\int_{S^1} \gamma^*\omega$$

is the *winding number* of γ. (One can show that this is always an integer, and that two loops can be deformed into each other if and only if they have the same winding number.)

Example 6.5 (Linking number). Let $f,g :\ S^1 \to \mathbb{R}^3$ be two smooth maps whose images don't intersect, that is, with $f(z) \neq g(w)$ for all $z,w \in S^1$ (we regard S^1 as the unit circle in $\mathbb{C}$). Define a new map

$$F :\ S^1 \times S^1 \to S^2,\ \ (z,w) \mapsto \frac{f(z)-g(w)}{||f(z)-g(w)||}.$$

On S^2, we have a 2-form ω of total integral 1. It is the pullback of

$$\frac{1}{4\pi}\big(x\mathrm{d}y\wedge\mathrm{d}z - y\mathrm{d}x\wedge\mathrm{d}z + z\mathrm{d}x\wedge\mathrm{d}y\big) \in \Omega^2(\mathbb{R}^3)$$

to the 2-sphere. The integral

$$L(f,g) = \int_{S^1\times S^1} F^*\omega$$

is called the *linking number* of f and g. (One can show that this is always an integer.) Note that if it is possible to deform one of the loops, say f, into a constant loop through loops that are always disjoint from g, then the linking number is zero. In his case, we consider f,g as 'unlinked'.

Example 6.6. Let M be a compact, connected oriented manifold. There exists a differential form ω on M such that $\int_M \omega = 1$. (Note that ω cannot be exact, since otherwise the integral would be zero, by Stokes.) Given another compact, oriented manifold N of the *same* dimension (e.g., M itself), and a smooth map $F :\ N \to M$, we can define the *degree of F*

$$\deg(F) = \int_N F^*\omega.$$

The degree is invariant under deformations of F. It turns out that it is independent of the choice of ω, and is always an integer. In the preceding example, the linking number was defined as the degree of a map $S^1 \times S^1 \to S^2$ obtained from f, g.

6.12 Volume forms

A top degree differential form $\Gamma \in \Omega^m(M)$ is called a *volume form* if it is non-vanishing everywhere: $\Gamma_p \neq 0$ for all $p \in M$. In a local coordinate chart (U, φ), this means that

$$(\varphi^{-1})^*\Gamma = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f(x) \neq 0$ for all $x \in \varphi(U)$.

Example 6.7. The Euclidean space $\mathbb{R}^n$ has a *standard volume form* $\Gamma_0 = dx^1 \wedge \cdots \wedge \mathrm{d}x^n$. Suppose $S \subseteq \mathbb{R}^n$ is a submanifold of dimension $n-1$, and X a vector field that is *nowhere tangent* to S. Let $i:\ S \to \mathbb{R}^n$ be the inclusion. Then

$$\Gamma := i^*\big(\iota_X \Gamma_0\big) \in \Omega^{n-1}(S)$$

is a volume form. For instance, if S is given as a level set $f^{-1}(0)$, where 0 is a regular value of f, then the gradient vector field

$$\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}$$

has this property.

Exercise: Verify the claim that $\Gamma := i^*\big(\iota_X \Gamma_0\big)$ is a volume form.

Example 6.8. Let $i:\ S^n \to \mathbb{R}^{n+1}$ be the inclusion of the standard n-sphere. Let $X = \sum_{i=0}^n x^i \frac{\partial}{\partial x^i}$. Then

$$\iota_X(\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n) = \sum_{i=0}^n (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n$$

pulls back to a volume form on S^n.

Lemma 6.7. *A volume form $\Gamma \in \Omega^m(M)$ determines an orientation on M, by taking as the oriented charts those charts (U, φ) such that $(\varphi^{-1})^*\Gamma = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ with $f > 0$ everywhere on $\Phi(U)$.*

Proof. We have to check that the condition is consistent. Suppose (U,φ) and (V,ψ) are two charts, where $(\varphi^{-1})^*\Gamma = f\, \mathrm{d}x^1\wedge\cdots\wedge \mathrm{d}x^m$ and $(\psi^{-1})^*\Gamma = g\, \mathrm{d}y^1\wedge\cdots\wedge \mathrm{d}y^m$ with $f>0$ and $g>0$. If $U\cap V$ is non-empty, let $F=\psi\circ\varphi^{-1}:\ \varphi(U)\to\psi(V)$ be the transition function. Then

$$F^*(\psi^{-1})^*\Gamma|_{U\cap V} = (\varphi^{-1})^*\Gamma|_{U\cap V},$$

hence

$$g(F(x))\, J(x)\, \mathrm{d}x^1\wedge\cdots\wedge \mathrm{d}x^m = f(x)\, \mathrm{d}x^1\wedge\cdots\wedge \mathrm{d}x^m.$$

where J is that Jacobian determinant of the transition map $F=\psi\circ\varphi^{-1}$. Hence $f = J\,(g\circ F)$ on $\varphi(U\cap V)$. Since $f>0$ and $g>0$, it follows that $J>0$. Hence the two charts are oriented compatible. □

Theorem 6.5. *A manifold M is orientable if and only if it admits a volume form. In this case, any two volume forms compatible with the orientation differ by an everywhere positive smooth function:*

$$\Gamma' = f\Gamma, \quad f>0.$$

Proof. As we saw above, any volume form determines an orientation. Conversely, if M is an oriented manifold, there exists a volume form compatible with the orientation: Let $\{(U_\alpha,\varphi_\alpha)\}$ be an oriented atlas on M. Then each

$$\Gamma_\alpha = \varphi_\alpha^*(\mathrm{d}x^1\wedge\cdots\wedge \mathrm{d}x^m)\in\Omega^m(U_\alpha)$$

is a volume form on U_α; on overlaps $U_\alpha\cap U_\beta$ these are related by the Jacobian determinants of the transition functions, which are *strictly positive* functions. Let $\{\chi_\alpha\}$ be a locally finite partition of unity subordinate to the cover $\{U_\alpha\}$, see Appendix A.4. The forms $\chi_\alpha\Gamma_\alpha$ have compact support in U_α, hence they extend by zero to global forms on M (somewhat imprecisely, we use the same notation for this extension). The sum

$$\Gamma = \sum_\alpha \chi_\alpha\Gamma_\alpha \in \Omega^m(M)$$

is a well-defined volume form. Indeed, near any point p at least one of the summands is non-zero; and if other summands in this sum are non-zero, they differ by a positive function.

For a compact manifold M with a given volume form $\Gamma\in\Omega^m(M)$, one can define the volume of M,

$$\mathrm{vol}(M) = \int_M \Gamma.$$

Here the orientation used in the definition of the integral is taken to be the orientation given by Γ. Thus $\mathrm{vol}(M)>0$.

Note that volume forms are always closed, for degree reasons (since $\Omega^{m+1}(M)=0$). But on a compact manifold, they cannot be exact:

Theorem 6.6. *Let M be a compact manifold with a volume form $\Gamma \in \Omega^m(M)$. Then Γ cannot be exact.*

Proof. We have $\mathrm{vol}(M) = \int_M \Gamma > 0$. But if Γ were exact, then Stokes' theorem would give $\int_M \Gamma = 0$.

Of course, the compactness of M is essential here: For instance, $\mathrm{d}x$ is an exact volume form on $\mathbb{R}$.